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# ON SELF-INTERSECTION NUMBER OF A SECTION ON A RULED SURFACE

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***To Professor K. Ono for his sixtieth birthday***

Let $E$ be a non-singular projective curve of genus $g \geq 0$, $\boldsymbol{P}$ the projective line and let $F$ be the surface $E \times \boldsymbol{P}$. Then it is well known that a ruled surface $F^*$ which is birational to $F$ is biregular to a surface which is obtained by successive elementary transformations from $F$ (for the notion of an elementary transformation, see [3]). The main purpose of the present article is to prove the following

THEOREM 1. *For any such $F^*$, there is a section (i.e., an irreducible curve $s$ on $F$ such that $(s, l) = 1$ for a fibre $l$ of $F^*$) such that its self-intersection number $(s, s)$ is not greater than $g$.*

In classifying ruled surface $F^*$, as was noted by Atiyah [1], it is important to know the minimum value of self-intersection numbers $(s, s)$ of sections of $F^*$.[1] Our Theorem 1 is important in the respect.

The following is a key to our proof of Theorem 1:

THEOREM 2. *Let $d$ be a non-negative rational integer. If $Q_1, \cdots, Q_{g+2d+1}$ are points[2] of $F$, then there is a positive divisor $D$ of $F$ such that* (i) *$D$ goes through $Q_1, \cdots, Q_{g+2d+1}$ and* (ii) *$D$ is linearly equivalent to $E \times P + \sum_{i=1}^{g+d} R_i \times \boldsymbol{P}$ with a $P \in \boldsymbol{P}$ and suitable $R_i \in E$.*

In connection with this Theorem 2, we prove the following theorem too:

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1) Atiyah proved that the minimum value is not greater than $2g-1$ if $g>0$. On the other hand, it was remarked by M. Maruyama that there is an $F$ (for every $E$) which carries only sections $s$ such that $(s, s) \geqq g$ (see [2]).

2) In this theorem, these $Q_i$ need not be ordinary points, namely, some of these $Q_i$ may be infinitely near points of some ordinary points. For the definition of the term "go through" in such a case, see [3].

THEOREM 3. *Let $Q_1^*, \cdots, Q_{t+1}^*$ be independent generic points of $F$ over a field of definition $k$ of $F$. Let $S^*$ be the set of positive divisors $D$ of $F$ such that* (i) *$D$ goes through $Q_1^*, \cdots, Q_{t+1}^*$ and* (ii) *$D$ is linearly equivalent to $E \times P + \sum_{i=1}^t R_i \times \boldsymbol{P}$ with a $P \in \boldsymbol{P}$ and suitable $R_i \in E$. If $t \leq g$, then $S^*$ is not empty and $S^*$ dose not contain any algebraic family of positive dimension.*

In appendix, we add some remarks on dimensions of algebraic families.

## 1. Some preliminary results, notation.

Since the case where $g = 0$ is obvious, we assume that $g \geq 1$. $P$ (or $P'$) denotes a point of $\boldsymbol{P}$. $R$ (or $R_i$, $R'_j$, $R^*_t$, etc.) denotes a point of $E$. $Q$ (or $Q_i$, $Q'_j$, etc.) denotes a point of $F$. $k$ is a field of definition for $E$ and $F$, and for the sake of simplicity, we assume that $k$ is algebraically closed. $L(R_1, \cdots, R_s)$ is the complete linear system $|E \times P + \sum_{i=1}^s R_i \times \boldsymbol{P}|$. Specializations are understood with reference to $k$. For fundamentals on specializations of cycles, see [4] and [5].

LEMMA 1. *Let $d$ be the dimension of the complete linear system $|\sum_{i=1}^s R_i|$ on $E$. Let $\sum_{i=1}^s R_i^*$ be a generic member of the linear system over a field containing $k$ and let $C^*$ be a generic member of $L(R_1, \cdots, R_s)$ over $k(R_1^*, \cdots, R_s^*)$. Then*

(i) $\dim L(R_1, \cdots, R_s) = 2d + 1$,

(ii) $\text{trans.}\deg_k k(C^*) = d + 1 + \text{trans.}\deg_k k(R_1^*, \cdots, R_s^*)$,

(iii) *if $\dim |\sum_{i=1}^s R'_i| = d$ and if $(R'_1, \cdots, R'_s)$ is a specialization of $(R_1^*, \cdots, R_s^*)$ then every member of $L(R'_1, \cdots, R'_s)$ is a specialization of $C^*$ over the specialization $(R_1^*, \cdots, R_s^*) \to (R'_1, \cdots, R'_s)$.*

*Proof.* Consider $E' = E \times P$. Then $\dim \mathrm{Tr}_{E'} L(R_1, \cdots, R_s) = d = \dim(L(R_1, \cdots, R_s) - E')$, from which (i) follows readily. Now, consider loci $T$ and $U$ of $(C^*, R_1^*, \cdots, R_s^*)$ and $C^*$ respectively, over $k$. Then $\dim T = \text{trans.}\deg_k k(R_1^*, \cdots, R_s^*) + \text{trans.}\deg_{k(R_1^*, \ldots, R_s^*)} k(C^*)$, and on the other hand, letting $p$ denote the natural projection from $T$ onto $U$, we have $\dim p^{-1}(C^*) = \dim |\sum_{i=1}^s R_i| = d$. Therefore $\text{trans.}\deg_k k(C^*) = \dim U = \dim T - d = d+1+\text{trans.}\deg_k k(R_1^*, \cdots, R_s^*)$, which proves (ii). As for (iii), we consider a specialization of $(C^*, R_1^*, \cdots, R_s^*, L(R_1, \cdots, R_s))$ over the specialization $(R_1^*, \cdots, R_s^*) \to (R'_1, \cdots, R'_s)$. $E \times P + \sum_i R_i^* \times \boldsymbol{P}$ is specialized to $E \times P' + \sum R'_i \times \boldsymbol{P}$, which must be a member of the specialization $L^*$ of $L(R_1, \cdots, R_s)$. Since $\dim L^* = \dim L(R_1, \cdots, R_s) = d = \dim L(R'_1, \cdots, R'_s)$ and since all

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members of $L^*$ are linearly equivalent to each other,[3] we see that $L^* = L(R'_1, \cdots, R'_s)$. Thus Lemma 1 is proved.

LEMMA 2. *Let $V$ be a surface defined over $k$. If $M_1, \cdots, M_n$ are points of $V$ and if* trans. $\deg_k k(M_1, \cdots, M_n) \geq 2n - \alpha$, *then suitable $n - \alpha$ points among $M_1, \cdots, M_n$ are independent generic points of $V$ over $k$.*

*Proof.* We use induction argument on $n$. (1) If $M_n$ is a generic point of $V$ over $k(M_1, \cdots, M_{n-1})$, then trans. $\deg_k k(M_1, \cdots, M_{n-1}) \geq 2(n-1) - \alpha$. Then, by our induction assumption, there are $n - 1 - \alpha$ independent generic points among $M_1, \cdots, M_{n-1}$ and we see the assertion in this case. (2) Otherwise, we have trans. $\deg_k k(M_1, \cdots, M_{n-1}) \geq 2(n-1) - (\alpha - 1)$, and we completes the proof by our induction assumption.

## 2. Proof of Theorem 2.

Let $R_1^*, \cdots, R_{g+d}^*$ be independent generic points of $E$ over $k$ and let $C^*$ be a generic member of $L(R_1^*, \cdots, R_{g+d}^*)$ over $k(R_1^*, \cdots, R_{g+d}^*)$. Let $Q_1^*, \cdots, Q_{2g+2d+1}^*$ be independent generic points of $C^*$ over $k(C^*)$. Then by Lemma 1, trans. $\deg_k k(C^*, Q_1^*, \cdots, Q_{2g+2d+1}^*)$ = trans. $\deg_k k(C^*) + 2g + 2d + 1 = d + 1 + d + g + 2g + 2d + 1 = 3g + 4d + 2 = 2(2g + 2d + 1) - g$. Now we consider locus $T$ of $(C^*, Q_1^*, \cdots, Q_{2g+2d+1}^*)$ and the natural projection pr from $T$ into the $(2g + 2d + 1)$-ple product $F''$ of $F$. Since the self-intersection number $(C^*, C^*)$ of $C^*$ is equal to $2g + 2d$, we see that pr is generically a one-one correspondence between $T$ and pr $T$, which shows that $\dim T = \dim$ pr $T$. Therefore, applying Lemma 2 with $n = 2g + 2d + 1$, we see that there are $g + 2d + 1$ independent generic points of $F$ among $Q_1^*, \cdots, Q_{2g+2d+1}^*$. This proves Theorem 2 in the case where $Q_1, \cdots, Q_{g+2d+1}$ are independent generic points of $F$. New we complete the proof making use of specializations.

## 3. Proof of Theorem 1.

As was noted at the beginning, $F^*$ is obtained by successive elementary transformations with centers, say, $P_1, \cdots, P_m$ from $F$. If $m \leq g$, then the proper transform of an $E \times P$ has self-intersection number $\leq g$. Therefore we assume that $m > g$. Then there is $d$ such that $m = g + 2d$ or $m = g + 2d + 1$. By virtue of Theorem 2, there is a positive divisor $D$ of $F$ such that (i)

[3] Note that if $D$ and $D'$ are divisors which are linearly equivalent to each other, and if they are specialized to $D_1$ and $D'_1$ under the same specialization, then $D_1$ is linearly equivalent to $D'_1$.

$D$ goes through $P_1, \cdots, P_m$ and (ii) $D$ is linearly equivalent to $E \times P + \sum_{i=1}^{g+d} R_i \times \boldsymbol{P}$. Then the proper transform $D'$ of $D$, or more precisely, the divisor of $F^*$ which is the transform of $D - \sum P_i$, has self-intersection number $2g + 2d - m$, which is either $g$ or $g - 1$. $D'$ has a section $s$ of $F^*$ as a component, and $(s, s) \leq g$. This completes our proof of Theorem 1.

## 4. Proof of Theorem 3.

Let $P$ and $R_i (i = 1, \cdots, t)$ be such that $Q_i^* \in R_i \times \boldsymbol{P}$ and $Q_{t+1}^* \in E \times P$. Then $E \times P + \sum_{i=1}^{t} R_i \times \boldsymbol{P}$ is in $S^*$, and therefore $S^*$ is not empty. Assume now that there is an irreducible algebraic family $S$ of positive dimension contained in $S^*$. Let $C$ be a generic member of $S$ over $k(Q_1^*, \cdots, Q_{t+1}^*)$ and let $R_i'$ be such that $C \in L(R_1', \cdots, R_t')$. Let $\sum_{i=1}^{t} R_i''$ be a generic member of $|\sum_i R_i'|$ over $k(Q_1^*, \cdots, Q_{t+1}^*, R_1', \cdots, R_t')$ and let $C''$ be a generic member of $L(R_1', \cdots, R_t')$ over $k(Q_1^*, \cdots, Q_{t+1}^*, R_1', \cdots, R_t', R_1'', \cdots, R_t'')$. Let $U$ be the locus of $C''$ over $k$ and set $d = \dim |\sum_{i=1}^{t} R_i'|$. Lemma 1 shows that $\dim U = \text{trans.}\deg_k k(C'') = d + 1 + \text{trans.}\deg_k k(R_1'', \cdots, R_t'')$. Set $u = \text{trans.}\deg_k k(R_1'', \cdots, R_t'')$. Then we may assume that $R_1'', \cdots, R_u''$ are independent generic points of $E$ over $k$. Since $t \leq g$, $\dim |\sum_{i=1}^{u} R_i''| = 0$, whence $d = \dim |\sum_{i=1}^{t} R_i''| \leq t - u$. Thus we have that $\dim U \leq t - u + 1 + u = t + 1$. Since $U$ is defined over $k$ and since $Q_1^*, \cdots, Q_{t+1}^*$ are independent generic points, $\dim S \leq t + 1 - (t + 1) = 0$. This completes our proof of Theorem 3.

## Appendix

Our proof of Theorem 2 above really gives a proof of the following fact:

THEOREM A1. *Let $\mathfrak{F}$ be an algebraic family of positive divisors on a projective variety $V$. If $\dim \mathfrak{F} \geq d$ and if $P_1, \cdots, P_d$ are points of $V$, then there is a member $D$ of $\mathfrak{F}$ such that $P_i \in D$ for all $i$.*

If $\mathfrak{F}$ is a linear system, then, for a point $P$ of $V$, $\{D \in \mathfrak{F} | P \in D\}$ forms a hyperplane of $\mathfrak{F}$ if $\mathfrak{F}$ is viewed as a projective space of dimension $d$. Therefore if $\mathfrak{F}$ is a linear system, then Theorem A1 is obvious and is well known. But, in the general case, the same reasoning cannot be given. Furthermore, if $\mathfrak{F}$ is an algebraic family of $r$-cycles ($\neq$ divisors), then the dimension defect by the condition to go through one point is not uniform. For instance, let $V$ be the projective space of dimension $n$ and let $\mathfrak{F}$ be the family of $m$ points which are colinear $(m \geq 3)$, then $\dim \mathfrak{F} = 2(n-1) + m$.

For $\mathfrak{F}' = \{D \in \mathfrak{F} | P \in D\}$ (where $P$ is a point of $V$), $\dim \mathfrak{F}' = \dim \mathfrak{F} - n$. For $\mathfrak{F}'' = \{D \in \mathfrak{F}' | P' \in D\}$ (where $P'$ is a point of $V$ which is different from $P$), $\dim \mathfrak{F}'' = \dim \mathfrak{F}' - n$. But then, if $P''$ is a point of $V$ which is different from $P, P'$, (i) if $P''$ is in outside of the line going through $P, P'$, then $\mathfrak{F}^* = \{D \in \mathfrak{F}'' | \mathfrak{P}'' \in D\}$ is empty, (ii) otherwise, $\dim \mathfrak{F}^* = \dim \mathfrak{F}'' - 1$.

Here we shall discuss such dimension defect in the general case. Our result will give another proof of Theorem A1 above.

From now on, let $V$ be a projective variety of dimension $n$ and let $\mathfrak{F}$ be an (irreducible) algebraic family of positive $r$-cycles on $V$. We fix an algebraically closed, common field of definition $k$ for $V$ and $\mathfrak{F}$. Let $C^*$ be a generic member of $\mathfrak{F}$ over $k$ and let $P$ be a point of $V$. Denote by $\mathfrak{F} - P$ the set $\{C \in \mathfrak{F} | P \in C\}$.

Assume that there is a member $C$ of $\mathfrak{F} - P$. Then there is a point $P^*$ of $C^*$ such that $(C^*, P^*)$ is specialized to $(C, P)$. Let $U$ be the locus of $P^*$ over $k$. Then

THEOREM A2. *There is an algebraic family* $\mathfrak{F}'$ *such that* (1) $C \in \mathfrak{F}' \subseteq \mathfrak{F} - P$ *and* (2) $\dim \mathfrak{F}' = \dim \mathfrak{F} + \dim (U \cap C^*) - \dim U$.

*Proof.* To begin with, we may assume that $P^*$ is a generic point of an arbitrarily fixed component of $C^* \cap U$ over $k(C^*)$, whence we may assume that $\dim (U \cap C^*) = \text{trans.} \deg_{k(C^*)} k(C^*, P^*)$. Let $W$ and $T$ be the locus of $C^*$ over $k(P^*)$ and the locus of $(C^*, P^*)$ over $k$ respectively. Then $\dim U + \dim W = \text{trans.} \deg k(P^*) + \text{trans.} \deg_{k(P^*)} k(P^*, C^*) = \dim T = \dim F + \dim (U \cap C^*)$. Thus $\dim W = \dim F + \dim (U \cap C^*) - \dim U$. Consider a specialization $W \to W'$ over $(C^*, P^*) \to (C, P)$. Then, since $C^* \in W$, we have $C \in W'$. Thus it is enough to set $\mathfrak{F}' = W'$.

From our Theorem A2, we get the following result immediately:

Let $C_i^*$ $(i = 1, \cdots, t)$ be all of the irreducible components of $C^*$ and let $P_i^*$ be a generic point of $C_i^*$ over $k(C_1^*, \cdots, C_t^*)$. Let $V_i$ be the locus of $P_i^*$ over $k$ for each $i$. Then

THEOREM A3. *For* $P \in V$, *we have*

(1) *if* $P$ *is not in any of* $V_i$, *then* $\mathfrak{F} - P$ *is empty,*

(2) *otherwise, let* $p$ *be the maximum of* $\dim U_i$ *where* $U_i$ *ranges over all* $V_i$ *which goes through* $P$, *then the dimension of every component of* $F - P$ *is not less than* $\dim \mathfrak{F} + r - p$.

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Now, our Theorem A1 is a corollary to this.

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