ON SMALLEST RADICAL AND SEMI-SIMPLE CLASSES

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Introduction. In a recent paper [5] one of us has given a sufficient condition to be satisfied by a given property of radical classes within a universal class $\mathscr{W}$ in order that, for any subclass $\mathscr{M}$ of $\mathscr{W}$, there should be a smallest radical class having the given property and containing $\mathscr{M}$. The sufficient condition is that the class of all radical classes with the given property can be characterised as the class of all radical classes fixed by an admissible function F (see Section 1 below). In this paper a necessary and sufficient condition is derived and the corresponding result for semi-simple classes is also presented. These results are given in Section 2.

In Section 3 we apply the semi-simple construction to show that, given any subclass $\mathscr{M}$ of $\mathscr{W}$, there is a largest radical $\mathscr{P}$ such that both $\mathscr{P}$ and its semi-simple class $\mathrm{S}\mathscr{P}$ are hereditary and $\mathrm{S}\mathscr{P} \supseteq \mathscr{M}$. An example is given to show that there is, in general, no largest radical $\mathscr{P}$ such that $\mathscr{P}$ is hereditary and $\mathrm{S}\mathscr{P} \supseteq \mathscr{M}$. Finally, in Section 4, an example is given to show that there is, in general, no smallest radical class $\mathscr{P}$ such that $\mathrm{S}\mathscr{P}$ is hereditary and $\mathscr{P} \supseteq \mathscr{M}$.

1. Definitions. Let $\mathscr{W}$ be a universal class; that is, $\mathscr{W}$ is hereditary and homomorphically closed. Denote by $\mathscr{T}$ the class of all subclasses of $\mathscr{W}$, by $\mathscr{R}$ the class of all radical subclasses of $\mathscr{W}$, and by $\mathscr{Y}$ the class of all semi-simple subclasses of $\mathscr{W}$.

If $\mathscr{M} \in \mathscr{T}$, then a class $\mathscr{M}' \supseteq \mathscr{M}$ is called an *s-completion* of $\mathscr{M}$ when $\mathscr{M}'$ has the property:

(a) If $R \in \mathscr{M}'$, then every non-zero ideal of R has a non-zero homomorphic image in $\mathscr{M}'$.

If $\mathscr{M}$ is an *s*-completion of itself, then we shall say that $\mathscr{M}$ is *s-complete*. We recall that every semi-simple class in $\mathscr{W}$ is *s*-complete and that if $\mathscr{M}$ is *s*-complete, then there is a smallest semi-simple class in $\mathscr{W}$, containing $\mathscr{M}$ [1, p. 6]. The corresponding radical class, the upper $\mathscr{M}$-radical class, is denoted by $\mathrm{U}\mathscr{M}$. In particular, if $\mathscr{M} \in \mathscr{Y}$, $\mathrm{U}\mathscr{M}$ is the radical class determined by $\mathscr{M}$.

If a function $\mathrm{F}: \mathscr{R} \to \mathscr{T}$ is such that

A.1. for all $\mathscr{P} \in \mathscr{R}$, $\mathscr{P} \subseteq \mathrm{F}\mathscr{P}$;

A.2. if $\mathscr{P}_1, \mathscr{P}_2 \in \mathscr{R}$ and $\mathscr{P}_1 \subseteq \mathscr{P}_2$, then $\mathrm{F}\mathscr{P}_1 \subseteq \mathrm{F}\mathscr{P}_2$;

A.3. if $\{\mathscr{P}_\alpha : \mathscr{P}_\alpha \in \mathscr{R}\}$ is defined for all ordinals α and $\mathscr{P}_\alpha \subseteq \mathscr{P}_\beta$ for $\alpha \leqq \beta$, then $\mathrm{F}\mathscr{P} \subseteq \bigcup \mathrm{F}\mathscr{P}_\alpha$, where $\mathscr{P} = \bigcup_\alpha \mathscr{P}_\alpha$;

then, following [5], F is said to be *admissible*. If a function $\mathrm{F}: \mathscr{R} \to \mathscr{T}$ satisfies A.1 and A.2 above, then we shall say that F is *non-inductive admissible*, which will be abbreviated to *n-admissible*. Note that in A.3, from all $\mathscr{P}_\alpha \in \mathscr{R}$ it follows without difficulty that $\mathscr{P} \in \mathscr{R}$. Also note by A.2 that A.3, in fact, implies that $\mathrm{F}\mathscr{P} = \bigcup_\alpha \mathrm{F}\mathscr{P}_\alpha$.

Here, in Theorem 1, it is shown that, for each $\mathscr{M} \in \mathscr{T}$, *n*-admissibility of $\mathrm{F}: \mathscr{R} \to \mathscr{T}$ is a necessary and sufficient condition for the existence of a smallest radical class $\bar{\mathscr{M}} \supseteq \mathscr{M}$ and

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such that $\mathrm{F}\bar{\mathcal{M}} = \bar{\mathcal{M}}$. This improves Theorem 1 of [5], where it was shown that admissibility of F is sufficient to imply the existence of $\bar{\mathcal{M}}$.

If a function $\mathrm{F}:\mathcal{T} \to \mathcal{T}$ is such that

S.A.1. for $\mathcal{M} \in \mathcal{T}$, $\mathrm{F}\mathcal{M}$ is an s-completion of $\mathcal{M}$;

S.A.2. if $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{T}$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathrm{F}\mathcal{M}_1 \subseteq \mathrm{F}\mathcal{M}_2$;

S.A.3. if $\{\mathcal{M}_\alpha : \mathcal{M}_\alpha \in \mathcal{Y}\}$ is defined for all ordinals α and $\mathcal{M}_\alpha \subseteq \mathcal{M}_\beta$ for $\alpha \leqq \beta$, then $\mathrm{F}\mathcal{M} \subseteq \bigcup_\alpha \mathrm{F}\mathcal{M}_\alpha$, where $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$;

then F is said to be *s-admissible*. An *ns-admissible* function $\mathrm{F}:\mathcal{T} \to \mathcal{T}$ is one which satisfies S.A.1 and S.A.2. Again note that in S.A.3, from all $\mathcal{M}_\alpha \in \mathcal{Y}$ it follows without difficulty that $\mathcal{M} \in \mathcal{Y}$.

In [5, Theorem 2] it was shown that if F is an s-admissible function, then for each $\mathcal{M} \in \mathcal{T}$ there is a smallest semi-simple class $\bar{\mathcal{M}} \supseteq \mathcal{M}$ and such that $\mathrm{F}\bar{\mathcal{M}} = \bar{\mathcal{M}}$. In fact, as we shall show in Theorem 2, ns-admissibility of F is necessary and sufficient for the existence of $\bar{\mathcal{M}}$.

We conclude this section with some remarks about our notation. If $A \in \mathcal{W}$, then $B \leqq A$ will denote that B is an ideal of A; if $B \leqq A$ but $B \neq A$, then this will be denoted by $B < A$. Small capitals are used to denote operators on classes even though we are not usually dealing with closure operators. The lower radical class determined by $\mathcal{M} \in \mathcal{T}$ is written $\mathrm{L}\mathcal{M}$. If $\mathcal{P} \in \mathcal{R}$, then $\mathrm{S}\mathcal{P}$ is the semi-simple class determined by $\mathcal{P}$, and $P(A)$ is the $\mathcal{P}$-radical of A. Finally, when the elements of a class are listed it is understood that what is meant is the class of all isomorphic copies of the rings listed.

2. Main theorems.

THEOREM 1. *Let $\mathcal{V} \subseteq \mathcal{R}$ define a property of radical classes in $\mathcal{W}$. There exists, for each $\mathcal{M} \in \mathcal{T}$, a smallest class $\bar{\mathcal{M}} \in \mathcal{V}$ with $\mathcal{M} \subseteq \bar{\mathcal{M}}$ if and only if there is an n-admissible function* F *such that $\mathcal{V} = \{\mathcal{P} \in \mathcal{R} : \mathrm{F}\mathcal{P} = \mathcal{P}\}$.*

Proof. Suppose that F is an n-admissible function, $\mathcal{V} = \{\mathcal{P} \in \mathcal{R} : \mathrm{F}\mathcal{P} = \mathcal{P}\}$ and $\mathcal{M} \in \mathcal{T}$. Since the class $\mathcal{W} \in \mathcal{V}$ and $\mathcal{W} \supseteq \mathcal{M}$, there is a non-empty class $\bar{\mathcal{M}}$ which is the intersection of all the classes $\mathcal{N} \in \mathcal{V}$ such that $\mathcal{N} \supseteq \mathcal{M}$. The proof in [2] that the intersection of a set of radical classes is a radical class applies also for a class of radical classes. Hence $\bar{\mathcal{M}}$ is a radical class. By A.2, $\bar{\mathcal{M}} \subseteq \mathcal{N}$ implies that $\mathrm{F}\bar{\mathcal{M}} \subseteq \mathrm{F}\mathcal{N} = \mathcal{N}$ and so $\mathrm{F}\bar{\mathcal{M}} \subseteq \bar{\mathcal{M}}$. Therefore, from A.1, $\mathrm{F}\bar{\mathcal{M}} = \bar{\mathcal{M}}$ so $\bar{\mathcal{M}} \in \mathcal{V}$. Finally, if $\mathcal{K} \supseteq \mathcal{M}$ and $\mathcal{K} \in \mathcal{V}$, then $\mathcal{K} \supseteq \bar{\mathcal{M}}$ and so $\bar{\mathcal{M}}$ is the smallest class in $\mathcal{V}$ which contains $\mathcal{M}$.

Conversely, if, given $\mathcal{P} \in \mathcal{R}$, $\bar{\mathcal{P}}$ is the smallest class in $\mathcal{V}$ with $\mathcal{P} \subseteq \bar{\mathcal{P}}$, define a function $\mathrm{F}:\mathcal{R} \to \mathcal{T}$ by setting $\mathrm{F}\mathcal{P} = \bar{\mathcal{P}}$. Then F is an n-admissible function and $\mathcal{V} = \{\mathcal{P} \in \mathcal{R} : \mathrm{F}\mathcal{P} = \mathcal{P}\}$.

REMARK 1. If F is an n-admissible function, then, setting $\mathcal{M}_1 = \mathrm{L}\mathcal{M}$, where $\mathcal{M} \in \mathcal{T}$, and defining

$$\mathcal{M}_\beta = \begin{cases} \mathrm{L}\left(\bigcup_{\alpha<\beta} \mathcal{M}_\beta\right) & \text{if } \beta \text{ is a limit ordinal,} \\ \mathrm{LF}\mathcal{M}_\alpha & \text{if } \beta = \alpha + 1, \end{cases}$$

we obtain an ascending chain of radical classes whose union $\mathcal{M}^*$ is a radical class. An easy induction argument shows that $\mathcal{M}^* \subseteq \bar{\mathcal{M}}$ and if F is admissible, then $\mathcal{M}^* = \bar{\mathcal{M}}$ [5, Theorem 1]. For n-admissible F we need not have $\mathcal{M}^* = \bar{\mathcal{M}}$ as the following example shows.

Let Φ_α be the quotient field of the polynomial ring $\Phi[A_\alpha]$, where A_α is a set of commuting indeterminates of cardinality $\aleph_\alpha$, Φ is a finite field and α is any ordinal. The fields Φ_α, Φ_β have different cardinalities for $\alpha \neq \beta$, and so are non-isomorphic.

Let $\mathcal{W} = \{0, \Phi, \Phi_\alpha : \alpha \text{ any ordinal}\}$. Every ring in $\mathcal{W}$ is simple so $\mathcal{R} = \mathcal{T} = \mathcal{Y}$. Define

$$\mathrm{F}\{\mathcal{W} - \{\Phi\}\} = \mathrm{F}\mathcal{W} = \mathcal{W}$$

and otherwise

$$\mathrm{F}\mathcal{M} = \mathcal{M} \cup \Phi_\gamma,$$

where γ is the least ordinal such that $\Phi_\gamma \notin \mathcal{M}$. Then F is an n-admissible function which is not admissible. If $\mathcal{M} = \{0, \Phi_0\}$, then $\mathcal{M}^* = \{0, \Phi_\alpha : \alpha \text{ any ordinal}\}$ but $\bar{\mathcal{M}} = \mathcal{W}$, so $\bar{\mathcal{M}} \neq \mathcal{M}^*$.

THEOREM 2. *Let $\mathcal{X} \subseteq \mathcal{Y}$ define a property of semi-simple classes. There exists, for each $\mathcal{M} \in \mathcal{T}$, a smallest class $\bar{\mathcal{M}} \in \mathcal{X}$ with $\bar{\mathcal{M}} \supseteq \mathcal{M}$ if and only if there is an ns-admissible function* F *such that $\mathcal{X} = \{\mathcal{Q} \in \mathcal{Y} : \mathrm{F}\mathcal{Q} = \mathcal{Q}\}$.*

Proof. Suppose that F is an ns-admissible function $\mathcal{X} = \{\mathcal{Q} \in \mathcal{Y} : \mathrm{F}\mathcal{Q} = \mathcal{Q}\}$ and $\mathcal{M} \in \mathcal{T}$. Since $\mathcal{W} \in \mathcal{X}$ and $\mathcal{M} \subseteq \mathcal{W}$, there is a non-empty class $\bar{\mathcal{M}}$ which is the intersection of all the classes $\mathcal{N} \in \mathcal{X}$ such that $\mathcal{M} \subseteq \mathcal{N}$.

By S.A.2, $\bar{\mathcal{M}} \subseteq \mathcal{N}$ implies that $\mathrm{F}\bar{\mathcal{M}} \subseteq \mathrm{F}\mathcal{N}$ and so $\mathrm{F}\bar{\mathcal{M}} \subseteq \bar{\mathcal{M}}$. Hence, from S.A.1, $\mathrm{F}\bar{\mathcal{M}} = \bar{\mathcal{M}}$ and $\bar{\mathcal{M}}$ is s-complete. Therefore $\bar{\mathcal{M}}$ satisfies condition (a) of Section 1. Furthermore, if $A \in \mathcal{W} \backslash \bar{\mathcal{M}}$, there is a class $\mathcal{N} \in \mathcal{X}$ such that $\mathcal{N} \supseteq \bar{\mathcal{M}}$ but $A \notin \mathcal{N}$. By the semi-simplicity of the class $\mathcal{N}$, there is $(0) \neq B \leqq A$ such that no non-zero homomorphic image of B belongs to $\mathcal{N}$ and so no non-zero homomorphic image of B belongs to $\bar{\mathcal{M}}$. Thus $\bar{\mathcal{M}}$ satisfies the condition:

(b) If every $(0) \neq B \leqq A \in \mathcal{W}$ can be mapped homomorphically onto a non-zero ring in $\bar{\mathcal{M}}$, then $A \in \bar{\mathcal{M}}$.

Now any class which satisfies both (a) and (b) is semi-simple [1, Theorem 2], so $\bar{\mathcal{M}} \in \mathcal{X}$ and $\bar{\mathcal{M}} \supseteq \mathcal{M}$. Finally, if $\mathcal{Q} \in \mathcal{X}$ and $\mathcal{Q} \supseteq \mathcal{M}$, then $\mathcal{Q} \supseteq \bar{\mathcal{M}}$ and so $\bar{\mathcal{M}}$ is the smallest class in $\mathcal{X}$ which contains $\mathcal{M}$.

For the converse, define a function $\mathrm{F} : \mathcal{T} \to \mathcal{T}$ by setting $\mathrm{F}\mathcal{M} = \bar{\mathcal{M}}$. Then F is ns-admissible and $\{\mathcal{Q} \in \mathcal{Y} : \mathrm{F}\mathcal{Q} = \mathcal{Q}\} = \mathcal{X}$.

REMARK 2. If F is an ns-admissible function, then setting $\mathcal{M}_1 = \mathrm{SUF}\mathcal{M}$, where $\mathcal{M} \in \mathcal{T}$, and defining

$$\mathcal{M}_\beta = \begin{cases} \bigcup_{\alpha < \beta} \mathcal{M}_\alpha & \text{if } \beta \text{ is a limit ordinal,} \\ \mathrm{SUF}\mathcal{M}_\alpha & \text{if } \beta = \alpha + 1, \end{cases}$$

we obtain an ascending chain of semi-simple classes whose union $\mathcal{M}^*$ is a semi-simple class. An easy induction argument shows that $\mathcal{M}^* \subseteq \bar{\mathcal{M}}$ and, if F is s-admissible, $\mathcal{M}^* = \bar{\mathcal{M}}$ [5, Theorem 2]. For an ns-admissible function F we need not have $\mathcal{M}^* = \bar{\mathcal{M}}$, as the example of Remark 1 shows.

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From these improved forms of Theorems 1 and 2 of [5] it is clear that Theorems 5 and 6 of [5] can also be improved by replacing the conditions of admissibility and s-admissibility by n-admissibility and ns-admissibility respectively.

3. Applications of the semi-simple construction. Given $\mathcal{M} \in \mathcal{T}$, the hereditary closure of $\mathcal{M}$ is defined to be the class of all rings in $\mathcal{W}$ isomorphic to accessible subrings of $\mathcal{M}$-rings. The hereditary closure of $\mathcal{M}$ is denoted by $\mathrm{I}\mathcal{M}$ and is the smallest hereditary class containing $\mathcal{M}$. Since every hereditary class is s-complete, the class $\mathrm{I}\mathcal{M}$ is s-complete. A class $\mathcal{M}$ is hereditary if and only if $\mathrm{I}\mathcal{M} = \mathcal{M}$. It is easily seen that I is an admissible function defined on $\mathcal{T}$ and we have from Theorem 2 that, given $\mathcal{M} \in \mathcal{T}$, there is a smallest hereditary semi-simple class $\mathcal{S} \supseteq \mathcal{M}$, [4, Theorem 2].

To show that there is a smallest semi-simple class $\mathcal{S} \supseteq \mathcal{M}$ such that the radical determined by $\mathcal{S}$ is strongly hereditary, that is both $\mathcal{S}$ and its radical class are hereditary, we use a procedure which is essentially that used by Rjabuhin in [6] to construct, within the universal class of all associative rings, a largest hereditary radical whose semi-simple class contains a given class of rings.

Given $\mathcal{M} \in \mathcal{T}$, we define $\mathrm{J}\mathcal{M}$ to be the class consisting of the ring 0 and all isomorphic copies of rings R/J obtained as follows:

(α) $B \leqq A \leqq R \in \mathcal{W}$,
(β) $(0) \neq A/B \in \mathcal{M}$,
(γ) $J \leqq R$ maximal with respect to the property $J \cap A \subseteq B$.

Then $\mathrm{J}\mathcal{M} \supseteq \mathcal{M}$ and if $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathrm{J}\mathcal{M}_1 \subseteq \mathrm{J}\mathcal{M}_2$. In the following lemmas we establish some properties of the function J.

LEMMA 1. *If $\mathcal{M}$ is s-complete, then $\mathrm{J}\mathcal{M}$ is s-complete.*

Proof. Suppose that $(0) \neq I/J \leqq R/J \in \mathrm{J}\mathcal{M}$. In the $\mathcal{M}$-ring A/B, where A and B are as in the definition of the class $\mathrm{J}\mathcal{M}$, there is the non-zero ideal

$$\frac{(I \cap A)+B}{B} \cong \frac{I \cap A}{I \cap B}.$$

Therefore, since $\mathcal{M}$ is s-complete, there is $K < I \cap A$ such that $I \cap B \subseteq K$ and $(I \cap A)/K \in \mathcal{M}$. Also

$$J \cap (I \cap A) = I \cap (J \cap A) \subseteq I \cap B \subseteq K,$$

so there is an ideal J^* of I such that $J^* \supseteq J$ and J^* is maximal with respect to the property $J^* \cap (I \cap A) \subseteq K$. Since $K \neq I \cap A$, the ideal $J^* \neq I$.

Hence, from the definition of the class $\mathrm{J}\mathcal{M}$, the ring I/J^*, which is a non-zero homomorphic image of I/J, is an element of $\mathrm{J}\mathcal{M}$. Thus the class $\mathrm{J}\mathcal{M}$ is s-complete.

LEMMA 2. *Let $\mathcal{P} \in \mathcal{R}$ and $\mathrm{S}\mathcal{P} = \mathcal{M}$. If $\mathcal{P}$ is strongly hereditary, then $\mathrm{J}\mathcal{M} = \mathcal{M}$.*

Proof. Let P/J be the $\mathcal{P}$-radical of the ring $R/J \in \mathrm{J}\mathcal{M}$, where $B \leqq A \leqq R \in \mathcal{W}$, $(0) \neq A/B \in \mathcal{M}$, and J is as in (γ) above.

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Since $\mathscr{M}$ is hereditary, the ideal $((P\cap A)+B)/B$ of A/B belongs to $\mathscr{M}$ and so $(P\cap A)/(P\cap B)\in\mathscr{M}$. But, because $\mathscr{P}$ is hereditary, the ideal $((P\cap A)+J)/J$ of P/J belongs to $\mathscr{P}$ and so $(P\cap A)/(J\cap A)\in\mathscr{P}$. Now $J\cap A\subseteq P\cap B$, so $(P\cap A)/(P\cap B)$ is a homomorphic image of the $\mathscr{P}$-ring $(P\cap A)/(J\cap A)$. Therefore

$$\frac{P\cap A}{P\cap B}\in\mathscr{P}\cap \mathrm{S}\mathscr{P}$$

and $P\cap A=P\cap B\subseteq B$. Finally, by the maximality of J, $P=J$ and $R/J\in\mathscr{M}$, which completes the proof.

Lemma 3. *Let $\mathscr{M}$ be a hereditary class such that $\mathrm{J}\mathscr{M}=\mathscr{M}$. Then $\mathscr{P}=\mathrm{U}\mathscr{M}$ is hereditary.*

Proof. Let $(0)\neq A\leqq R\in\mathscr{P}$ and suppose that $A\notin\mathscr{P}$. Then there is $B\leqq A$ such that $(0)\neq A/B\in\mathscr{M}$. If $J\leqq R$ and is maximal with respect to the property $J\cap A\subseteq B$, then $R/J\in\mathrm{J}\mathscr{M}=\mathscr{M}\subseteq\mathrm{S}\mathscr{P}$. But $R/J\in\mathscr{P}$, so $R=J$, which implies that $A=B$. This is a contradiction, so $A\in\mathscr{P}$ and $\mathscr{P}$ is hereditary.

With the aid of these lemmas and Theorem 2, we are now able to prove the main result of this section.

Theorem 3. *Given $\mathscr{M}\in\mathscr{T}$, there is a smallest semi-simple class $\mathscr{J}\supseteq\mathscr{M}$ such that the radical determined by $\mathscr{J}$ is strongly hereditary.*

Proof. Let $\mathscr{X}$ be the class of all semi-simple classes whose corresponding radicals are strongly hereditary. Given $\mathscr{M}\in\mathscr{T}$, we define $\mathrm{F}\mathscr{M}=\mathrm{J}(\mathrm{I}\mathscr{M})$. It is immediate from Lemma 1 that F is an ns-admissible function. Put

$$\mathscr{X}'=\{\mathscr{Q}\in\mathscr{Y}:\mathrm{F}\mathscr{Q}=\mathscr{Q}\}.$$

From Lemma 2, $\mathscr{X}'\supseteq\mathscr{X}$. On the other hand, if $\mathscr{Q}\in\mathscr{X}'$ and $\mathscr{P}$ is its corresponding radical class, then both $\mathrm{I}\mathscr{Q}=\mathscr{Q}$ and $\mathrm{J}\mathscr{Q}=\mathscr{Q}$. Hence $\mathscr{Q}$ is a hereditary class such that $\mathrm{J}\mathscr{Q}=\mathscr{Q}$ and then, from Lemma 3, $\mathrm{U}\mathscr{Q}=\mathscr{P}$ is also hereditary. Thus $\mathscr{P}$ is a strongly hereditary radical and $\mathscr{Q}\in\mathscr{X}$. Therefore $\mathscr{X}'=\mathscr{X}$ and the result follows from Theorem 2.

It might be conjectured that, given $\mathscr{M}\in\mathscr{T}$, there is a smallest semi-simple class $\mathscr{Q}\supseteq\mathscr{M}$ and such that the radical determined by $\mathscr{Q}$ is hereditary. This is false as the following example shows.

Example 1. Let K be the algebra over $GF(p)$ with generators e, x, y, z and multiplication determined by the table:

	e	x	y	z
e	e	e	e	x
x	e	0	0	e
y	e	0	y	z
z	x	e	0	y

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Let E be the subring generated by e and X the subring generated by x and e. Then $E \leqq X \leqq K$ and X is the only ideal in K other than (0) and K itself. The rings X/E, K/X and E are non-isomorphic and simple.

Let $\mathcal{W} = \{K, X, E, K/X, X/E, 0\}$, $\mathcal{M} = \{0, K\}$, $\mathcal{Q}_1 = \{0, K, X, E\}$ and $\mathcal{Q}_2 = \{0, K, X/E\}$. Then $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are minimal s-completions of $\mathcal{M}$ so, in general, there is not a smallest s-completion of a class $\mathcal{M}$. This confirms a conjecture made in [**4**]. Furthermore $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the semi-simple classes determined respectively by the radical classes $\{0, K/X, X/E\}$ and $\{0, E, K/X\}$, each of these being hereditary.

4. Smallest radicals. It is known [**3**, Corollary] that, given $\mathcal{M} \in \mathcal{T}$, there is a smallest hereditary radical class $\mathcal{H} \supseteq \mathcal{M}$ and a smallest strongly hereditary radical class $\mathcal{D} \supseteq \mathcal{M}$. The existence of these radicals can be established from Theorem 1 using the n-admissible functions I and IG, where G, as in [**5**], is given by

$$\mathrm{G}\mathcal{P} = \{K \in \mathcal{W} : J \leqq I \leqq A \in \mathcal{W} \text{ with } J \in \mathcal{P} \text{ and } K \text{ the ideal of } A \text{ generated by } J\}$$

and $\mathcal{P} \in \mathcal{R}$. The function G is itself n-admissible, and it is not difficult to see that $\mathrm{G}\mathcal{P} = \mathcal{P}$ if and only if, given $I \leqq A \in \mathcal{W}$, we have $P(I) \leqq A$. We shall denote the smallest such radical class containing $\mathcal{M} \in \mathcal{T}$ by $\mathcal{G}$. It is clear that $\mathcal{H} \subseteq \mathcal{D}$ and, from [**1**, Lemmas 68 and 69], $\mathcal{G} \subseteq \mathcal{D}$. All three radicals are, in general, distinct as the next example shows.

EXAMPLE 2. By a construction of Rjabuhin [7] there are rings A_1, A_2, A_3, A_4 such that the only proper ideal of A_{i+1} is A_i for $i = 1, 2, 3$. Also A_1 and $B_{i+1} = A_{i+1}/A_i$ are non-isomorphic simple rings.

Let $\mathcal{W} = \{0, A_1, A_2, A_3, A_4, B_2, B_3, B_4\}$, $\mathcal{M} = \{0, A_2, B_2\}$, $G = \{0, A_2, A_3, B_2, B_3\}$ and $\mathcal{H} = \{0, A_1, A_2, B_2\}$. Then $\mathcal{W}$ is a universal class, $\mathcal{M}$ is a radical class in $\mathcal{W}$, $\mathcal{H}$ is the smallest hereditary radical class containing $\mathcal{M}$, $\mathcal{G}$ is the smallest radical class containing $\mathcal{M}$ and such that, given $I \leqq A \in \mathcal{W}$, we have $G(I) \leqq A$, and $\mathcal{H} \cup \mathcal{G} = \mathcal{D}$ is the smallest strongly hereditary radical containing $\mathcal{M}$. Each of these assertions is easily checked so we omit the proofs.

The semi-simple classes corresponding to the radical classes $\mathcal{G}$ and $\mathcal{D}$ are always hereditary and again one might conjecture that, given $\mathcal{M} \in \mathcal{T}$, there is a smallest radical class $\mathcal{P} \supseteq \mathcal{M}$ with hereditary semi-simple class. Using Example 1, we can show that this is, in general, false. Let $\mathcal{W}$ be as in Example 1, $\mathcal{M} = \{0, E\}$, $\mathcal{P}_1 = \{0, X, E, X/E\}$ and $\mathcal{P}_2 = \{0, E, K, K/X\}$. Then $\mathcal{M}$ is a radical class in $\mathcal{W}$ and both $\mathcal{P}_1$ and $\mathcal{P}_2$ are minimal radical classes containing $\mathcal{M}$ and having hereditary semi-simple class. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are incomparable, there is no smallest radical containing $\mathcal{M}$ and having hereditary semi-simple class. Again each of these assertions is easily checked and the proofs are omitted.

REFERENCES

1. N. Divinsky, *Rings and Radicals* (London, 1965).
2. W. G. Leavitt, Sets of radical classes, *Publ. Math. Debrecen* **14** (1967), 321–324.
3. W. G. Leavitt, Strongly hereditary radicals, *Proc. Amer. Math. Soc.* **21** (1969), 703–705.
4. W. G. Leavitt, Hereditary semi-simple classes, *Glasgow Math. J.* **11** (1970), 7–8.

5. W. G. Leavitt, Radical and semi-simple classes with specified properties, *Proc. Amer. Math. Soc.* **24** (1970), 680–687.

6. Ju. M. Rjabuhin, On the imbedding of radicals, *Isv. Akad. Nauk. Mold. SS* **11** (1963), 34–40.

7. Ju. M. Rjabuhin, Lower radicals of rings, *Mat. Zametki* **2** (1967), 239–244.

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