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STRONGLY AND WEAKLY NON-POISED H-B INTERPOLATION PROBLEMS

R. DEVORE, A. MEIR, AND A. SHARMA

1. Introduction. The problem of Hermite-Birkhoff interpolation is to determine: what $n + 1$ interpolatory conditions imposed on a polynomial $P(x)$ of degree n and its derivatives determine the polynomial uniquely. It is customary now to indicate the imposed conditions by means of a matrix $E = (e_{ij})$ which is called the incidence matrix for the problem. E has n columns indexed from 0 to n, which correspond to the derivatives and has m rows indexed from 1 to m, which correspond to the real points $x_1, \ldots, x_m$, where the data are given. The entries in E are all 0 or 1 and there are exactly $n + 1$ ones in E. An entry $e_{ij} = 1$ indicates that $P^{(j)}(x_i)$ is prescribed. An entry $e_{ij} = 0$ indicates that no condition is imposed on $P^{(j)}(x_i)$.

If for each ordered m-tuple of real points $x_1 > x_2 > \ldots > x_m$ (called the nodes) the problem given by E has a unique solution, then E is said to be order-poised. Clearly E is order-poised if and only if the corresponding homogeneous interpolation problem has only the identically zero solution for every ordered m-tuple $x_1 > \ldots > x_m$. If we let

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

and write the conditions imposed by E at $x_1, \ldots, x_m$ as equations in $a_0, \ldots, a_n$ then a necessary and sufficient condition that E be order-poised is that the determinant

$$D(x_1, \ldots, x_m; E) = \left\| \frac{x_i^{\nu-j}}{(\nu - j)!} \right\| \neq 0 \tag{1.1}$$

where $\nu = 0, 1, \ldots, m$ and for each $i (1 \leqq i \leqq m)$ only those j's occur for which $e_{ij} = 1$.

A first analysis of the problem relies on the Pólya constants $M_k (k = 0, 1, \ldots, n)$ defined by

$$M_k = \sum_{i=1, j \leqq k}^{m} e_{ij}, \qquad k = 0, 1, \ldots, n. \tag{1.2}$$

A necessary condition for E to be order-poised is that $M_k \geqq k + 1$, $k = 0, 1, \ldots, n$. This is called the Pólya condition. It is important to recall that

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when $m = 2$, the Pólya condition is actually necessary and sufficient for a 2-point interpolation problem to be order-poised. However, it is easy to construct examples for $m > 2$ which show that the Pólya condition is not sufficient. For a historical survey of Hermite-Birkhoff interpolation problems, we refer to [**6**].

The term *sequence* denotes a string of ones in some row of E : $e_{i\alpha} = \ldots = e_{i\beta} = 1$, with $e_{i,\alpha-1} = 0$ or $\alpha = 0$ and $e_{i,\beta+1} = 0$ or $\beta = n$. Such a sequence is called *even* (*odd*) if there is an even (odd) number of ones in the sequence. The sequence is said to be supported if $\alpha \neq 0$ and there exists $i_1 < i < i_2$ and $j_1, j_2 < \alpha$ such that $e_{i_1,j_1} = e_{i_2,j_2} = 1$.

Atkinson and Sharma [**1**] have shown that if E satisfies the Pólya condition and has no supported odd sequences, then E is order-poised. Their analysis also led them to conjecture that if E has supported odd sequences, then E is non-poised. In fact Lorentz [**4**] has shown that if some row of E has exactly one supported odd sequence (and possibly other even sequences) then E is non-poised. On the other hand, Lorentz and Zeller [**5**] have pointed out that

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.3}$$

is order-poised, while

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.4}$$

is non-poised. These examples show that even for three point interpolation problems the situation becomes more complicated when E contains more than one supported odd sequence. It appears that the poisedness of a problem depends also on the location of the odd sequences.

In a recent paper, Karlin and Karon [**3**] have introduced a method of coalescing the nodes which under certain conditions reduces the problem of non-poisedness of a matrix E to that of a suitable three row matrix Γ. More precisely they show [**3**, Theorem 7, p. 20] that if the determinant of the reduced matrix Γ changes sign, then the original problem corresponding to E is non-poised. This motivates us to call a three row matrix Γ *strongly non-poised* if the determinant $D(1, x, -1; \Gamma)$ changes sign in $(-1, 1)$ and *weakly non-poised* if the determinant vanishes but does not change sign. We shall say that an incidence matrix E has the property (H), if for some integer ν $(1 < \nu < m)$, the two submatrices E_1 and E_2 of E formed by the first $\nu - 1$ and the last $m - \nu$ rows satisfy the Pólya condition. It is easy to see that if E possesses (H), then E can be coalesced to a three row matrix Γ with Hermite data in the first and last row.

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For example, the matrix

$$E = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 & 0 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 1 & \end{bmatrix} \tag{1.5}$$

can be coalesced to

$$\Gamma = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \mathbf{0} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{bmatrix} \tag{1.6}$$

by coalescing the first three rows and the last two rows. Thus it follows from Karlin and Karon's result that if the determinant corresponding to (1.6) changes sign (which will follow from our Theorem 1) then (1.5) is also non-poised.

The above considerations underscore the importance of studying three point interpolation problems. The object of this paper is to consider the particular case (motivated by the examples (1.3), (1.4), (1.5) and (1.6)) where Γ is a three row matrix with p Hermite data in the first row, q Hermite data in the last row and with two non-zero entries in the middle row. Our results yield simple criteria for poisedness and non-poisedness for a large class of incidence matrices and in particular, we give complete characterisation of poisedness in the cases $q = p$, $q = p + 1$ and $p = 1$. Our results also shed light on how the positioning of the ones in the second row effects the poisedness of the problem. For example, our Theorem 3 will show why (1.3) is poised and (1.4) is non-poised.

2. A sufficient condition for non-poisedness. We shall suppose that $p \leqq q$ and that Γ is a three row matrix with p Hermite data in the first and q Hermite data in the third row. We also suppose that Γ has exactly two non-zero entries. Without loss of generality, we may assume that $x_1 = 1$, $x_2 = \alpha$, $x_3 = -1$. Set

$$Q(x) = (x-1)^p(x+1)^q, \quad P(x) = (Ax+B)Q(x). \tag{2.1}$$

We suppose that the 1's in the second row of Γ occur in the ith and jth columns. Then the interpolation problem corresponding to Q is poised, if and only if $P^{(i)}(\alpha) = P^{(j)}(\alpha) = 0$ implies $A = B = 0$, for every α, $-1 < \alpha < 1$. Since

$$P^{(i)}(\alpha) = (A\alpha + B)Q^{(i)}(\alpha) + iAQ^{(i-1)}(\alpha),$$

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we see that a necessary and sufficient condition for poisedness is that the determinant

$$\Delta(x) = \begin{vmatrix} ((x-1)Q)^{(i)} & Q^{(i)} \\ ((x-1)Q)^{(j)} & Q^{(j)} \end{vmatrix} = iQ^{(i-1)}(x)Q^{(j)}(x) - jQ^{(j-1)}(x)Q^{(i)}(x) \tag{2.2}$$

does not vanish in $(-1, 1)$.

Since it is known that if $j = i + 1$, the problem is poised, we shall restrict our attention to $j \geqq i + 2$. We shall show that under certain conditions, we can guarantee that $\Delta(x)$ vanishes (respectively changes sign) in $(-1, 1)$ and hence Γ is weakly (respectively strongly) non-poised.

In the sequel we shall need the following lemmas:

LEMMA 1. *The zeros of $Q^{(\nu)}(x)$ in $(-1, 1)$ are simple. If Z_ν denotes the number of these zeros, then*

$$Z_\nu = \begin{cases} \nu, & \nu \leqq p \\ p, & p \leqq \nu \leqq q \\ p + q - \nu, & q \leqq \nu. \end{cases}$$

Also, if $\bar{Z}_\nu$ denotes the number of distinct zeros of $Q^{(\nu)}(x)$ in $[-1, 1]$, then $\bar{Z}_\nu = Z_{\nu+1} + 1$.

Proof. For $\nu \leqq p$, $Q^{(\nu)}(x) = C_\nu(x-1)^\alpha(x+1)^\beta P_\nu^{(\alpha,\beta)}(x)$ where $\alpha = p - \nu \geqq 0$, $\beta = q - \nu \geqq 0$, C_ν is a constant and $P_\nu^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree ν, which is known to have real and simple zeros in $(-1, 1)$. Thus the lemma is proved for $\nu \leqq p$. For $\nu > p$, the proof follows from successive use of Rolle's theorem and a simple accounting for the zeros of $Q^{(\nu)}(x)$.

The second part of the lemma easily follows from the values of Z_ν.

LEMMA 2. *Suppose $j \geqq i + 2$. Then a necessary and sufficient condition that*

$$|Z_i - Z_j| \leqq 1 \quad \text{and} \quad |Z_{i-1} - Z_{j-1}| \leqq 1 \tag{2.3}$$

is that at least one of the following holds:

$$p \leqq i < j \leqq q, \tag{2.4}$$

$$i + j = p + q + 1. \tag{2.5}$$

Proof. The proof depends on checking 6 cases.

Case 1: $j \leqq p + 1$. In this case $i \leqq p - 1$ and $j - 1 \leqq p$, so that

$$|Z_{i-1} - Z_{j-1}| = |(i-1) - (j-1)| = j - i \geqq 2.$$

Case 2: $p + 2 \leqq j \leqq q + 1$ and $i \leqq p - 1$. Then

$$|Z_{i-1} - Z_{j-1}| = |(i-1) - p| \geqq 2.$$

Case 3: $p \leqq i < j \leqq q + 1$. In this case

$$|Z_i - Z_j| \leqq |p - (p-1)| = 1.$$
$$|Z_{i-1} - Z_{j-1}| \leqq |(p-1) - p| = 1.$$

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Case 4: $q+1 \leqq j$ and $i \leqq p$. In this case

$$|Z_i - Z_j| = |i - (p+q-j)| = |(i+j) - p - q|$$

and

$$|Z_{i-1} - Z_{j-1}| = |(i-1) - (p+q-j+1)| = |(i+j) - (p+q+2)|.$$

Thus both these numbers are $\leqq 1$ if and only if $i+j = p+q+1$.

Case 5: $q+2 \leqq j$ and $p+1 \leqq i \leqq q$. In this case

$$|Z_i - Z_j| = |p - (p+q-j)| = |j-q| \geqq 2.$$

Case 6: $q+1 \leqq i < j$. In this case

$$|Z_i - Z_j| = |(p+q-i) - (p+q-j)| = |j-i| \geqq 2.$$

LEMMA 3. *Let* $Q(x) = (x-1)^p(x+1)^q$. *Suppose* $0 \leqq i < p$, $i+j = p+q+1$. *Then we have*

$$Q^{(i)}(x) = (x-1)^{p-i}(x+1)^{q-i} \cdot 2^i \cdot i!\, P_i^{(p-i,q-i)}(x) \tag{2.6}$$

$$((x-1)Q)^{(i)} = (x-1)^{p+1-i}(x+1)^{q-i} \cdot 2^i \cdot i!\, P_i^{(p+1-i,q-i)}(x) \tag{2.7}$$

$$Q^{(j)}(x) = 2^{i-1} \cdot j!\, P_{i-1}^{(q+1-i,p+1-i)}(x) \tag{2.8}$$

$$((x-1)Q)^{(j)} = 2^j \cdot j!\, P_i^{(q-i,p+1-i)}(x) \tag{2.9}$$

where $P_n^{(\alpha,\beta)}(x)$ *denotes the classical Jacobi polynomial defined by*

$$2^n \cdot n!\, P_n^{(\alpha,\beta)}(x) = (x-1)^{-\alpha}(x+1)^{-\beta}D^{(n)}\{(x-1)^{n+\alpha}(x+1)^{n+\beta}\}. \tag{2.10}$$

Proof. Formulae (2.6) and (2.7) are immediate consequences of (2.10). In order to obtain (2.8) and (2.9) we use first (2.10) and then the identity

$$P_n^{(-\alpha,-\beta)}(x) = \left(\frac{x-1}{2}\right)^{\alpha}\left(\frac{x+1}{2}\right)^{\beta} P_{n-\alpha-\beta}^{(\alpha,\beta)}(x), \tag{2.11}$$

where $1 \leqq \alpha \leqq n$, $1 \leqq \beta \leqq n-\alpha$. (2.11) can be easily derived from formula (4.22.2) in [7].

3. Principal results.

We shall prove the following results:

THEOREM 1. *Suppose* $j \geqq i+2$. *If* $i+j \neq p+q+1$ *and either* $i < p$ *or* $j > q$ *holds, then* Γ *is strongly non-poised.*

COROLLARY 1. *If* $i < p$, $j \leqq q+1$ *and* $j \geqq i+2$, *then* Γ *is strongly non-poised.*

COROLLARY 2. *If* $i \geqq p+1$, $j > q$ *and* $j \geqq i+2$, *then* Γ *is strongly non-poised.*

THEOREM 2. *Suppose* $j \geqq i+2$ *and* $i+j = p+q+1$. *If*

$$\frac{p+1}{q-p} \leqq i \leqq p \tag{3.1}$$

then Γ *is strongly non-poised.*

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COROLLARY 3. *If* $q \geqq 2p + 1$ *and* $j \geqq i + 2$, *then* Γ *is strongly non-poised when* $1 \leqq i \leqq p$ *or* $j > q$ (poised cases can possibly occur when $p < i < j \leqq q$).

THEOREM 3. *Suppose* $q = p$. *Then* Γ *is order-poised if* $i + j = 2p + 1$ *or* $j = i + 1$. *In all other cases* Γ *is strongly non-poised.*

THEOREM 4. *Suppose* $q = p + 1$. *If* $i + j \neq 2p + 2$, *then* Γ *is strongly non-poised for* $j \geqq i + 2$ *and is poised for* $j = i + 1$. *If* $i + j = 2p + 2$, *then* Γ *is weakly non-poised for* $i \geqq 1$ *and is poised for* $i = 0$.

THEOREM 5. *Let* $p = 1$ *and let*

$$\rho(i, j) = (q + 2)(i + j - 1)^2 - 4ij(q + 1). \tag{3.2}$$

Then Γ *is poised if* $\rho(i, j) < 0$ *and is strongly (respectively weakly) non-poised if* $\rho(i, j) > 0$ *(respectively* $= 0$*).*

Remark. The condition $\rho(i, j) < 0$ has a simple geometrical interpretation. It means that all (i, j) which give rise to poised matrices lie inside the ellipse $(q + 2)(x + y - 1)^2 - 4xy(q + 1) = 0$.

THEOREM 6. *Suppose* $i = 1$, $p \geqq 2$ *and* $q \geqq p + 2$. *Then* Γ *is order-poised if* $j = 2$ *and in all other cases* Γ *is strongly non-poised.*

Remark. The case $i = p = 1$ is covered by Theorem 5 and the case $i = 1$, $q = p + 1$ by Theorem 4.

It follows from Theorem 1 that besides the known case of $j = i + 1$, the only other poised cases may occur when either

(i) $p \leqq i < j \leqq q$

or

(ii) $i \leqq p$ and $i + j = p + q + 1$.

Regarding (ii), observe that Theorem 3 shows that if $q = p$ all these cases are order-poised. On the other hand if $q = p + 1$ or $q \geqq 2p + 1$, then Theorem 4 and Corollary 3 respectively show that all these cases are non-poised. In view of this we suspect that if $q > p$ then all cases covered by (ii) are non-poised.

The cases covered by (i) seem to be the most difficult to settle. In fact our only result in this direction is Theorem 5 where we consider the special case $p = 1$. The necessary and sufficient condition for poisedness already depends on a quadratic inequality in i and j.

The proofs of these theorems follow in the next section. As will be seen, we use different techniques in the various cases to decide whether $\Delta(x)$ changes sign or not. We do not know of any general device which would settle all cases.

4. *Proof of Theorem* 1. By the hypothesis of the theorem we see from Lemma 2 that either

$$|Z_{i-1} - Z_{j-1}| \geqq 2 \quad \text{or} \quad |Z_i - Z_j| \geqq 2.$$

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First suppose that $|Z_{i-1} - Z_{j-1}| \geqq 2$. Then either $Z_{i-1} \geqq Z_{j-1} + 2$ or $Z_{j-1} \geqq Z_{i-1} + 2$. We suppose that $Q^{(i-1)}$ has two more zeros than $Q^{(j-1)}$ in $(-1, 1)$. The other case is treated similarly. If one of the zeros (say ξ) of $Q^{(i-1)}$ is also a zero of $Q^{(j-1)}$, then by an easy calculation,

$$\Delta(\xi + \epsilon) = \epsilon(i - j)Q^{(i)}(\xi)Q^{(j)}(\xi) + o(\epsilon).$$

From the simplicity of the zeros, $Q^{(i)}(\xi)Q^{(j)}(\xi) \neq 0$, so that $\Delta(x)$ changes sign at ξ. If none of the zeros of $Q^{(i-1)}$ coincide with a zero of $Q^{(j-1)}$, then there exist two zeros of $Q^{(i-1)}$, say ξ_1, ξ_2 such that $Q^{(j-1)}$ has no zeros in $[\xi_1, \xi_2]$. Then

$$\Delta(\xi_1) = -j\, Q^{(j-1)}(\xi_1)Q^{(i)}(\xi_1)$$
$$\Delta(\xi_2) = -j\, Q^{(j-1)}(\xi_2)Q^{(i)}(\xi_2).$$

Since $Q^{(j-1)}(x)$ is of the same sign in $[\xi_1, \xi_2]$ and since $\operatorname{sgn} Q^{(i)}(\xi_1) = -\operatorname{sgn} Q^{(i)}(\xi_2)$, we see that

$$\operatorname{sgn} \Delta(\xi_1) = -\operatorname{sgn} \Delta(\xi_2)$$

so that $\Delta(x)$ changes sign at some interior point in (ξ_1, ξ_2).

Now suppose $|Z_i - Z_j| \geqq 2$. Then by Lemma 1, $|\bar{Z}_{i-1} - \bar{Z}_{j-1}| \geqq 2$, so that $Q^{(i-1)}$ has two more zeros than $Q^{(j-1)}$ in $[-1, 1]$ or vice-versa. Assume the former. Again if any of the interior zeros of $Q^{(i-1)}$ is also a zero of $Q^{(j-1)}$, we are done. If not, there is, as before a pair ξ_1, ξ_2 of consecutive zeros of $Q^{(i-1)}$ such that $Q^{(j-1)}$ does not vanish on $[\xi_1, \xi_2]$. If $[\xi_1, \xi_2] \subset (-1, 1)$ we return to the case treated above. Suppose that $\xi_1 = -1$ (the case $\xi_2 = 1$ is similar). If $\xi_1 = -1$ is a simple zero the argument is exactly as in the preceding paragraph. If $\xi_1 = -1$ is a multiple zero of $Q^{(i-1)}$ of order $\nu \geqq 2$, then $Q^{(i)}$ has a multiple zero at $\xi_1 = -1$ of order $\nu - 1$, and the above argument applies to $\Delta(x)/(x + 1)^{\nu-1}$.

This completes the proof of Theorem 1. Corollaries 1 and 2 follow immediately from Theorem 1 on observing that when $i < p$ and $j \geqq q + 1$ (or if $i \geqq p + 1$ and $j > q$) then $i + j \neq p + q + 1$.

Proof of Theorem 2. Since $i \leqq p$ and $i + j = p + q + 1$, we have by integration by parts

$$\int_{-1}^{1} Q^{(i-1)}(x)Q^{(j)}(x)dx = (-1)^{i-1}(p + q)! \int_{-1}^{1} Q(x)dx$$
$$= (-1)^{p+i-1}(p + q)! \int_{-1}^{1} (1 - x)^p(1 + x)^q dx.$$

Similarly

$$\int_{-1}^{1} Q^{(i)}(x)Q^{(j-1)}(x)dx = (-1)^{i+p}(p + q)! \int_{-1}^{1} (1 - x)^p(1 + x)^q dx.$$

Hence

$$\operatorname{sgn}\left(\int_{-1}^{1} \Delta(x)dx\right) = (-1)^{p+i-1}. \tag{4.1}$$

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On the other hand

$$\int_{-1}^{1} Q^{(i-1)}(x)Q^{(j)}(x)(1-x)dx$$
$$= (-1)^{i+p-1}(p+q-1)! \int_{-1}^{1} \{(p+q)(1-x) + (i-1)(p(1-x) - q(1+x))\}(1-x)^p(1+x)^q dx,$$

and

$$\int_{-1}^{1} Q^{(i)}(x)Q^{(j-1)}(x)(1-x)dx$$
$$= (-1)^{i+p}(p+q-1)! \int_{-1}^{1} \{(p+q)(1-x) + i(p(1-x) - q(1+x))\}(1-x)^p(1+x)^q dx,$$

which lead to the explicit value

$$\int_{-1}^{1} \Delta(x)(1-x)dx = (-1)^{p+i-1} \cdot 2^{p+q+1} \frac{p!q!}{p+q+2}[p+1-i(q-p)]. \tag{4.2}$$

From (4.1) and (4.2), we see that under the hypothesis (3.1) either

$$\int_{-1}^{1} \Delta(x)dx \quad \text{and} \quad \int_{-1}^{1} \Delta(x)(1-x)dx$$

have opposite signs or the latter integral vanishes. In either case $\Delta(x)$ must change sign in $(-1, 1)$. This completes the proof of Theorem 2.

Proof of Theorem 3. If $j = i + 1$, it is known that Γ is order-poised. When $j \geqq i + 2$ and $i + j \neq 2p + 1$, it follows from Theorem 1 that Γ is strongly non-poised.

If $j \geqq i + 2$ and $i + j = 2p + 1$, then $0 \leqq i \leqq p - 1$. Using (2.7) and (2.9), we obtain

$$\begin{aligned} \Delta(x) &= i\, Q^{(i-1)}(x)Q^{(j)}(x) - j\, Q^{(j-1)}(x)Q^{(i)}(x) \\ &= (-1)\, 2^{2i-2}(x^2-1)^{p-i} \cdot i!j! \\ &\quad \times [(1-x^2)(P_{i-1}^{(\alpha,\alpha)}(x))^2 + 4(P_i^{(\alpha-1,\alpha-1)}(x))^2] \end{aligned} \tag{4.3}$$

where we set $\alpha = p + 1 - i$. Since

$$\tfrac{1}{2}(2p+2-i)P_{i-1}^{(\alpha,\alpha)}(x) = \frac{d}{dx} P_i^{(\alpha-1,\alpha-1)}(x),$$

we see that the two squared terms on the right side of (4.3) do not vanish simultaneously, thus $\Delta(x) \neq 0$ in $(-1, 1)$. Hence the problem is poised for $0 \leqq i \leqq p - 1$, $i + j = 2p + 1$. This completes the proof of Theorem 3.

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Proof of Theorem 4. The case $j = i + 1$ is known to be poised. When $i = 0$ and $j = 2p + 2$, the vanishing of the $(2p + 2)$th derivative reduces the problem to a Hermite problem and therefore it is poised. If $i = 0$ and $i + j \neq 2p + 2$, then by Theorem 1, Γ is strongly non-poised. This leaves only the case when $1 \leqq i \leqq p$ and $i + j = 2p + 2$.

We shall show that in this case $\Delta(x)$ vanishes but does not change sign in $(-1, 1)$. Indeed, on using Lemma 3 with $q = p + 1$ we get

$$\Delta(x) = \begin{vmatrix} ((x-1)Q)^{(i)} & Q^{(i)} \\ ((x-1)Q)^{(j)} & Q^{(j)} \end{vmatrix} = 2^{2i-1} j!i!\,(x^2-1)^{p-i}(x+1) \times \begin{vmatrix} (x-1)P_i^{(\alpha,\alpha)} & P_i^{(\alpha-1,\alpha)} \\ 2P_i^{(\alpha,\alpha)} & P_{i-1}^{(\alpha+1,\alpha)} \end{vmatrix}$$

$$= 2^{2i-1} i!j!\,(x^2-1)^{p-i}(x+1)P_i^{(\alpha,\alpha)} \cdot \Delta_1(x)$$

where we set $\alpha = p + 1 - i$ and $\Delta_1(x) = (x-1)P_{i-1}^{(\alpha+1,\alpha)} - 2P_i^{(\alpha-1,\alpha)}$. Now, on using the known recurrence relations for Jacobi polynomials (see [**8**, p. 166, Formula (4.16)]), we get

$$(1-x)P_{i-1}^{(\alpha+1,\alpha)} = P_{i-1}^{(\alpha,\alpha)} - \frac{i}{p+1} P_i^{(\alpha,\alpha)}$$

$$2P_i^{(\alpha-1,\alpha)} = \frac{2p+2-i}{p+1} P_i^{(\alpha,\alpha)} - P_{i-1}^{(\alpha,\alpha)},$$

which shows that

$$\Delta_1(x) = \frac{2p+2-2i}{p+1} P_i^{(\alpha,\alpha)}.$$

Hence,

$$\Delta(x) = C \cdot (x^2-1)^{p-i}(x+1)(P_i^{(\alpha,\alpha)})^2, \; C > 0$$

and so $\Delta(x)$ vanishes in $(-1, 1)$ but $(-1)^{p-i}\Delta(x) \geqq 0$ there.

Proof of Theorem 5. It is more convenient now to consider $x_1 = 1$, $x_3 = 0$ so that $Q(x) = x^q(x - 1)$. Then for $i < j \leqq q + 2$,

$$\Delta(x) = \frac{x^{2q-i-j+1}(q!)^2}{(q+2-j)!(q+2-i)!} \Delta_1(x)$$

where

$$\begin{aligned} \Delta_1(x) &= i(q+2-j)R(x) - j(q+2-i)S(x) \\ R(x) &= \{(q+1)x - (q+2-i)\}\{(q+1)x - (q+1-j)\} \\ S(x) &= \{(q+1)x - (q+2-j)\}\{(q+1)x - (q+1-i)\}. \end{aligned}$$

The discriminant of the quadratic $\Delta_1(x)$ is

$$(q+2)^2(2q+3-i-j)^2 - 4(q+1)(q+2)(q+2-i)(q+2-j) \equiv (q+2)\rho(i,j).$$

So the problem is poised if $\rho(i, j)$ is negative. On the other hand, evaluating

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$S(x)$ and $R(x)$ at $x_0 = (2q + 3 - i - j)/2(q + 1)$ (clearly $0 < x_0 < 1$), we have $S(x_0) = -(j - i - 1)^2/4$, $R(x_0) = -(i - j - 1)^2/4$, so that

$$4\Delta_1(x_0) = (j - i) \cdot \rho(i, j).$$

Since $\Delta_1(1 - i/(q + 1)) < 0$, it follows that if $\rho(i, j) > 0$, then $\Delta_1(x)$ and hence $\Delta(x)$ changes sign between x_0 and $1 - i/(q + 1)$. Hence $\rho(i, j) > 0$ implies that the problem is strongly non-poised. If $\rho(i, j) = 0$ then $\Delta_1(x_0) = 0$ and so $\Delta_1(x) = C(x - x_0)^2$, whence $\Delta(x)$ vanishes in $(0, 1)$ without changing sign. This completes the proof of Theorem 5.

Proof of Theorem 6. Since $i = 1 < p$, (2.4) is not satisfied. Hence all cases where $3 \leqq j \neq p + q$ are strongly non-poised on account of Theorem 1. If $j = p + q$, we have

$$\Delta(x) = (p + q)!(x - 1)^{p-1}(x + 1)^{q-1}[(q - p)^2 - 1 + x^2 - (p + q)^2x^2],$$

so that $\Delta(x)$ changes sign at $\xi = \{(q - p)^2 - 1\}^{1/2}/\{(q + p)^2 - 1\}^{1/2} < 1$. Hence in this case also, E is strongly non-poised.

5. Interpolation at more than three points. It is possible to apply the results of Section 3 to obtain non-poisedness criteria for certain multipoint interpolation problems by using the coalescing procedure mentioned in the introduction. Suppose E is an incidence matrix satisfying property (H) of Section 1. Let p and q be the number of non-zero entries in E_1 and E_2 respectively and suppose that the νth row of E has exactly two non-zero entries in the ith and jth columns. Then we have

THEOREM 7. *Let E be an incidence matrix as described above and let Γ be the three row matrix with p and q Hermite data in the first and third rows respectively and with exactly two non-zero entries in the ith and jth columns of the second row. If Γ is strongly non-poised then E is non-poised.*

Proof of Theorem 7. The proof follows from Theorem 7 of Karlin and Karon [**3**]. We know that since E satisfies (H), it can be coalesced into Γ. It is enough to show that the determinant $D(x_1, x_2, x_3; \Gamma)$ given by (1.1) with $x_1 = 1$, $x_2 = x$, $x_3 = -1$ changes sign in $[-1, 1)$. Let a new basis $\{\phi_j(x)\}$ for polynomials of degree $\leqq p + q + 1$ be $1, x - 1, \ldots, (x - 1)^p, (x - 1)^p(x + 1), \ldots, (x - 1)^p(x + 1)^q, (x - 1)^p(x + 1)^{q+1}$. If we express the polynomial $P(x)$ in the form

$$\sum_{k=0}^{p+q+1} b_k\phi_k(x)$$

and the conditions of the homogeneous interpolation problem for as linear equations in b_k's, we easily see that the determinant $\tilde{D}(x)$ of the system is a constant multiple of the determinant $\Delta(x)$ given by (2.2).

On the other hand, since $\{\phi_j(x)\}$ is a basis for polynomials of degree $\leqq p + q + 1$, there exists a non-singular matrix A such that if

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$X = (1, x, \ldots, x^{p+q+1})$ and $\Phi = (\phi_0, \ldots, \phi_{p+q+1})$, then $X \cdot A = \Phi$. Hence it follows that $\tilde{D}(x) = ||A|| \cdot D(+1, x, -1; \Gamma)$. Therefore $D(1, x, -1; \Gamma)$ changes sign in $(-1, 1)$ whenever $\tilde{D}(x)$ and therefore also whenever $\Delta(x)$ does. This completes the proof.

Remark. Theorem 7 allows us to conclude non-poisedness for multipoint problems reducible to strongly non-poised three point problems covered by any of the Theorems 1–6. For example, matrix E given by (1.5) is non-poised because the coalesced matrix Γ of (1.6) is by Theorem 1, strongly non-poised.

Added in proof. After this paper had been submitted, we learned from Professor G. G. Lorentz that he has obtained among other results a Theorem which is equivalent to ours for the case $p = q$ (i.e., Theorem 3).

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Oakland University,
Rochester, Michigan;
University of Alberta,
Edmonton, Alberta

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