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Baker-Type Estimates for Linear Forms in the Values of q-Series

Keijo Väänänen and Wadim Zudilin

Abstract. We obtain lower estimates for the absolute values of linear forms of the values of generalized Heine series at non-zero points of an imaginary quadratic field $\mathbb{I}$, in particular of the values of q-exponential function. These estimates depend on the individual coefficients, not only on the maximum of their absolute values. The proof uses a variant of classical Siegel's method applied to a system of functional Poincaré-type equations and the connection between the solutions of these functional equations and the generalized Heine series.

1 Introduction

Let $\mathbb{I}$ denote the field of rational numbers or an imaginary quadratic field. In the present paper we are interested in linear independence measures for the values of the function

(1) $$\phi(z) = 1 + \sum_{n=1}^{\infty} \frac{q^{-sn(n-1)/2}}{\mathcal{P}(1)\mathcal{P}(q^{-1})\cdots\mathcal{P}(q^{-(n-1)})} z^n,$$

where s is a positive integer, q is an integer in $\mathbb{I}$ with $|q| > 1$, and the polynomial $\mathcal{P}(z) \in \mathbb{I}[z]$ of degree $\leq s$ satisfies the conditions $\mathcal{P}(0) \neq 0$ and $\mathcal{P}(q^{-k}) \neq 0$ for $k = 0, 1, \dots$. Two interesting special cases are the Tschakaloff function [Tsch]

$$T_q(z) = \sum_{n=0}^{\infty} q^{-n(n+1)/2} z^n$$

and the q-exponential function

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q-1)\cdots(q^n-1)} = \prod_{n=1}^{\infty}\Big(1 + \frac{z}{q^n}\Big).$$

There are many results on linear independence measures of the values of $T_q(z)$, for these we refer to [Bu]. Already Stihl [St] was able to obtain linear independence measure for the values of $\phi(z)$, if $\mathcal{P}(z) = (1 - a_1 z)\cdots(1 - a_t z)$ with non-zero $a_i \in \mathbb{I}$ and $t < s$. From Bézivin [Be] we obtain linear independence of the values of $\phi(z)$ also in the case $\deg \mathcal{P} = s$, in particular, of the values of $E_q(z)$, but his proof is based

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on Borel–Dwork-type rationality criteria, see [A], and is not quantitative (at least until now). The first quantitative linear independence measure for the values of general $\phi(z)$ was obtained in [Va2]. This paper uses Siegel's method applied to a system of functional equations of Poincaré-type and the connection between the solutions of these functional equations and $\phi(z)$ applied already in [AKV]. Another essential ingredient is the use of Padé-type approximations of the second kind for these solutions.

A variant of Siegel's method can be used to get for the values of Siegel E- and G-functions, a linear independence measure depending not only on the maximum of the absolute values of the coefficients but on individual coefficients. Baker [Ba] was the first to obtain such a result for the values of exponential function, and there are a lot of later works of this type, see, *e.g.*, [Fe1, Fe2, Ma, Va1, So, Zu]. Such measures are not known for the values of q-series, and our aim in the present work is to give a linear independence measure depending on individual coefficients of the linear form in the values of $\phi(z)$. Our approach is mainly based on the ideas used in [Va2] and [So]. More precisely, we prove the following general result.

Theorem 1 *Suppose that $\alpha_1, \dots, \alpha_m$ are non-zero elements of $\mathbb{I}$ satisfying $\alpha_i \neq \alpha_j q^l$, $l \in \mathbb{Z}$, for all $i \neq j$. Further, suppose that either $\deg \mathcal{P}(z) < s$ or $\deg \mathcal{P}(z) = s$ and $\alpha_i \neq \mathcal{P}_s q^n$, $i = 1, \dots, m$; $n = 1, 2, \dots,$ where $\mathcal{P}_s$ is the leading coefficient of $\mathcal{P}(z)$. Then for any given $\epsilon > 0$, there exists a positive constant $C = C(\epsilon)$ such that for all integers $l_0, l_1, \dots, l_m$ of $\mathbb{I}$, not all zero, we have*

$$|l_0 + l_1\phi(\alpha_1) + \cdots + l_m\phi(\alpha_m)| > C(\bar{l}_1 \cdots \bar{l}_m)^{-\mu(m,s)-\epsilon}, \tag{2}$$

where $\bar{l}_i = \max\{1, |l_i|\}$, $i = 1, \dots, m$, and

$$\mu(m,s) = \frac{4s\rho_0^2 + 4(s+2)\rho_0 + (s+17)}{4\rho_0 - 13m}$$

with

$$\rho_0 = \rho_0(m,s) = \frac{13m}{4} + \sqrt{\Big(\frac{13m}{4}\Big)^2 + \frac{13m(s+2)+s+17}{4s}}. \tag{3}$$

Easy verification shows that

$$\mu(m+1,s) - \mu(m,s) < 13s \quad \text{and} \quad \mu(1,s) < 15s + 5 \quad \text{for } m \geq 1 \text{ and } s \geq 1,$$

hence $\mu(m,s) < 13ms + 2s + 5$ for all $m \geq 1$ and $s \geq 1$.

As a special corollary of Theorem 1, in which the exponent on the right of (2) can be sharpened, we have the following result for the values of the q-exponential function.

Theorem 2 *Suppose that $\alpha_1, \dots, \alpha_m$ are non-zero elements of $\mathbb{I}$ satisfying $\alpha_i \neq -q^n$, $i = 1, \dots, m$; $n = 1, 2, \dots,$ and $\alpha_i \neq \alpha_j q^l$, $l \in \mathbb{Z}$, for all $i \neq j$. Then there exists a positive constant C' such that for all integers $l_0, l_1, \dots, l_m$ of $\mathbb{I}$, not all zero, we have*

$$|l_0 + l_1 E_q(\alpha_1) + \cdots + l_m E_q(\alpha_m)| > C'(\bar{l}_1 \cdots \bar{l}_m)^{-(24m+11)/2}. \tag{4}$$

Theorems 1 and 2 improve the corresponding results of [Va2] in the case of archimedian valuation and the field $\mathbb{I}$. Our theorems also partly sharpen the results of [St] when $t < s$. We also note that it would be possible to consider non-integral $q \in \mathbb{I}$, if the denominator is sufficiently small in comparision to $|q|$, but for the sake of simplicity we assume that q is an integer.

Remark 3 As shown in Section 6 below, the exponent on the right of (2) in Theorem 1 can be also sharpened for the values of the Tschakaloff function. Namely, assuming that non-zero elements $\alpha_1, \dots, \alpha_m$ of $\mathbb{I}$ satisfy $\alpha_i \neq \alpha_j q^l$, $l \in \mathbb{Z}$, for all $i \neq j$, with some positive constant C'' we have the estimate

$$(5) \qquad |l_0 + l_1 T_q(\alpha_1) + \cdots + l_m T_q(\alpha_m)| > C''(\bar{l}_1 \cdots \bar{l}_m)^{-(17m+9)/2},$$

where $l_0, l_1, \dots, l_m$ are any non-trivial integers of $\mathbb{I}$. But the estimate (5) is weaker than the earlier results obtained by using explicit Padé approximations (see [Bu, St]).

2 A Difference Equation

We shall consider the q-difference equation

$$(6) \qquad \alpha z^s f(z) = \mathcal{P}(z) f(qz) + \mathcal{Q}(z),$$

where s is a positive integer, $\alpha \in \mathbb{I}$ is non-zero, and $\mathcal{P}(z), \mathcal{Q}(z) \in \mathbb{I}[z]$ satisfy $\mathcal{P}(0) \neq 0$, $\mathcal{Q}(z) \not\equiv 0$, and $t = \deg \mathcal{P} \leq s$. Let us write an analytic solution (at $z = 0$) $f(z)$ of (6) as a power series

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu.$$

By denoting

$$\mathcal{P}(z) = \sum_{i=0}^{t} \mathcal{P}_i z^i, \quad \mathcal{Q}(z) = \sum_{i=0}^{u} \mathcal{Q}_i z^i$$

and using (6) we then obtain

$$(7) \qquad \mathcal{P}_0 q^\nu f_\nu = -\sum_{i=1}^{t} \mathcal{P}_i q^{\nu-i} f_{\nu-i} - \mathcal{Q}_\nu, \qquad \nu = 0, 1, \dots, s-1,$$

$$\mathcal{P}_0 q^\nu f_\nu = \alpha f_{\nu-s} - \sum_{i=1}^{t} \mathcal{P}_i q^{\nu-i} f_{\nu-i} - \mathcal{Q}_\nu, \quad \nu \geq s,$$

where we agree that $f_\nu = 0$ for all $\nu < 0$ and $\mathcal{Q}_\nu = 0$ for all $\nu > u$. By (7) it follows that

$$(8) \qquad F_\nu := \mathcal{P}_0^{\nu+1} q^{\nu(\nu+1)/2} f_\nu \in \mathbb{Z}[\alpha, \mathcal{P}_i, \mathcal{Q}_i, q], \quad \nu = 0, 1, \dots,$$

and the degree of F_ν with respect to $\alpha, \mathcal{P}_i, \mathcal{Q}_i$ is $\leq \nu + 1$ and with respect to q is $\leq \nu(\nu+1)/2$. Furthermore the recursive formulae (7) also imply, as proved in [AKV], that

$$|f_\nu| \leq C_1^{\nu+1}, \tag{9}$$

where C_1 (as also $C_2, C_3, \dots$ later) is a positive constant depending on s, q, α (or α_i later), $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ (or $\mathcal{Q}_i(z)$ later). We also note that by using (6) the function $f(z)$ can be continued meromorphically to $\mathbb{C}$.

The functional equation (6) implies

$$f(z) = -\sum_{n=1}^{\infty} \frac{q^{-sn(n-1)/2}\mathcal{Q}(zq^{-n})}{\mathcal{P}(zq^{-1})\cdots\mathcal{P}(zq^{-n})}(\alpha z^s)^{n-1},$$

if $\mathcal{P}(zq^{-k}) \neq 0$, $k = 1, 2, \dots$. If $\mathcal{Q}(z) = -\mathcal{P}(z)$, then $f(q) = \phi(\alpha)$ in (1), and thus we can consider linear independence of $\phi(\alpha_1), \dots, \phi(\alpha_m)$ by considering a system of difference equations of the type(6). In particular,

(i) $s = 1, \mathcal{P}(z) = q - z$

gives the q-exponential function $E_q(z)$, while

(ii) $s = 1, \mathcal{P}(z) \equiv q$

gives the Tschakaloff function $T_q(z)$. Note that in these two cases we have

$$F_\nu = q\alpha \prod_{j=1}^{\nu-1}(\alpha + q^j) \quad \text{in (i)}, \tag{10}$$

$$F_\nu = q\alpha^\nu \quad \text{in (ii)}.$$

Still another consequence of the difference equation is the iteration equation

$$(\alpha z^s)^k q^{uk} f(zq^{-k}) = X_k(z,q) f(z) + Y_k(z,q), \tag{11}$$

where (see [AKV, Lemma 3])

$$X_k(z,q) = q^{sk(k+1)/2+uk} \prod_{j=1}^{k} \mathcal{P}(zq^{-j})$$

is independent of α and $\mathcal{Q}(z)$, and

$$Y_k(z,q) = \sum_{j=1}^{k} (\alpha z^s)^{j-1} q^{s(k(k+1)/2 - j(j-1)/2)+uk} \mathcal{Q}(zq^{-j}) \prod_{l=j+1}^{k} \mathcal{P}(zq^{-l}).$$

Further, we have

$$|X_k(z,q)| \leq C_2^k |q|^{sk(k+1)/2} \max\{1, |z|\}^{C_3 k}. \tag{12}$$

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3 An Analytic Construction

Let $\alpha_1, \dots, \alpha_m \in \mathbb{I}$ and consider a system

$$\alpha_i z^s f_i(z) = \mathcal{P}(z) f_i(qz) + \mathcal{Q}_i(z), \qquad i = 1, \dots, m, \tag{13}$$

of difference equations, where $\mathcal{Q}_i(z) \in \mathbb{I}[z]$, $\mathcal{Q}_i(z) \not\equiv 0$. Let

$$f_i(z) = \sum_{\nu=0}^{\infty} f_{i\nu} z^{\nu}, \quad i = 1, \dots, m,$$

be the analytic (at $z = 0$) solution of (13). We shall construct Padé-type approximations of the second kind for these functions.

Let $n_1, \dots, n_m$ be positive integers, $N = n_1 + \cdots + n_m$, and choose δ, $0 < \delta < 1/m$, such that

$$n_i \geq \delta N, \quad i = 1, \dots, m. \tag{14}$$

We are looking for a polynomial

$$P(z) = \sum_{\mu=0}^{N} \frac{p_\mu z^\mu}{q^{\mu(\mu-1)/2}} \not\equiv 0 \tag{15}$$

with integer coefficients $p_\mu \in \mathbb{I}$, such that for all $i = 1, \dots, m$ the expansion

$$P(z) f_i(z) = \sum_{k=0}^{\infty} q_{ik} z^k$$

satisfies the conditions $q_{ik} = 0$ for $k = N+1, N+2, \dots, N + n_i - [\delta N] - 1$. We have

$$\begin{aligned} P(z) f_i(z) &= \sum_{k=0}^{\infty} \sum_{\substack{\nu=0 \\ \nu \geq k-N}}^{k} \frac{f_{i\nu} p_{k-\nu}}{q^{(k-\nu)(k-\nu-1)/2}} z^k \\ &= \sum_{k=0}^{\infty} \sum_{\substack{\nu=0 \\ \nu \geq k-N}}^{k} \frac{F_{i\nu} p_{k-\nu}}{\mathcal{P}_0^{\nu+1} q^{k(k-1)/2 + \nu(\nu-k+1)}} z^k, \end{aligned}$$

where, analogously to (8), $F_{i\nu} = \mathcal{P}_0^{\nu+1} q^{\nu(\nu+1)/2} f_{i\nu}$. Thus the condition $q_{ik} = 0$ for $k > N$ is equivalent to

$$\sum_{\nu=k-N}^{k} \mathcal{P}_0^{k-\nu} q^{(\nu+1)(k-\nu)} F_{i\nu} p_{k-\nu} = 0. \tag{16}$$

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We now choose natural numbers A and B in such a way that the numbers $A\alpha_i$ and the coefficients of $B\mathcal{P}(z)$ and $B\mathcal{Q}_i(z)$ for $i = 1, \dots, m$ are integers in $\mathbb{I}$. Multiplying the equation (16) by $(AB^2)^k$ we thus obtain a linear equation in p_μ with integer coefficients from $\mathbb{I}$ satisfying, by (8) and (9),

$$|\text{coefficients}| \le C_4^k \max_{k-N\le\nu\le k} \left\{ |q|^{\nu(\nu+1)/2+(\nu+1)(k-\nu)} \right\} \le C_5^k |q|^{k^2/2}.$$

We need the condition $q_{ik} = 0$ for $k = N+1, N+2, \dots, N+n_i - [\delta N] - 1$, and for these k we have

$$\begin{aligned} k &\le N + n_i - \delta N = N + (N - n_1 - \cdots - n_{i-1} - n_{i+1} - \cdots - n_m) - \delta N \\ &\le 2N - m\delta N \end{aligned}$$

by (14). Thus the absolute values of the coefficients are bounded by

$$C_6^N |q|^{(2N-m\delta N)^2/2}.$$

The number of linear equations $q_{ik} = 0$ is equal to

$$\sum_{i=1}^m (n_i - [\delta N] - 1) = N - m([\delta N] + 1),$$

and the number of indeterminates p_μ is $N + 1$. Therefore Siegel's lemma (see, *e.g.*, [Sh, Chapter 3, Lemma 13]) yields the existence of integers $p_\mu \in \mathbb{I}$, not all zero, such that

$$(17) \qquad |p_\mu| \le C_7^N |q|^{\gamma_1 N^2}, \quad \gamma_1 = \gamma_1(\delta) = \frac{(2 - m\delta)^2(1 - m\delta)}{2m\delta}.$$

By using (10), we see that in the special cases (i) and (ii) we can replace $\gamma_1(\delta)$ in (17) by

$$(18) \qquad \gamma_1^{(\mathrm{i})}(\delta) = \frac{(3 - 2m\delta)(1 - m\delta)}{2m\delta} \quad \text{and} \quad \gamma_1^{(\mathrm{ii})}(\delta) = \frac{1}{2}\gamma_1(\delta),$$

respectively.

Let us define

$$Q_i(z) = \sum_{k=0}^N q_{ik} z^k, \quad i = 1, \dots, m.$$

Since, for $k \le N$,

$$q_{ik} = \sum_{\nu=0}^k \frac{f_{i\nu} p_{k-\nu}}{q^{(k-\nu)(k-\nu-1)/2}} = \sum_{\nu=0}^k \frac{F_{i\nu} p_{k-\nu} q^{\nu(k-\nu)}}{\mathcal{P}_0^{\nu+1} q^{k(k-1)/2+\nu}},$$

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it follows that the polynomials

$$q^{N(N+1)/2}(AB^2)^{N+1}Q_i(z)$$

have integer coefficients in $\mathbb{I}$.

By (9) and (17), for all $k > N$, the following estimates hold:

$$|q_{ik}| = \Big| \sum_{\nu=k-N}^{k} \frac{f_{i\nu} p_{k-\nu}}{q^{(k-\nu)(k-\nu-1)/2}} \Big| \le C_1^{k+1} C_7^N |q|^{\gamma_1 N^2} \Big| \sum_{\nu=k-N}^{k} \frac{1}{|q|^{(k-\nu)(k-\nu-1)/2}} \Big|$$
$$\le C_8^k |q|^{\gamma_1 N^2}.$$

By defining

$$R_i(z) = P(z) f_i(z) - Q_i(z), \qquad i = 1, \dots, m,$$

we then obtain, for all $|z| < (2C_8)^{-1}$,

$$(19) \qquad |R_i(z)| = \Big| \sum_{k=N_i}^{\infty} q_{ik} z^k \Big| \le |q|^{\gamma_1 N^2} \sum_{k=N_i}^{\infty} (C_8 |z|)^k \le C_9^N |q|^{\gamma_1 N^2} |z|^{N_i},$$

where $N_i = N + n_i - [\delta N]$, $i = 1, \dots, m$.

We have thus proved the following

Lemma 4 *There exists a polynomial*

$$P(z) = \sum_{\mu=0}^{N} \frac{p_\mu z^\mu}{q^{\mu(\mu-1)/2}} \not\equiv 0$$

with integers $p_\mu \in \mathbb{I}$ satisfying (17) *such that the polynomials*

$$q^{N(N-1)/2} P(z), \quad q^{N(N+1)/2}(AB^2)^{N+1} Q_i(z)$$

have integer coefficients in $\mathbb{I}$ and the forms $R_i(z)$ satisfy the estimates (19) *for all $|z| <$ $(2C_8)^{-1}$.*

4 An Iteration Process

Let

$$P_0(z) = P(z), \quad Q_{0i}(z) = Q_i(z), \quad R_{0i}(z) = R_i(z),$$

and define further

$$(20) \quad P_j(z) = z^s P_{j-1}(qz), \quad Q_{ji}(z) = -\alpha_i^{-1} \big(\mathcal{P}(z) Q_{j-1,i}(qz) + \mathcal{Q}_i(z) P_{j-1}(qz) \big),$$

where $i = 1, \dots, m$, $j = 1, 2, \dots$. If

$$R_{ji}(z) = P_j(z) f_i(z) - Q_{ji}(z),$$

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then from the functional equations (13) it follows that

$$R_{ji}(z) = \alpha_i^{-1}\mathcal{P}(z)R_{j-1,i}(qz), \quad i = 1,\dots,m,\ j = 1,2,\dots. \tag{21}$$

We are interested in the determinant

$$\Delta(z) = \det\begin{pmatrix} P_0(z) & Q_{01}(z) & \dots & Q_{0m}(z) \\ P_1(z) & Q_{11}(z) & \dots & Q_{1m}(z) \\ \dots & \dots & \dots & \dots \\ P_m(z) & Q_{m1}(z) & \dots & Q_{mm}(z) \end{pmatrix}$$
$$= (-1)^m \cdot \det\begin{pmatrix} P_0(z) & R_{01}(z) & \dots & R_{0m}(z) \\ P_1(z) & R_{11}(z) & \dots & R_{1m}(z) \\ \dots & \dots & \dots & \dots \\ P_m(z) & R_{m1}(z) & \dots & R_{mm}(z) \end{pmatrix}.$$

Assume now that none of the functions $f_i(z)$ is a polynomial and that $\alpha_i \neq \alpha_j q^l$, $l \in \mathbb{Z}$, for all $i \neq j$. Furthermore, let $\alpha \neq 0$ be an element of $\mathbb{I}$ satisfying $\mathcal{P}(\alpha q^{-k}) \neq 0$ for $k = 1,2,\dots$. Then (see [Va2, Lemma 3]) $\Delta(z) \not\equiv 0$. Since

$$\operatorname{ord}_{z=0}\Delta(z) \geq N_1 + \dots + N_m \geq (m+1)N - m\delta N$$

and

$$\deg_z \Delta(z) \leq (m+1)N + S\frac{m(m+1)}{2},$$

where $S = \max\{s, \deg \mathcal{Q}_i(z)\}$, we deduce that for each $\rho > m\delta$, there exists an integer k satisfying (see [Va2, Section 5])

$$(\rho - m\delta)N - S\frac{m(m+1)}{2} < k \leq \rho N \tag{22}$$

and

$$\Delta(\alpha q^{-k}) \neq 0. \tag{23}$$

Let us take

$$D_k = (AB)^{N+1}(A_1B)^m A_2^{N+Sm} q^{N(N+1)/2+k(N+Sm)},$$

A_1 and A_2 are nonzero rational integers such that $A_1\alpha_i^{-1}$ and $A_2\alpha$ are integers in $\mathbb{I}$. By Lemma 1 and the recursions (20) it then follows that the numbers

$$D_k P_j(\alpha q^{-k}), \quad D_k Q_{ji}(\alpha q^{-k})$$

are integers in $\mathbb{I}$. Furthermore, by (15), (17) and (20),

$$|P_j(\alpha q^{-k})| = |q^{j(j-1)/2}\alpha q^{-k}|^s\,|P(\alpha q^{j-k})| \leq C_{10}^N |q|^{\gamma_1 N^2}, \quad j = 0,1,\dots,m, \tag{24}$$

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and by (19) and (21),

$$|R_{ji}(\alpha q^{-k})| = |\alpha_i^{-j}\mathcal{P}(\alpha q^{-k})\cdots\mathcal{P}(\alpha q^{-k+j-1})|\,|R_i(\alpha q^{j-k})| \le C_{11}^N|q|^{\gamma_1 N^2 - kN_i}, \qquad j = 0, 1, \dots, m, \tag{25}$$

if $2C_8|\alpha|\,|q|^m < |q|^k$.

We now denote $u = \max_{1\le i\le m}\{\deg \mathcal{Q}_i(z)\}$ and use (11) to obtain

$$\begin{aligned}\hat{r}_{ji} &= (\alpha_i\alpha^s)^k q^{uk} R_{ji}(\alpha q^{-k})\\ &= X_k(\alpha, q)P_j(\alpha q^{-k})f_i(\alpha) + \big(Y_k(\alpha, q)P_j(\alpha q^{-k}) - (\alpha_i\alpha^s)^k q^{uk} Q_{ji}(\alpha q^{-k})\big)\\ &=: \hat{p}_j f_i(\alpha) - \hat{q}_{ji}.\end{aligned} \tag{26}$$

Assume now that k satisfies (22) and (23). Let

$$r_{ji} = (BA_2^s)^k D_k \hat{r}_{ji} =: p_j f_i(\alpha) - q_{ji}.$$

Then all p_j, q_{ji} are integers in $\mathbb{I}$ and by the above consideration and (25) and (26) we obtain

$$\begin{aligned}|r_{ji}| &\le C_{12}^N|q|^{N^2(\gamma_1+1/2)-(\rho-m\delta)N(N_i-N)} \le C_{12}^N|q|^{\gamma_2 N^2-(\rho-m\delta)Nn_i},\\ \gamma_2 &= \gamma_2(\delta,\rho) = \gamma_1(\delta) + \frac{1}{2} + \delta(\rho - m\delta),\end{aligned} \tag{27}$$

provided that $|q|^{(\rho-m\delta)N} > C_{13}$, and by the estimates (12) and (24) we have

$$\begin{aligned}|p_j| &\le C_{14}^N|q|^{N^2(\gamma_1+1/2+\rho+s\rho^2/2)} = C_{14}^N|q|^{\gamma_3 N^2},\\ \gamma_3 &= \gamma_3(\delta,\rho) = \gamma_1(\delta) + \rho + \frac{1}{2}(1 + s\rho^2).\end{aligned} \tag{28}$$

Finally, we note that

$$\det\begin{pmatrix} p_0 & q_{01} & \cdots & q_{0m}\\ p_1 & q_{11} & \cdots & q_{1m}\\ \cdots & \cdots & \cdots & \cdots\\ p_m & q_{m1} & \cdots & q_{mm}\end{pmatrix} = \big((BA_2^s)^k D_k\big)^{m+1} X_k(\alpha, q)\Delta(\alpha q^{-k})(q^u\alpha^s)^{mk}\prod_{i=1}^m \alpha_i^k \ne 0. \tag{29}$$

5 A Number-Theoretical Result

We shall require a more useful notation $\rho_0 = \rho - m\delta$. Suppose that

$$\rho_0 > \frac{m(\gamma_1 + 1/2)}{1 - m\delta}.$$

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Then

$$\rho_0 - m\gamma_2 = \rho_0 - m\Big(\gamma_1 + \frac{1}{2} - \delta\rho_0\Big) = \rho_0(1 - m\delta) - m\Big(\gamma_1 + \frac{1}{2}\Big) > 0.$$

Next take an arbitrary number $\epsilon > 0$ satisfying

$$\epsilon < \frac{\rho_0 - m\gamma_2}{2m},$$

so that we have $\rho_0 - m(\gamma_2 + 2\epsilon) > 0$, and define

$$\lambda_0 = \max\big\{\epsilon^{-1}\log_{|q|}\max\{2m, C_{12}, C_{14}\}, \quad 1 + \epsilon^{-1}(m+1)\rho_0,\ m + \rho_0^{-1}\log_{|q|} C_{13}\big\}.$$

Then, for any $\lambda > \lambda_0$, we have

$$\max\{2m, C_{12}, C_{14}\} < |q|^{\epsilon\lambda}, \tag{30}$$

$$(m+1)\rho_0 < \epsilon(\lambda - 1), \tag{31}$$

$$C_{13} < |q|^{\rho_0(\lambda - m)}. \tag{32}$$

Set $L_0 = |q|^{\lambda_0^2(\rho_0 - m(\gamma_2 + 2\epsilon))}$ and consider an arbitrary linear form

$$\ell = l_0 + l_1 f_1(\alpha) + \cdots + l_m f_m(\alpha), \tag{33}$$

with integer coefficients $l_i \in \mathbb{I}$, not all zero, satisfying the condition $L = \bar{l}_1\bar{l}_2\cdots\bar{l}_m > L_0$, where $\bar{l}_i = \max\{1, |l_i|\}$ for $i = 1, \dots, m$. Define

$$\lambda = \sqrt{\frac{\log_{|q|} L}{\rho_0 - m(\gamma_2 + 2\epsilon)}} > \lambda_0 \tag{34}$$

(thanks to the definition of L_0) and

$$n_i = \Big[\frac{\log_{|q|}\bar{l}_i + \lambda^2(\gamma_2 + 2\epsilon)}{\rho_0\lambda}\Big], \quad i = 1, \dots, m. \tag{35}$$

Since

$$\sum_{i=1}^{m}\frac{\log_{|q|}\bar{l}_i + \lambda^2(\gamma_2 + 2\epsilon)}{\rho_0\lambda} = \frac{\log_{|q|} L + m\lambda^2(\gamma_2 + 2\epsilon)}{\rho_0\lambda} = \lambda,$$

we deduce that

$$\lambda - m < N = n_1 + \cdots + n_m \le \lambda. \tag{36}$$

In addition,

$$\begin{aligned} n_i &> \frac{\log_{|q|}\bar{l}_i + \lambda^2(\gamma_2 + 2\epsilon)}{\rho_0\lambda} - 1 \ge \frac{\gamma_2 + 2\epsilon}{\rho_0}\lambda - 1 \\ &\ge \gamma_2\frac{N}{\rho_0} = \Big(\gamma_1 + \frac{1}{2} + \delta\rho_0\Big)\frac{N}{\rho_0} > \delta N \end{aligned}$$

as required.

We now choose k satisfying (22) or equivalently, (23) and

$$\rho_0 N - S\frac{m(m+1)}{2} < k \le (\rho_0 + m\delta)N.$$

For the given linear form (33) there exists, by (29), an index $j \in \{0, 1, \dots, m\}$ such that

$$\Lambda = l_0 p_j + l_1 q_{j1} + \cdots + l_m q_{jm} \neq 0.$$

Since Λ is an integer in $\mathbb{I}$, we have $|\Lambda| \ge 1$. By denoting for brevity $p = p_j$, $q_i = q_{ji}$ and $r_i = r_{ji}$, we then obtain by (27), (28), (30) and (36)

$$|p| \le C_{14}^N |q|^{\gamma_3 N^2} < |q|^{(\gamma_3+\epsilon)\lambda^2},$$
$$|r_i| \le C_{12}^N |q|^{\gamma_2 N^2 - \rho_0 N n_i} < |q|^{(\gamma_2+\epsilon)\lambda^2 - \rho_0(\lambda - m)n_i};$$

note that we may use (27) by (32) and (36). Since

$$\bar{l}_i < |q|^{\rho_0 \lambda(n_i+1) - \lambda^2(\gamma_2+2\epsilon)}$$

by (35), we obtain for all $i = 1, \dots, m$

$$\begin{aligned}\bar{l}_i |r_i| &< |q|^{-\epsilon\lambda^2 + \rho_0\lambda(n_i+1) - \rho_0(\lambda-m)n_i} \\ &< |q|^{-\epsilon\lambda^2 + \rho_0\lambda + \rho_0 m\lambda} = |q|^{\lambda(-\epsilon\lambda + (m+1)\rho_0)} \\ &< |q|^{-\epsilon\lambda} \qquad \text{(by (31))} \\ &< \frac{1}{2m} \qquad \text{(by (30))}.\end{aligned}$$

By the relation

$$\begin{aligned}p\ell &= l_0 p + l_1 p f_1(\alpha) + \cdots + l_m p f_m(\alpha) \\ &= l_0 p + l_1(r_1 + q_1) + \cdots + l_m(r_m + q_m) \\ &= \Lambda + l_1 r_1 + \cdots + l_m r_m\end{aligned}$$

we thus derive an inequality

$$|p\ell| \ge |\Lambda| - \sum_{i=1}^{m} |l_i r_i| \ge 1 - \sum_{i=1}^{m} \bar{l}_i |r_i| > 1 - \sum_{i=1}^{m} \frac{1}{2m} = \frac{1}{2}.$$

Finally, using the definition (34) of λ we obtain

$$|\ell| > (2p)^{-1} > \frac{1}{2}|q|^{-(\gamma_3+\epsilon)\lambda^2} = \frac{1}{2}L^{-(\gamma_3+\epsilon)/(\rho_0 - m(\gamma_2+2\epsilon))}.$$

Since $\epsilon > 0$ is arbitrary, we can state the final result in the following form (we set $\delta_0 = m\delta$).

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Theorem 5 *Suppose that none of the functions $f_1(z), \dots, f_m(z)$ is a polynomial and that $\alpha_i \neq \alpha_j q^l$, $l \in \mathbb{Z}$, for all $i \neq j$. Let $\alpha \neq 0$ be an element of $\mathbb{I}$ satisfying $\mathcal{P}(\alpha q^{-k}) \neq 0$, $k = 1, 2, \dots$. Let*

$$\gamma_1(\delta_0) = \frac{(2-\delta_0)^2(1-\delta_0)}{2\delta_0}, \quad \gamma_2(\delta_0, \rho_0) = \gamma_1 + \frac{1}{2} + \frac{\delta_0\rho_0}{m},$$
$$\gamma_3(\delta_0, \rho_0) = \gamma_1 + \delta_0 + \rho_0 + \frac{1}{2}\big(1 + s(\delta_0+\rho_0)^2\big),$$

where $0 < \delta_0 < 1$, $\rho_0 > 0$ and, in addition,

$$\rho_0 > \frac{m(\gamma_1 + 1/2)}{1-\delta_0}.$$

Then for any $\epsilon_0 > 0$, there exists a positive constant $C_0 = C_0(\epsilon_0)$ such that for any integers $l_0, l_1, \dots, l_m$ of $\mathbb{I}$, not all zero, there holds the inequality

$$|l_0 + l_1 f_1(\alpha) + \cdots + l_m f_m(\alpha)| > C_0 \cdot (\bar{l}_1 \cdots \bar{l}_m)^{-\gamma_3/(\rho_0 - m\gamma_2) - \epsilon_0},$$

where $\bar{l}_i = \max\{1, |l_i|\}$ for $i = 1, \dots, m$.

Remark 6 From the above proof we can see that the construction used here does not work in the p-adic case. For the p-adic case it is obviously necessary to change the construction in such a way that the dependence on the individual n_i is in the polynomials Q_i instead of the forms R_i (see [Va1]).

6 Proof of Theorems 1 and 2

In the proof of Theorem 1 we use the fact $f_i(q) = \phi(\alpha_i)$ if $\mathcal{Q}_i(z) = -\mathcal{P}(z)$, $i = 1, \dots, m$ (see Section 2). By the assumptions of Theorem 1 it follows that the corresponding $f_i(z) \notin \mathbb{I}[z]$, for the details we refer to[AKV]. Thus we may apply Theorem 3 to get a result with

$$\frac{\gamma_3}{\rho_0 - m\gamma_2} = \frac{4 - 7\delta_0(1-\delta_0) - \delta_0^3 + 2\delta_0\rho_0 + \delta_0 s(\delta_0+\rho_0)^2}{2\delta_0(1-\delta_0)\rho_0 - m(4 - 7\delta_0 + 5\delta_0^2 - \delta_0^3)} \tag{38}$$

instead of $\mu(m, s)$. We now choose $\delta_0 = 1/2$; then $\rho_0 = \rho_0(m, s)$ given in (3) admits the minimum value $\mu(m, s)$ for the above expression (38) (in the case $\delta_0 = 1/2$). This proves Theorem 1.

Theorem 2 and Remark 1 follow by noting that the choices (i) and (ii) give the functions $E_q(z)$ and $T_q(z)$, respectively. Thanks to (18), we may replace $\gamma_1(\delta_0)$ in (37) by

$$\gamma_1^{(\mathrm{i})}(\delta_0) = \frac{(3-2\delta_0)(1-\delta_0)}{2\delta_0}$$

in the case (i) and by

$$\gamma_1^{(\mathrm{ii})}(\delta_0) = \frac{1}{2}\gamma_1(\delta_0)$$

in the case (ii). Next take $\delta_0^{(i)} = 2/5$, $\delta_0^{(ii)} = 1/3$ and the corresponding values $\rho_0^{(i)}(m)$, $\rho_0^{(ii)}(m)$ that minimize the exponent $\gamma_3/(\rho_0 - m\gamma_2)$. Let $\mu^{(i)}(m)$ and $\mu^{(ii)}(m)$ denote the minimal exponents. It follows easily that

$$\mu^{(i)}(1) = 17.14\cdots \quad \text{and} \quad \mu^{(i)}(m+1) - \mu^{(i)}(m) < 12 \qquad \text{for } m \geq 1,$$
$$\mu^{(ii)}(1) = 12.98\cdots \quad \text{and} \quad \mu^{(ii)}(m+1) - \mu^{(ii)}(m) < 8.5 \qquad \text{for } m \geq 1,$$

and as a consequence we arrive at the desired estimates (4) and (5). The proof of Theorem 2 and Remark 1 is complete.

Remark 7 Taking $\delta_0^{(i)} = 4/9$ we arrive at the better exponent $11.79m + 5.27$ for $m \geq 10$ in (4).

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