ON C^∞ FUNCTIONS ANALYTIC ON SETS OF SMALL MEASURE

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1. Introduction. The original motivation for this work was the problem of determining whether the signum function of a real valued continuous function defined on the real line is Riemann integrable. This problem is considered in § 2 where an example of an infinitely differentiable function is presented which possesses a non-Riemann integrable signum function. Moreover, it is shown that, for any $\epsilon > 0$, it is possible to construct such an example for which the set of points of analyticity has Lebesgue measure which is less than ϵ. This appears to be a more interesting property than the one originally sought.

In § 3 the work of the previous paragraph is used to construct a function which is infinitely differentiable on the entire real line but which is nowhere analytic.

2. Let $\epsilon > 0$. We construct a function f_ϵ which has the following properties:

(i) f_ϵ is infinitely differentiable on the entire real line;

(ii) the signum function of f_ϵ is non-Riemann integrable on any interval the length of which exceeds ϵ;

(iii) the set of points of analyticity of f_ϵ has measure which is less than ϵ.

First consider the function g_a which is defined by

$$g_a(x) = \exp\left\{\frac{a^2}{x^2 - a^2}\right\}, \quad \text{if } x \in (-a, a),$$

$$g_a(x) = 0 \qquad\qquad , \quad \text{if } x \notin (-a, a).$$

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It is clear that g_a is everywhere infinitely differentiable and that g_a is analytic except at the points a and $-a$.

Let $\{r_n\}$ be an enumeration of the rational numbers. Let $G = G(\epsilon) = \bigcup_{n=1}^{\infty} (r_n - \epsilon 2^{-n-1}, r_n + \epsilon 2^{-n-1})$. Then G is open and has Lebesgue measure which is less than ϵ. Furthermore, since G is a superset of the set of rational numbers, we have $\overline{G} = R$, the real line. We may express G as the countable union of pairwise disjoint open intervals $\{I_n\}$ where we write $I_n = (b_n - a_n, b_n + a_n)$, where $a_n > 0$.

We define the function f_ϵ by

$$f_\epsilon(x) = \sum_{n=1}^{\infty} \frac{g_{a_n}(x - b_n)}{n^2 K_n},$$

where the constants K_n are defined by

$$K_n = \sup \{ |g_{a_n}^{(r)}(x)| : 0 \leq r \leq n, \ x \in R \}.$$

It is clear, from the definition of the functions g_a, that the constants $\{K_n\}$ are finite and non-zero. Furthermore, we see that

$$\left| \frac{g_{a_n}(x - b_n)}{K_n} \right| \leq 1, \quad \text{for every } n = 1, 2, 3, \ldots, \text{ and for every } x \in R.$$

Thus the series which represents f_ϵ converges for every x and f_ϵ is well defined. We note the following:

(a) $f_\epsilon(x) \geq 0$, for all $x \in R$, and $f_\epsilon(x) = 0$, if and only if $x \notin G$; thus the signum function of f_ϵ is the characteristic function of G.

Now we show that $f_\epsilon \in C^\infty(R)$. Let r be any integer and consider the formal r^{th} derivative of f_ϵ:

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$$f_\epsilon^{(r)}(x) = \sum_{n=1}^{\infty} \frac{g_{a_n}^{(r)}(x - b_n)}{n^2 K_n}$$

$$= \sum_{n=1}^{r-1} \frac{g_{a_n}^{(r)}(x - b_n)}{n^2 K_n} + \sum_{n=r}^{\infty} \frac{g_{a_n}^{(r)}(x - b_n)}{n^2 K_n} .$$

Now the finite sum above is a linear combination of functions each of which is bounded on R. (This follows from the definition of the functions g_a.) From the definition of the constants $\{K_n\}$ we have for $n \geq r$ and for all $x \in R$,

$$\left| \frac{g_{a_n}^{(r)}(x - b_n)}{K_n} \right| \leq 1.$$

Thus all but a finite number of terms of the above series are dominated, for every x, by the terms of the convergent series $\Sigma \frac{1}{n^2}$. By the Weierstrass M-test, the series representing $f_\epsilon^{(r)}(x)$ is uniformly convergent on R. It follows that $f_\epsilon^{(r)}(x)$ exists for every x, and since r is arbitrary, $f_\epsilon \in C^\infty(R)$.

The following is clear from the above discussion:

(b) for every fixed r, $f_\epsilon^{(r)}(x)$ is uniformly bounded in x;

(c) if $x \notin G$, then $f_\epsilon^{(r)}(x) = 0$, for $r = 1, 2, 3, \ldots$.

(To prove (c) we note that if $x \notin G$ then $x \notin I_n$, for any n, thus $g_{a_n}^{(r)}(x - b_n) = 0$, for $r = 0, 1, 2, 3, \ldots$, and so $f_\epsilon^{(r)}(x) = 0$.)

From (c), we see that the formal Taylor expansion of f_ϵ about any point $a \notin G$ is identically zero. However, by (a), we see, since G is everywhere dense in R, that f_ϵ is not identically zero on any open set. Thus f_ϵ is not analytic on $\complement G$. However if $a \in G$ then $a \in I_{n_o}$,

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for some n_o, whence $f_\epsilon(x) = \frac{1}{n_o^2 K_{n_o}} \exp\left\{\frac{a_{n_o}^2}{(x-b_{n_o})^2 - a_{n_o}^2}\right\}$ in some neighborhood of a. Thus $f_\epsilon(x)$ is analytic at $a \in G$. Thus we have

(d) f_ϵ is analytic at the point a if and only if $a \in G$.

Since the Lebesgue measure of G is less than ϵ, property (iii) has been demonstrated.

Since by (a), sgn f_ϵ is the characteristic function of G, and since $\overline{G} = R$, it follows that sgn $f_\epsilon(x) = 1$ at any point at which sgn f_ϵ is continuous. Since sgn f_ϵ is clearly continuous on G, and since sgn $f_\epsilon(x) = 0$, if $x \notin G$, we see that G is precisely the set of points of continuity of sgn f_ϵ. Since G has Lebesgue measure which is less than ϵ, it follows from a well known characterization of Riemann integrable functions that f_ϵ is not Riemann integrable on any interval the length of which exceeds ϵ.

Since the zeros of an analytic function are isolated, the signum function of an analytic function is Riemann integrable. Thus in some sense, the example constructed here is the best possible with respect to properties (i) and (ii).

3. <u>A nowhere analytic C^∞ function</u>. Let us write $f_n = f_{1/n}$, where $f_{1/n}$ is the function constructed in §2 with $\epsilon = 1/n$. From (b), we see that $A_n^r = \sup\{|f_n^{(r)}(x)| : x \in R\}$ is finite and non-zero for every $n = 1, 2, 3, \ldots$ and $r = 0, 1, 2, 3 \ldots$.

Let $B_n = \max\{A_s^r : 0 \leq r \leq n \text{ and } 1 \leq s \leq n\}$. We define a function f by

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{n^2 B_n}.$$

We will show that $f \in C^\infty(R)$ but that f is nowhere analytic.

We consider the formal r^{th} derivative of f,

$$f^{(r)}(x) = \sum_{n=1}^{\infty} \frac{f_n^{(r)}(x)}{n^2 B_n} .$$

As in the previous paragraph, we argue that all but a finite number of terms of the above series are dominated by the corresponding terms of those of $\Sigma \frac{1}{n^2}$. Thus, as in §2, we see that $f \in C^{\infty}(R)$.

Let us now consider the sets $G_n = \bigcup_{k=1}^{\infty} \left(r_k - \frac{2^{-k-1}}{n}, r_k + \frac{2^{-k-1}}{n} \right)$ associated with the functions f_n. Clearly $\{G_n\}$ is a decreasing sequence. We have $mG_n < \frac{1}{n}$, and thus $m\left(\bigcap_{n=1}^{\infty} G_n\right) = 0$, where m denotes Lebesgue measure. Thus $\mathcal{C}\left(\bigcap_{n=1}^{\infty} G_n\right)$ is everywhere dense in R.

Now we will show that f is not analytic at any point $a \in \mathcal{C}(\bigcap_{n=1}^{\infty} G_n)$. Thus, since the set of points at which a function is analytic is open, it will follow that f is nowhere analytic.

Let $a \in \mathcal{C}(\bigcap_{n=1}^{\infty} G_n)$. Then $a \notin \bigcap_{n=1}^{\infty} G_n$. If $a \notin G_n$, for every n, then, by (c), we have $f_n^{(r)}(a) = 0$, for $r = 0, 1, 2, 3, \ldots$ and $n = 1, 2, 3 \ldots$. Thus the Taylor expansion of f about a is identically zero. But then f is not analytic at a since f is not identically zero in any neighborhood of a. Now, if there exists n such that $a \in G_n$, then since $\{G_n\}$ is a decreasing sequence, there is an integer n_o such that $a \in G_n$, if $n \leq n_o$, but $a \notin G_n$, if $n > n_o$. Thus, by (c), $f_n^{(r)}(a) = 0$, if $n > n_o$ and for $r = 0, 1, 2, 3, \ldots$. We therefore have

$$f^{(r)}(a) = \sum_{n=1}^{n_o} \frac{f_n^{(r)}(a)}{n^2 B_n} .$$

Now the derivatives above are also those of the function

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$$g(x) = \sum_{n=1}^{n_o} \frac{f_n(x)}{n^2 B_n},$$

evaluated at $x = a$. Each f_n is analytic on G_n; thus g is certainly analytic on $\bigcap_{n=1}^{n_o} G_n = G_{n_o}$. In particular then g is analytic at a. Now suppose f is analytic at $x = a$. Then we have

$$g(x) = f(x),$$

for x in some neighborhood H of a since all respective derivatives are equal at a. Thus

$$\sum_{n=n_o+1}^{\infty} \frac{f_n(x)}{n^2 B_n} = 0,$$

for $x \in H$. Since each term of the above series is non-negative, we have $f_{n_o+1}(x) = 0$, for all $x \in H$. It follows that

$$\{x: f_{n_o+1}(x) > 0\} = G_{n_o+1}$$

is not everywhere dense. This is a contradiction, since G_{n_o+1} contains the set of rational numbers.

Thus f is not analytic on $C(\bigcap_{n=1}^{\infty} G_n)$, whence f is nowhere analytic.

It is also not difficult to see that f also satisfies property (ii) of §2, where we set $\epsilon = 1$.

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