

SIGNATURE-BASED REPRESENTATIONS FOR THE RELIABILITY OF SYSTEMS WITH HETEROGENEOUS COMPONENTS

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Abstract

Signature-based representations of the reliability functions of coherent systems with independent and identically distributed component lifetimes have proven very useful in studying the ageing characteristics of such systems and in comparing the performance of different systems under varied criteria. In this paper we consider extensions of these results to systems with heterogeneous components. New representation theorems are established for both the case of components with independent lifetimes and the case of component lifetimes under specific forms of dependence. These representations may be used to compare the performance of systems with homogeneous and heterogeneous components.

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1. Introduction

The notion of system signatures was introduced by Samaniego [25] as a useful tool in providing a characterization of the class of coherent systems which satisfy the IFR (increasing failure rate) closure property for systems with independent and identically distributed (i.i.d.) component lifetimes. The *signature* of a coherent system whose n components have i.i.d. lifetimes was defined as the n -dimensional probability vector whose i th element is $s_i = P(T = X_{i:n})$, where T denotes the lifetime of the system and $(X_{1:n}, \dots, X_{n:n})$ is the vector of order statistics of the component lifetimes $X_1, \dots, X_n \sim F$. The distribution F is assumed to be continuous with support set $(0, \infty)$. Under the i.i.d. assumption on component lifetimes, the signature vector $\mathbf{s} = (s_1, \dots, s_n)$ is a distribution-free measure of the system's design. The system's reliability function $\bar{F}_T(t) = P(T > t)$ may be represented as

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \bar{F}_{i:n}(t), \quad (1.1)$$

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where $\bar{F}_{i:n}(t) = P(X_{i:n} > t)$. The monograph by Samaniego [26] provides an extensive survey of system signatures and their applications.

Since (1.1) may also be written as

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \sum_{j=n-i+1}^n \binom{n}{j} \{\bar{F}(t)\}^j \{F(t)\}^{n-j},$$

we may express the reliability of the system at time t through the reliability polynomial $h(p)$ (see [2, p. 21]) given by

$$h(p) = \sum_{i=1}^n s_i \sum_{j=n-i+1}^n \binom{n}{j} p^j (1-p)^{n-j}, \tag{1.2}$$

where $p = \bar{F}(t)$. This polynomial may also be written in the form $h(p) = \sum_{i=1}^n d_i p^i$, where $\mathbf{d} = (d_1, \dots, d_n)$ is generally called the system's domination vector (see [27]). Explicit formulae for the relationships $\mathbf{d} = g(\mathbf{s})$ and $\mathbf{s} = g^{-1}(\mathbf{d})$ connecting the signature and domination vectors may be found in [26]. In the sequel, we will utilize the reliability polynomial $h(p)$ primarily in the form (1.2). The coefficients of this polynomial depend only on the system structure. The polynomial $h(p)$ is strictly increasing for $p \in (0, 1)$, with $h(0) = 0$ and $h(1) = 1$.

The representation in (1.1) continues to hold when the components are dependent with a joint exchangeable distribution; see [22] and the references therein. It also holds for *mixed systems*, that is, for stochastic mixtures of coherent systems (see [26]). However, this representation is not necessarily true when the component lifetimes have different distributions, as demonstrated in Example 5.1 of [22].

Other useful representations for coherent systems with exchangeable components are

$$\bar{F}_T(t) = \sum_{i=1}^n a_i \bar{F}_{1:i}(t) \tag{1.3}$$

and

$$\bar{F}_T(t) = \sum_{i=1}^n b_i \bar{F}_{i:i}(t), \tag{1.4}$$

where $\bar{F}_{1:i}(t) = P(X_{1:i} > t)$ and $\bar{F}_{i:i}(t) = P(X_{i:i} > t)$ are the reliability functions of the series system lifetime $X_{1:i} = \min(X_1, \dots, X_i)$ and the parallel system lifetime $X_{i:i} = \max(X_1, \dots, X_i)$, respectively. The vectors of coefficients $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, which depend only on the system structure, are called *minimal signatures*, respectively; see [19]. While some of the elements of \mathbf{a} and \mathbf{b} may be negative, the vectors obey the constraints $\sum_{i=1}^n a_i = 1$ and $\sum_{i=1}^n b_i = 1$, respectively. When the component lifetimes are i.i.d., representation (1.3) reduces to

$$\bar{F}_T(t) = \sum_{i=1}^n a_i \{\bar{F}(t)\}^i,$$

where \bar{F} is the common reliability function of the components. However, representations (1.3) and (1.4) do not necessarily hold when the components have different distributions, as

demonstrated in Example 5.1 of [22]. Related work on system signatures includes [4], [5], [9], [10], [11], [14], [15], [20], [29], and [30].

Few representation results have been obtained in the literature for coherent systems with heterogeneous components. Generalized mixture representations based on minimal path sets and minimal cut sets were obtained by Esary and Proschan [7] (see also [2, p. 26]) in the case of independent components. Also, in this case, some comparison results were obtained in [14] and [21]. In the case of dependent heterogeneous components, representations and bounds for k -out-of- n systems (order statistics) were obtained in [24, Chapter 5], and comparisons between parallel and series systems were given in [17] and [18].

In this paper, representations similar to those in (1.1), (1.3), and (1.4) are obtained for coherent systems with heterogeneous components. We consider the case of independent components in Section 2 and the case of dependent components in Section 3. These representations allow us to compare the performance of a given system under differing assumptions on component lifetimes—i.i.d. lifetimes and independent but heterogeneous lifetimes. We also use these representations to obtain some ordering properties.

Throughout the paper, when we say that a function g is increasing or decreasing, we mean that $g(x) \leq g(y)$ or $g(x) \geq g(y)$ for $x \leq y$, respectively.

2. Signature representations for systems with independent components

Let $T = \phi(X_1, \dots, X_n)$ be the lifetime of a coherent system with independent component lifetimes X_1, \dots, X_n with respective reliability functions $\bar{F}_1, \dots, \bar{F}_n$, each assumed to be continuous and with supports contained in $(0, \infty)$, where ϕ is the structure function of the system (see [2]). If $X_{1:n} < \dots < X_{n:n}$ are the order statistics obtained from X_1, \dots, X_n then it is evident that $P(T = X_{i:n})$ depends on the distributions of X_1, \dots, X_n . While the reliability function of T may still be written as a sum similar to that in (1.1), that is, as

$$\bar{F}_T(t) = \sum_{i=1}^n P(T = X_{i:n}) P(X_{i:n} > t \mid T = X_{i:n}),$$

this representation is not useful in the analysis and comparisons of systems due to the dependence of $P(T = X_{i:n})$ and $P(X_{i:n} > t \mid T = X_{i:n})$ on both $\{\bar{F}_i, i = 1, \dots, n\}$ and ϕ . In what follows, we will develop an alternative representation of \bar{F}_T which, quite unexpectedly, relies on the system’s distribution-free signature vector s (generally defined under an i.i.d. assumption on component lifetimes) defined in the previous section, notwithstanding the fact that the system under study is assumed to have heterogeneous rather than identical components. Under the i.i.d. assumption on component lifetimes, the signature of a coherent system does not depend on the common component lifetime distribution and may be computed as $s_i = |A_i|/n!$ for $i = 1, \dots, n$, where $|A_i|$ is the cardinality of the set A_i of permutations σ of the set $\{1, \dots, n\}$ such that $\phi(X_1, \dots, X_n) = X_{i:n}$ whenever $X_{\sigma(1)} < \dots < X_{\sigma(n)}$ (see [26]).

If a coherent system with lifetime T has independent components then its reliability function can be written as

$$\bar{F}_T(t) = H(\bar{F}_1(t), \dots, \bar{F}_n(t)), \tag{2.1}$$

where $H(p_1, \dots, p_n)$ is a multinomial expression (called the *structure reliability function* in [7]) which is linear in each p_i and whose coefficients depend only on the system’s design. In analogy with (1.2), at a fixed time t , the reliability of the system at time t may be written as $H(p_1, \dots, p_n)$, where $p_i = \bar{F}_i(t)$ for $i = 1, \dots, n$. This polynomial is strictly increasing

in $(0, 1)^n$ and is such that $H(0, \dots, 0) = 0$ and $H(1, \dots, 1) = 1$. Clearly, the reliability polynomial in (1.2) is simply $h(p) = H(p, \dots, p)$. Using representation (2.1), we can extend representation (1.1) as follows.

As defined above, the signature of a coherent system is a ‘topological invariant’ which can be treated without reference to the stochastic behavior of the system’s components. While it is generally defined in terms of the order statistics of an i.i.d. sample from a common component distribution F , the signature vector depends solely on the system’s design and does not depend on the distribution of component lifetimes. For example, the signature vectors of all four-component systems are displayed in [26] (see also [14] and [29]). It is true that the application and interpretation of system signatures have largely been restricted to the case of systems with i.i.d. component lifetimes. For example, the representation of a system’s reliability function in terms of the system’s signature, as in (1.1), requires the assumption of i.i.d. component lifetimes. The following result shows that the signature of a given system may be used in an analogous fashion in representing the reliability function of the system when component lifetimes are independent but not homogeneous. The representation is seen to involve a single distribution G that is functionally dependent on the individual lifetime distributions of the system’s components as well as on the system’s design.

Theorem 2.1. *Consider an n -component system with signature vector $s = (s_1, \dots, s_n)$, and let h and H be the system’s reliability polynomial and structure reliability function, respectively. Suppose that the lifetimes X_1, \dots, X_n of the components of this system are independent but are not identically distributed, with X_i having the continuous distribution function F_i and reliability function $\bar{F}_i = 1 - F_i$ for $i = 1, \dots, n$. Then the reliability function of the system lifetime T may be expressed as*

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \bar{G}_{i:n}(t), \tag{2.2}$$

where $\bar{G}_{i:n}(t) = P(Y_{i:n} > t)$ and $Y_{1:n} < \dots < Y_{n:n}$ are the order statistics obtained from the i.i.d. random variables Y_1, \dots, Y_n with common reliability function

$$\bar{G}(t) = h^{-1}(H(\bar{F}_1(t), \dots, \bar{F}_n(t))). \tag{2.3}$$

Proof. Recall that the reliability polynomial h is a strictly increasing continuous function in $(0, 1)$ with $h(0) = 0$ and $h(1) = 1$. Hence, its inverse function h^{-1} in $(0, 1)$ exists and is also strictly increasing in $(0, 1)$ with $h^{-1}(0) = 0$ and $h^{-1}(1) = 1$. We also know that H is a strictly increasing continuous function in $(0, 1)^n$ with $H(0, \dots, 0) = 0$ and $H(1, \dots, 1) = 1$. Hence, the function \bar{G} defined by (2.3) is right continuous and decreasing in t with $\bar{G}(0) = 1$ and $\bar{G}(\infty) = 0$, and is therefore a proper reliability function. If a system with signature s has components with i.i.d. lifetimes with the common distribution G whose reliability function is as given in (2.3), then it follows from (1.2) that the system’s reliability function is given by $h(\bar{G}(t))$. We thus find from (1.1) that this reliability function may be represented as

$$h(\bar{G}(t)) = \sum_{i=1}^n s_i \bar{G}_{i:n}(t).$$

Finally, we note from (2.3) that we have

$$h(\bar{G}(t)) = H(\bar{F}_1(t), \dots, \bar{F}_n(t)),$$

and, hence, due to (2.1), (2.2) holds.

This theorem can be used to obtain ordering properties similar to those given in [11] and [22], but for systems with heterogeneous components. We consider the following stochastic orders. Let X and Y be two random variables with respective absolutely continuous distribution functions F and G , hazard rate functions $r_F = F'/(1 - F)$ and $r_G = G'/(1 - G)$, and reversed hazard rate functions $\bar{r}_F = F'/F$ and $\bar{r}_G = G'/G$. Then the following statements hold:

- (i) X is said to be less than Y in the stochastic order (written as $X \leq_{st} Y$) when $F \geq G$;
- (ii) X is said to be less than Y in the hazard rate order (written as $X \leq_{hr} Y$) when $r_F \geq r_G$;
- (iii) X is said to be less than Y in the reversed hazard rate order (written as $X \leq_{rh} Y$) when $\bar{r}_F \leq \bar{r}_G$.

The main properties of these orders can be found in [28, Chapter 1]. To obtain our ordering results, we need the following lemma, which is a direct consequence of Theorems 4.5 and 5.5 of [3].

Lemma 2.1. *Let T and T^* be the lifetimes of two coherent systems with the same structure, reliability polynomial h , and i.i.d. components having common reliability functions \bar{F} and \bar{F}^* , respectively.*

- (i) *If $\bar{F} \leq_{hr} \bar{F}^*$ and $xh'(x)/h(x)$ is decreasing, then $T \leq_{hr} T^*$.*
- (ii) *If $\bar{F} \leq_{rh} \bar{F}^*$ and $(1 - x)h'(x)/(1 - h(x))$ is increasing, then $T \leq_{rh} T^*$.*

Theorem 2.2. *Let T and T^* be the lifetimes of two coherent systems with signatures $s = (s_1, \dots, s_n)$ and $s^* = (s_1^*, \dots, s_n^*)$, and both with independent component lifetimes. Let h and h^* be their reliability polynomials, and let H and H^* be their structure reliability functions. Let \bar{G} and \bar{G}^* be the reliability functions defined in (2.3).*

- (i) *If $\bar{G} \leq_{st} \bar{G}^*$ and $s \leq_{st} s^*$, then $T \leq_{st} T^*$.*
- (ii) *If $\bar{G} \leq_{hr} \bar{G}^*$, $s \leq_{hr} s^*$, and either $xh'(x)/h(x)$ or $x(h^*)'(x)/h(x)$ is decreasing, then $T \leq_{hr} T^*$.*
- (iii) *If $\bar{G} \leq_{rh} \bar{G}^*$, $s \leq_{rh} s^*$, and either $(1 - x)h'(x)/(1 - h(x))$ or $(1 - x)(h^*)'(x)/(1 - h^*(x))$ is increasing, then $T \leq_{rh} T^*$.*

Proof. The proof of (i) is immediate from Theorem 2.1.

We prove (ii) by assuming that $xh'(x)/h(x)$ is increasing. The proof of the other case is similar. Then we have

$$\begin{aligned} \bar{F}_T(t) &= \sum_{i=1}^n s_i \bar{G}_{i:n}(t) \quad (\text{from Theorem 2.1}) \\ &\leq_{hr} \sum_{i=1}^n s_i \bar{G}_{i:n}^*(t) \quad (\text{from } \bar{G} \leq_{hr} \bar{G}^* \text{ and Lemma 2.1}) \\ &\leq_{hr} \sum_{i=1}^n s_i^* \bar{G}_{i:n}^*(t) \quad (\text{from } s \leq_{hr} s^*) \\ &= \bar{F}_{T^*}(t) \quad (\text{from Theorem 2.1}), \end{aligned}$$

where the last inequality holds by Theorem 1.B.14 of [28].

The proof of (iii) can be obtained in a similar way by using Lemma 2.1(ii) and Theorem 1.B.52 of [28].

Example 2.1. If we consider the coherent system with lifetime $T = \max(\min(X_1, X_2), \min(X_3, X_4))$ and corresponding signature $s = (0, \frac{2}{3}, \frac{1}{3}, 0)$ (see system number 10 in Table 3.2 of [26, p. 25]), then its reliability function is given by

$$\begin{aligned} \bar{F}_T(t) &= P(\max(\min(X_1, X_2), \min(X_3, X_4)) > t) \\ &= P(\{\min(X_1, X_2) > t\} \cup \{\min(X_3, X_4) > t\}) \\ &= P(\min(X_1, X_2) > t) + P(\min(X_3, X_4) > t) - P(X_{1:4} > t) \\ &= \bar{F}_1(t)\bar{F}_2(t) + \bar{F}_3(t)\bar{F}_4(t) - \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)\bar{F}_4(t) \\ &= H(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t)), \end{aligned}$$

where $H(p_1, p_2, p_3, p_4) = p_1 p_2 + p_3 p_4 - p_1 p_2 p_3 p_4$. Also, $h(p) = 2p^2 - p^4$ and its minimal signature is $\mathbf{a} = (0, 2, 0, -1)$. Then,

$$h^{-1}(x) = \sqrt{1 - \sqrt{1 - x}}$$

and the reliability function defined in (2.3) is

$$\bar{G}(t) = \sqrt{1 - \sqrt{1 - \bar{F}_1(t)\bar{F}_2(t) - \bar{F}_3(t)\bar{F}_4(t) + \bar{F}_1(t)\bar{F}_2(t)\bar{F}_3(t)\bar{F}_4(t)}}. \tag{2.4}$$

In general, we know that $h(p)$ is a strictly increasing polynomial in $p \in [0, 1]$ from $h(0) = 0$ and $h(1) = 1$, and, hence, its inverse function $h^{-1}(x)$ exists for $x \in [0, 1]$. However, the explicit computation of $h^{-1}(x)$ is not always as easy as in the preceding example. In general, numerical methods must be employed to obtain an approximation of \bar{G} .

Note that the reliability function \bar{G} in (2.3) is a function such that

$$H(\bar{G}(t), \dots, \bar{G}(t)) = H(\bar{F}_1(t), \dots, \bar{F}_n(t)).$$

This reliability function is related to the well-known concept of *mean function* associated with a real-valued function (see [6], [8, p. 65], and [18]), which is defined as follows.

Definition 2.1. If $\psi : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a real-valued function, a *mean function* of ψ in S is a function $m_\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\psi(x_1, \dots, x_n) = \psi(z, \dots, z)$$

for all $(x_1, \dots, x_n) \in S$, where $z = m_\psi(x_1, \dots, x_n)$.

Clearly, a mean function of ψ exists when its diagonal section $\delta(z) = \psi(z, \dots, z)$ is continuous, $\inf_{\mathbf{x} \in S} \psi(\mathbf{x}) = \inf_{z \in D} \delta(z)$ and $\sup_{\mathbf{x} \in S} \psi(\mathbf{x}) = \sup_{z \in D} \delta(z)$, where $D = \{z \in \mathbb{R} : (z, \dots, z) \in S\}$. Moreover, the mean function is unique when the preceding properties hold and $\delta(z)$ is strictly monotone in D . For example, if $\psi(x_1, x_2, x_3) = x_1 x_2 x_3$ then $\delta(z) = z^3$ and its unique mean function in \mathbb{R}^3 is $m_\psi(x_1, x_2, x_3) = (x_1 x_2 x_3)^{1/3}$. However, the mean function in \mathbb{R}^2 of $\psi(x_1, x_2) = x_1 x_2$ does not exist.

Note that the mean function m_H in $[0, 1]^n$ of the structure reliability function H of an n -component system always exists since its diagonal section $h(z) = H(z, \dots, z)$ is a strictly increasing polynomial in $[0, 1]$ with $h(0) = 0$ and $h(1) = 1$. Moreover, it is given by

$$m_H(x_1, \dots, x_n) = h^{-1}(H(x_1, \dots, x_n))$$

and satisfies $m_H(z, \dots, z) = z$. Hence, the reliability function defined in (2.3) satisfies

$$\bar{G}(t) = m_H(\bar{F}_1(t), \dots, \bar{F}_n(t)).$$

Thus, this reliability function $\bar{G}(t)$ can appropriately be called the *mean reliability function* associated with the system and the component reliability functions.

For example, in the case of a series system, the mean function of H is the geometric mean

$$m_H(x_1, \dots, x_n) = (x_1 \cdots x_n)^{1/n}.$$

For the parallel system, the mean function is

$$m_H(x_1, \dots, x_n) = 1 - [(1 - x_1) \cdots (1 - x_n)]^{1/n}.$$

For the system considered in Example 2.1, the mean function is

$$m_H(x_1, x_2, x_3, x_4) = \sqrt{1 - \sqrt{1 - x_1x_2 - x_3x_4 + x_1x_2x_3x_4}},$$

and its mean reliability function is as given in (2.4).

Note that, from Theorem 2.2, the mean reliability function defined in (2.3) provides a vehicle for comparing the performance of a system whose components have i.i.d. lifetimes with distribution F and the same system with components whose lifetimes are independent but have heterogeneous distributions.

Next, we extend representations (1.3) and (1.4) to systems with heterogeneous components. The result is given in the following theorem and it is an immediate consequence of Theorem 2.1. First, we need to extend the definition of minimal and maximal signatures to systems with heterogeneous components as follows. If T is the lifetime of a coherent system with independent components, we define the minimal and maximal signatures $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of T as the minimal and maximal signatures of a system with the same structure as T but with i.i.d. component lifetimes with common reliability function $\bar{G}(t)$. It is useful to note once again that these signatures do not depend on the component distributions and that they can be obtained from the system signature $\mathbf{s} = (s_1, \dots, s_n)$.

Theorem 2.3. *If T is the lifetime of a coherent system with independent component lifetimes X_1, \dots, X_n , and minimal and maximal signatures $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, then*

$$\bar{F}_T(t) = \sum_{i=1}^n a_i \bar{G}_{1:i}(t) \tag{2.5}$$

and

$$\bar{F}_T(t) = \sum_{i=1}^n b_i \bar{G}_{i:i}(t),$$

where $\bar{G}_{1:i}(t) = \bar{G}^i(t)$ and $\bar{G}_{i:i}(t) = 1 - G^i(t)$ for $i = 1, \dots, n$, \bar{G} is as given in (2.3), and $G = 1 - \bar{G}$.

Let us consider an application of this theorem. Since the reliability polynomial may be written as $h(p) = \sum_{i=1}^n a_i p^i$, (2.5) can alternatively be expressed as

$$\bar{F}_T(t) = \sum_{i=1}^n a_i \{\bar{G}(t)\}^i. \tag{2.6}$$

Recall that the coefficients a_i are real numbers such that $\sum_{i=1}^n a_i = 1$ and, therefore, \bar{F}_T is a generalized mixture. Then, (2.6) shows that the class of all reliability functions of coherent systems with n independent components is included in the class of generalized mixtures obtained from powers \bar{G}^i of a reliability function \bar{G} for $i = 1, \dots, n$. Hence, the asymptotic behavior (when $t \rightarrow \infty$) of the ageing measures of T can be determined from the results established in [12], [13], and [16]. For example, from the results in [16], it is known that if the components are independent then the hazard rate function r_T of the system satisfies

$$\lim_{t \rightarrow \infty} \frac{r_T(t)}{r_G(t)} = i,$$

where r_G is the hazard rate function associated with $G = 1 - \bar{G}$ and $i = \min\{j : a_j \neq 0\}$.

3. Signature representations for systems with dependent components

In this section we extend the representations based on signatures given in the preceding section to coherent systems with component lifetimes that may be dependent. Let $T = \phi(X_1, \dots, X_n)$ be the lifetime of a coherent system with structure function ϕ and with component lifetimes (X_1, \dots, X_n) . The dependence between component lifetimes can be represented by the joint reliability (or survival) function

$$\bar{F}(x_1, \dots, x_n) = P(X_1 > x_1, \dots, X_n > x_n).$$

Using Sklar’s celebrated theorem (see [23, p. 46]), this reliability function can be written as

$$\bar{F}(x_1, \dots, x_n) = K(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)), \tag{3.1}$$

where \bar{F}_i is the reliability function of X_i and K is the survival copula. It is well known that the survival copula can be obtained from the distributional copula (i.e., the copula used to represent the joint distribution function) and vice versa.

For our purposes, we will also need the well-known representation of the lifetime of a coherent system based on its minimal path sets (see [2, p. 12]). A set $P \subseteq \{1, \dots, n\}$ is a *path set* of a coherent system if the system works when all the components in P work. A path set P is a *minimal path set* if it does not contain other path sets. If T is the lifetime of a coherent system with minimal path sets P_1, \dots, P_k , then

$$T = \max_{j=1, \dots, k} X_{P_j},$$

where $X_{P_j} = \min_{i \in P_j} X_i$ is the lifetime of the series system with components in P_j for $j = 1, \dots, k$ (see [2, p. 12]). Hence, it follows from the inclusion-exclusion formula (see [1] and [19]) that the reliability function of the system can be expressed as

$$\bar{F}_T(t) = \sum_{j=1}^k \bar{F}_{P_j}(t) - \sum_{i < j} \bar{F}_{P_i \cup P_j}(t) + \dots + (-1)^{k+1} \bar{F}_{P_1 \cup \dots \cup P_k}(t), \tag{3.2}$$

where \bar{F}_P is the reliability function of the series system lifetime $X_P = \min_{i \in P} X_i$. Note that $P_1 \cup \dots \cup P_k = \{1, \dots, n\}$. Furthermore, \bar{F}_P can be written as

$$\bar{F}_P(t) = \bar{F}(t_P),$$

where $t_P = (x_1, \dots, x_n)$ with $x_i = t$ for $i \in P$ and $x_i = 0$ for $i \notin P$. Taking into account this

representation for series system reliability functions and using Sklar’s representation (3.1) for the survival copula, (3.2) can be written as

$$\bar{F}_T(t) = W(\bar{F}_1(t), \dots, \bar{F}_n(t)), \tag{3.3}$$

where

$$W(x_1, \dots, x_n) = \sum_{j=1}^k K(\mathbf{x}_{P_j}) - \sum_{i < j} K(\mathbf{x}_{P_i \cup P_j}) + \dots + (-1)^{k+1} K(\mathbf{x}_{P_1 \cup \dots \cup P_k})$$

and $\mathbf{x}_P = (z_1, \dots, z_n)$ with $z_i = x_i$ for $i \in P$ and $z_i = 1$ for $i \notin P$. In particular, if the components are independent then the function W is equal to the structure reliability function H . If the system is a series system then W is equal to the survival copula K . However, in general, W is neither a polynomial nor a copula and it depends both on the system structure and on the survival copula (i.e., the dependence structure between the components). However, it should be noted that W does not depend on the component (marginal) distributions. We will refer to the function $W = W(\phi, K)$ as the *structure-dependence function*.

For example, let us consider the system with lifetime $T = \min(X_1, \max(X_2, X_3))$ with minimal path sets $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. Then, (3.1) and (3.2) yield

$$\begin{aligned} \bar{F}_T(t) &= \bar{F}_{\{1,2\}}(t) + \bar{F}_{\{1,3\}}(t) - \bar{F}_{\{1,2,3\}}(t) \\ &= \bar{F}(t, t, 0) + \bar{F}(t, 0, t) - \bar{F}(t, t, t) \\ &= K(\bar{F}_1(t), \bar{F}_2(t), 1) + K(\bar{F}_1(t), 1, \bar{F}_3(t)) - K(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t)) \\ &= W(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t)), \end{aligned}$$

where

$$W(x_1, x_2, x_3) = K(x_1, x_2, 1) + K(x_1, 1, x_3) - K(x_1, x_2, x_3).$$

In particular, if the components are independent, K is the product copula (i.e., $K(x_1, x_2, x_3) = x_1x_2x_3$ for $0 \leq x_i \leq 1$ and $i = 1, 2, 3$), and

$$W(x_1, x_2, x_3) = x_1x_2 + x_1x_3 - x_1x_2x_3 = H(x_1, x_2, x_3).$$

Now, we are ready to state the first representation theorem for coherent systems with heterogeneous dependent components.

Theorem 3.1. *If $T = \phi(X_1, \dots, X_n)$ is the lifetime of a coherent system with component lifetimes X_1, \dots, X_n having survival copula K and structure-dependence function W with right-continuous increasing mean function m_W , then T is equal in law to $T^* = \phi(Y_1, \dots, Y_n)$ with identically distributed component lifetimes Y_1, \dots, Y_n with joint reliability function*

$$P(Y_1 > x_1, \dots, Y_n > x_n) = K(\bar{G}_W(x_1), \dots, \bar{G}_W(x_n)), \tag{3.4}$$

where

$$\bar{G}_W(t) = m_W(\bar{F}_1(t), \dots, \bar{F}_n(t)). \tag{3.5}$$

Proof. Using the facts that m_W is a right-continuous increasing function, $W(0, \dots, 0) = 0$, and $W(1, \dots, 1) = 1$, we note that \bar{G}_W in (3.5) is a proper reliability function. Then, by using the fact that m_W is the mean function of W we obtain

$$\bar{F}_T(t) = W(\bar{F}_1(t), \dots, \bar{F}_n(t)) = W(\bar{G}_W(t), \dots, \bar{G}_W(t)) = \bar{F}_{T^*}(t),$$

where T^* is the lifetime of the system under consideration when its component lifetimes are

identically distributed with joint reliability function as in (3.4), with T representing this same system with dependent component lifetimes governed by K and W .

Note that when K is the product copula (i.e., the components are independent), then the function \bar{G}_W in (3.5) coincides with the function \bar{G} in (2.3). Moreover, the function m_W exists and it is an increasing continuous function when the copula is absolutely continuous with support $[0, 1]^n$. As mentioned above, the existence and properties of the mean function m_W are related to the concept of the diagonal section usually applied to copulas. The diagonal section associated to the function W is defined as

$$\delta_W(t) = W(t, \dots, t).$$

If W is the structure-dependence function associated with a coherent system then $\delta_W(t)$ is an increasing function with $\delta_W(0) = 0$ and $\delta_W(1) = 1$. If δ_W is a continuous function then

$$\delta_W^{-1}(x) = \inf\{t : \delta_W(t) > x\}$$

is an increasing right-continuous function. Then,

$$m_W(x_1, \dots, x_n) = \delta_W^{-1}(W(x_1, \dots, x_n))$$

is a mean function of W . Moreover, m_W is right continuous and increasing. Hence,

$$\bar{G}_W(t) = m_W(\bar{F}_1(t), \dots, \bar{F}_n(t))$$

is a proper reliability function with support included in $[0, \infty)$.

We can obtain additional signature representations when the survival copula K is exchangeable. A copula (or a function) K is *exchangeable* if

$$K(x_1, \dots, x_n) = K(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for any permutation σ of the set $\{1, \dots, n\}$. The new representation result is then as follows.

Theorem 3.2. *If T is the lifetime of a coherent system with signature $\mathbf{s} = (s_1, \dots, s_n)$ and with component lifetimes X_1, \dots, X_n having an exchangeable copula K and structure-dependence function W with right-continuous increasing mean function m_W , then*

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \bar{G}_{i:n}(t), \tag{3.6}$$

where $\bar{G}_{i:n}(t) = P(Y_{i:n} > t)$ and $Y_{1:n} < \dots < Y_{n:n}$ are the order statistics obtained from the random variables Y_1, \dots, Y_n with joint reliability function as in (3.4).

Proof. Note that from Theorem 3.1 we know that T is equal in law to a coherent system with component lifetimes having the joint reliability function in (3.4). Also, if K is exchangeable then the reliability function in (3.4) is also exchangeable. Moreover, we know that representation (3.6) holds in the case of systems with component lifetimes having joint exchangeable distributions; see [22]. Hence, we obtain the desired result.

As a consequence, we obtain the following ordering property.

Theorem 3.3. *Let T and T^* be the lifetimes of two coherent systems with signatures $\mathbf{s} = (s_1, \dots, s_n)$ and $\mathbf{s}^* = (s_1^*, \dots, s_n^*)$, and with component lifetimes having the same exchangeable survival copula K . If $\mathbf{s} \leq_{st} \mathbf{s}^*$, both systems satisfy the assumptions of Theorem 3.2, and the reliability functions \bar{G}_W and \bar{G}_{W^*} defined by (3.5) are such that $\bar{G}_W \leq \bar{G}_{W^*}$, then $T \leq_{st} T^*$.*

Proof. The proof follows from Theorem 3.2 and Theorems 6.B.14 and 6.B.23 of [28].

Using the preceding result, we can obtain similar representations based on series or parallel systems as follows.

Theorem 3.4. *If T is the lifetime of a coherent system with minimal and maximal signatures $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, respectively, and with component lifetimes X_1, \dots, X_n having an exchangeable copula K and structure-dependence function W with right-continuous increasing mean function m_W , then*

$$\bar{F}_T(t) = \sum_{i=1}^n a_i \bar{G}_{1:i}(t) \tag{3.7}$$

and

$$\bar{F}_T(t) = \sum_{i=1}^n b_i \bar{G}_{i:i}(t), \tag{3.8}$$

where $\bar{G}_{1:i}(t) = P(Y_{1:i} > t)$, $\bar{G}_{i:i}(t) = P(Y_{i:i} > t)$, $Y_{1:i} = \min(Y_1, \dots, Y_i)$, $Y_{i:i} = \max(Y_1, \dots, Y_i)$, and the random variables Y_1, \dots, Y_n have the joint reliability function in (3.4).

The proof is similar to that of Theorem 3.2. It is proved in [22] that representations (3.7) and (3.8) hold in the case of systems with component lifetimes having joint exchangeable distributions.

The next representation result shows that the lifetime of a coherent system with dependent components is equal in law to the lifetime of a system with the same structure but with i.i.d. components.

Theorem 3.5. *If T is the lifetime of a coherent system with signature $\mathbf{s} = (s_1, \dots, s_n)$ and with component lifetimes X_1, \dots, X_n having structure-dependence function W , then*

$$\bar{F}_T(t) = \sum_{i=1}^n s_i \bar{G}_{i:n}(t),$$

where $\bar{G}_{i:n}(t) = P(Y_{i:n} > t)$ and $Y_{1:n} < \dots < Y_{n:n}$ are the order statistics obtained from the i.i.d. random variables Y_1, \dots, Y_n with common reliability function

$$\tilde{G}(t) = h^{-1}(W(\bar{F}_1(t), \dots, \bar{F}_n(t))),$$

where h is the reliability polynomial.

The proof is immediate from (3.3) and the definition of the reliability polynomial. This result proves that if in some inferential procedures we only have information about the lifetimes of the system (i.e., a sample T_1, \dots, T_m of system lifetimes), then we cannot distinguish between a system with dependent components and the system with the same structure and i.i.d. components with common reliability function \tilde{G} . For further comments on such identifiability issues, see [4].

Finally, it should be mentioned that all the preceding mixture representations can also be obtained for mixed systems.

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