THE MAXIMUM IDEMPOTENT-SEPARATING CONGRUENCE ON AN INVERSE SEMIGROUP

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A congruence ρ on a semigroup will be called *idempotent-separating* if each ρ-class contains at most one idempotent. It is shown below that there exists a maximum such congruence μ on every inverse semigroup S. Two characterisations of μ are found, and it is shown (a) that $S/\mu \simeq E$, the semilattice of idempotents of S, if and only if E is contained in the centre of S; (b) that μ is the identical congruence on S if and only if E is self-centralising, in a sense explained below.

A congruence ρ on a semigroup S is called a *group* congruence if S/ρ is a group. It has been shown by Munn (**4**) that there exists a minimum such congruence σ on every inverse semigroup. In Section **3** of this paper necessary and sufficient conditions are given for $\sigma \cap \mu$ to be the identical congruence and for $\sigma \vee \mu$ (the smallest congruence containing both σ and μ) to be the universal congruence.

1. Definitions and preliminaries

I shall use the terminology of Clifford and Preston (**2**). Two elements a and a' of a semigroup will be called *inverses* of each other if

$$aa'a = a, \quad a'aa' = a'.$$

An *inverse semigroup* is a semigroup S in which every element has a unique inverse. In such a semigroup, idempotent elements commute:

$$ef = fe \text{ if } e^2 = e \text{ and } f^2 = f$$

((**2**), Section 1.9). The unique inverse of the element a is written a^{-1}. Then aa^{-1} and $a^{-1}a$ are idempotents, and so also are aea^{-1} and $a^{-1}ea$, where e is any idempotent in S. In fact α_a, defined by

$$e\alpha_a = a^{-1}ea, \qquad \text{.......................................(1)}$$

is a homomorphism of E, the subsemigroup of idempotents of S, into itself. We record also for future use that $e^{-1} = e$ if e is idempotent, and that

$$(a^{-1})^{-1} = a, \quad (ab)^{-1} = b^{-1}a^{-1}$$

for any a, b in S.

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If a and b are two elements of an inverse semigroup S, we write $a \leqq b$ (or $b \geqq a$) if

$$aa^{-1} = ab^{-1},$$

or if any one of the following equivalent conditions holds:

$$aa^{-1} = ba^{-1},\ a^{-1}a = a^{-1}b,\ a^{-1}a = b^{-1}a.$$

The relation $\leqq$ is a compatible order relation (**11**, 7) on S. The following observations—variously due to Vagner (**10**, **11**), Preston (**7**) and Šaĭn (**9**), and all readily verifiable—will be of use below. First,

$$a^{-1} \leqq b^{-1} \text{ if } a \leqq b. \qquad \text{.................................(2)}$$

Also, if e is any idempotent and if a and b are arbitrary elements of S, then

$$ea \leqq a,\quad ae \leqq a,\quad aeb \leqq ab. \qquad \text{..............................(3)}$$

The restriction of the order relation $\leqq$ to E, the subsemigroup of idempotents of S, is the natural semilattice order on E: that is,

$$e \leqq f \text{ if and only if } ef = fe = e.$$

Thus clearly $ef \leqq e$ and $ef \leqq f$ for any two idempotents e, f in S.

If H is an arbitrary subset of S, we denote by $H\omega$ the " closure " of H with respect to the above order relation: that is,

$$H\omega = \{a \in S\colon\ a \geqq h \text{ for some } h \text{ in } H\}.$$

Then $H \subseteq H\omega$ for any H. A subset K will be called *closed* if $K\omega = K$. Clearly $H\omega$ is closed for any H.

If E is the semilattice of idempotents of an inverse semigroup S, we define $E\zeta$, the *centraliser of E in S*, by

$$E\zeta = \{z \in S\colon\ ez = ze \text{ for every } e \text{ in } E\}.$$

Clearly $E \subseteq E\zeta$. If $E\zeta = S$, then the idempotents are central, and the semigroup is a union of groups (**1**). If $E\zeta = E$, we shall say that E is *self-centralising*. An example of an inverse semigroup whose semilattice of idempotents is self-centralising is the bicyclic semigroup ((**2**), Section 1.12).

A *congruence* ρ on a semigroup S is an equivalence relation which satisfies the condition that $ac\rho bc$ and $ca\rho cb$ for each c in S whenever $a\rho b$. If we denote the equivalence class containing a by $a\rho$, then we can (not quite trivially) restate this condition as follows:

$$(x\rho)(y\rho) \subseteq (xy)\rho$$

for all $x, y \in S$. Thus S/ρ can be given a semigroup structure in a natural way, and the mapping $\rho^\natural\colon S \to S/\rho$ defined by

$$x\rho^\natural = x\rho$$

is a homomorphism of S onto S/ρ.

It is often convenient to consider a relation on S as a subset of $S \times S$, and to write $(a, b) \in \rho$ rather than $a\rho b$. Thus, when ρ and σ are two congruences, the statement $\rho \subseteq \sigma$ and the expressions $\rho \cap \sigma$, $\rho \cup \sigma$ have the obvious set-theoretic meanings. It is easy to check that $\rho \cap \sigma$ is a congruence. On the other hand, $\rho \cup \sigma$ is not necessarily a congruence; we denote by $\rho \vee \sigma$ the smallest congruence containing ρ and σ. We note that

$$\iota_S = \{(x, x)\colon\ x \in S\} \text{ and } \omega_S = S \times S$$

are congruences, which we call respectively the *identical* and the *universal* congruence on S.

If ξ and η are relations on a set S, then we write $\xi \circ \eta$ for the relation consisting of all (x, y) in $S \times S$ for which there exists z in S such that $(x, z) \in \xi$ and $(z, y) \in \eta$.

2. The maximum idempotent-separating congruence

As a starting point for our investigations we have the following theorem, and a lemma on which the theorem depends, both due to Vagner (**10**) and Preston (**6**).

Theorem 2.1. *A homomorphic image of an inverse semigroup is an inverse semigroup.*

Lemma 2.2. *Let ρ be a congruence on an inverse semigroup S. Then the inverse image $e(\rho^\natural)^{-1}$ of an idempotent e in S/ρ contains an idempotent of S.*

At this stage we record one consequence of the theorem which will be particularly useful.

Corollary 2.3. *If ρ is a congruence on an inverse semigroup, then $(x, y) \in \rho$ if and only if $(x^{-1}, y^{-1}) \in \rho$.*

Proof. We denote the inverse semigroup by S. If $(x, y) \in \rho$, then the two elements $x\rho$ and $y\rho$ of S/ρ are equal. It is easy to verify that $x^{-1}\rho$ and $y^{-1}\rho$ are both inverses of $x\rho$ in S/ρ, and so $x^{-1}\rho = y^{-1}\rho$ since S/ρ is an inverse semigroup. That is $(x^{-1}, y^{-1}) \in \rho$. The converse follows from the fact that $(x^{-1})^{-1} = x$ and $(y^{-1})^{-1} = y$.

Theorem 2.4. *Let S be an inverse semigroup and let α_a be defined by* (1) *for any a in S. Then the relation μ defined by the rule that $(x, y) \in \mu$ if and only if $\alpha_x = \alpha_y$ is the maximum idempotent-separating congruence on S.*

Proof. It is immediate that μ is an equivalence relation. Now suppose that $(x, y) \in \mu$ and that z is an arbitrary element of S. Then from the supposition that $x^{-1}ex = y^{-1}ey$ for every idempotent e it follows immediately that $z^{-1}x^{-1}exz = z^{-1}y^{-1}eyz$ for every idempotent e: that is, $(xz, yz) \in \mu$. To show that $(zx, zy) \in \mu$, we note that $z^{-1}ez$ is an idempotent for every idempotent e, and so $x^{-1}(z^{-1}ez)x = y^{-1}(z^{-1}ez)y$ for every idempotent e. Thus $(zx, zy) \in \mu$ as required, and so μ is a congruence.

We next show that μ is idempotent-separating. Suppose that $(e, f) \in \mu$, where e and f are idempotents. Then, for every idempotent g, we have that $e^{-1}ge = f^{-1}gf$: that is, $eg = fg$. The equality holds in particular when

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$g = e$; hence $e = fe$. We similarly obtain that $ef = f$ by putting $g = f$. Since $ef = fe$, it follows that $e = f$. Thus μ is idempotent-separating.

Finally, let ν be an idempotent-separating congruence on S; we shall show that $\nu \subseteq \mu$. Suppose that $(x, y) \in \nu$. Then $(x^{-1}, y^{-1}) \in \nu$ by Corollary 2.3 and, since ν is a congruence, it follows that $(x^{-1}ex, y^{-1}ey) \in \nu$ for every idempotent e. But both $x^{-1}ex$ and $y^{-1}ey$ are idempotents, and so it follows that $x^{-1}ex = y^{-1}ey$ for every idempotent e, since ν is by assumption idempotent separating. Thus $(x, y) \in \mu$ and so $\nu \subseteq \mu$ as required. This completes the proof of Theorem 2.4.

An alternative characterisation of μ is provided by the next theorem.

Theorem 2.5. *Let S be an inverse semigroup with semilattice of idempotents E, let μ be the maximum idempotent-separating congruence on S, and let $E\zeta$ be the centraliser of E in S. Then $(x, y) \in \mu$ if and only if $x^{-1}x = y^{-1}y$ and $xy^{-1} \in E\zeta$. Dually, $(x, y) \in \mu$ if and only if $xx^{-1} = yy^{-1}$ and $x^{-1}y \in E\zeta$.*

Proof. It will be sufficient to prove the first of the two dual statements. Suppose first that $(x, y) \in \mu$, so that

$$x^{-1}ex = y^{-1}ey \quad \text{.................................(4)}$$

for every e in E. Then $(x^{-1}, y^{-1}) \in \mu$ by Corollary 2.3; that is, $xex^{-1} = yey^{-1}$ for every e in E. Hence

$$\begin{aligned} x^{-1}x &= x^{-1}x \,.\, x^{-1}x \,.\, x^{-1}x = x^{-1} \,.\, x(x^{-1}x)x^{-1} \,.\, x = x^{-1} \,.\, y(x^{-1}x)y^{-1} \,.\, x \\ &= y^{-1} \,.\, y(x^{-1}x)y^{-1} \,.\, y = y^{-1}y \,.\, x^{-1}x \,.\, y^{-1}y = x^{-1}x \,.\, y^{-1}y; \end{aligned}$$

and similarly $y^{-1}y = x^{-1}x \,.\, y^{-1}y$. Thus $x^{-1}x = y^{-1}y$. Also, premultiplying both sides of (4) by x and post-multiplying by y^{-1}, we have that

$$xx^{-1}exy^{-1} = xy^{-1}eyy^{-1}$$

for every e in E. Now,

$$xx^{-1}exy^{-1} = exx^{-1}xy^{-1} = exy^{-1}$$

and

$$xy^{-1}eyy^{-1} = xy^{-1}yy^{-1}e = xy^{-1}e,$$

and so $exy^{-1} = xy^{-1}e$ for every e in E: that is, $xy^{-1} \in E\zeta$.

Conversely, if $x^{-1}x = y^{-1}y$ and if $xy^{-1} \in E\zeta$, we have that $exy^{-1} = xy^{-1}e$ for every e in E. Premultiplying by x^{-1} and postmultiplying by y, we obtain

$$x^{-1}exy^{-1}y = x^{-1}xy^{-1}ey.$$

But $x^{-1}exy^{-1}y = x^{-1}exx^{-1}x = x^{-1}ex$, and similarly $x^{-1}xy^{-1}ey = y^{-1}ey$. Thus $x^{-1}ex = y^{-1}ey$ for every e in E, and so $(x, y) \in \mu$ as required. This completes the proof.

Remark. It has been shown by Munn (**5**) that for inverse semigroups (and for certain other classes of regular semigroups) the idempotent-separating congruences are precisely those contained in the equivalence relation $\mathscr{H}$ introduced by Green ((**3**); see also (**2**), Section 2.1), and that, in an arbitrary regular semigroup, the set of congruences contained in $\mathscr{H}$ forms a modular lattice with respect to the operations $\cap$ and $\vee$.

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Theorem 2.6. *Let S be an inverse semigroup with semilattice of idempotents E, and let μ be the maximum idempotent-separating congruence on S. Then $S/\mu \simeq E$ if and only if E is central in S.*

Proof. Since μ is idempotent-separating, it follows from Lemma 2.2 that S/μ is a semilattice if and only if each μ-class contains exactly one idempotent. Thus, if S/μ is a semilattice, we must have that $S/\mu \simeq E$.

Suppose first that each μ-class contains an idempotent. That is, for every x in S there exists an f in E such that $x^{-1}x = f^{-1}f$ and $xf^{-1} \in E\zeta$ (Theorem 2.5). Thus

$$x = xx^{-1}x = xf^{-1}f = xf = xf^{-1} \in E\zeta.$$

But this holds for any x in S and so $E\zeta = S$ as required.

Conversely, suppose that $E\zeta = S$. Then, in Theorem 2.5, the condition that $xy^{-1} \in E\zeta$ becomes superfluous, and we have simply that $(x, y) \in \mu$ if and only if $x^{-1}x = y^{-1}y$. It is now clear that $(x, x^{-1}x) \in \mu$ for every x in S, since $x^{-1}x = (x^{-1}x)^{-1}(x^{-1}x)$; hence every μ-class contains an idempotent. This completes the proof.

Theorem 2.7. *Let S be an inverse semigroup with semilattice of idempotents E, and let μ be the maximum idempotent-separating congruence on S. Then $\mu = \iota_S$, the identical congruence on S, if and only if E is self-centralising in S.*

Proof. Suppose first that $\mu = \iota_S$, and let $z \in E\zeta$. Then, if we write f for $z^{-1}z$, it is easy to see that $z^{-1}z = f^{-1}f(=f)$ and that $zf^{-1}(=z) \in E\zeta$. Thus $(z, f) \in \mu$ by Theorem 2.5 and so, since $\mu = \iota_S$, we have that $z = f \in E$. Thus $E\zeta = E$.

Conversely, suppose that $E\zeta = E$, and let $(x, y) \in \mu$. Then, by Theorem 2.5,

$$x^{-1}x = y^{-1}y, \quad xx^{-1} = yy^{-1}, \text{ and } xy^{-1}, \quad x^{-1}y \in E\zeta = E. \quad \text{.........(5)}$$

Since the element xy^{-1} is idempotent it must equal its inverse; i.e.

$$xy^{-1} = yx^{-1}. \quad \text{.....................................(6)}$$

Also, using the original characterisation of μ and the fact that (x^{-1}, y^{-1}) belongs to μ if (x, y) does, we have that

$$xx^{-1} = xx^{-1} . xx^{-1} = x(x^{-1}x)x^{-1} = y(x^{-1}x)y^{-1}$$
$$= yx^{-1} . xy^{-1} = (xy^{-1})^2 = xy^{-1}. \quad \text{......(7)}$$

Hence $$x = xx^{-1}x = xy^{-1}x = yx^{-1}x = yy^{-1}y = y \quad \text{.................(8)}$$

by (7), (6) and (5). Thus $\mu = \iota_S$, and the proof is complete.

If S is an arbitrary inverse semigroup, then S/μ can have no non-identical idempotent-separating congruences, for if ν were such a congruence, then the relation ν' on S defined by the rule that $(x, y) \in \nu'$ if and only if $(x\mu, y\mu) \in \nu$ would be an idempotent-separating congruence on S properly containing μ—a contradiction. Hence we have the following corollary to Theorem 2.7:

Corollary 2.8. *Let μ be the maximum idempotent-separating congruence on an arbitrary inverse semigroup S. Then the semilattice of idempotents of S/μ is self-centralising.*

Remark. It is easy to check that $\alpha_x\alpha_y = \alpha_{xy}$, so that the mapping α which sends x to α_x is a representation. The homomorphism α_x can alternatively be considered as a partial one-to-one mapping of E into itself thus: α_x maps $\{e \in E\colon\ e \leqq xx^{-1}\}$ in a one-to-one manner onto $\{e \in E\colon\ e \leqq x^{-1}x\}$. Considered in this way, the representation becomes identical to that described by Preston in ((7), Section 3). The condition for α to be faithful given by Theorem 2.7 above appears to be new. We also remark that the partial one-to-one mappings α_x are restrictions to E of the partial isomorphisms considered by Preston in (8).

3. The minimum group congruence

For an arbitrary inverse semigroup S, Munn (4) has given the following characterisation of σ, the minimum group congruence: $(x, y) \in \sigma$ if and only if there exists an idempotent e in S such that $ex = ey$. An alternative characterisation is provided by the next theorem.

Theorem 3.1. *Let S be an inverse semigroup with semilattice of idempotents E, and let σ be the minimum group congruence on S. Then $(x, y) \in \sigma$ if and only if $xy^{-1} \in E\omega$.*

Proof. Suppose first that $ex = ey$ for some e in E. Then $exy^{-1} = eyy^{-1} \in E$. Now $xy^{-1} \geqq exy^{-1}$ by (3), and so $xy^{-1} \in E\omega$.

Conversely, suppose that $xy^{-1} \in E\omega$. Then there exists f in E such that $xy^{-1} \geqq f$, i.e. such that $fxy^{-1} = f$. If we write e for $fxy^{-1}yx^{-1}$, then $e \in E$ and clearly $ef = e$. Also,

$$ex = e^2x = efxy^{-1}yx^{-1}x = efxx^{-1}xy^{-1}y = efxy^{-1}y = efy = ey.$$

Thus Theorem 3.1 is proved.

This characterisation of σ is the key to the proof of the next theorem.

Theorem 3.2. *Let σ be the minimum group congruence and μ the maximum idempotent-separating congruence on an inverse semigroup S with semilattice of idempotents E. Then $\sigma \cap \mu = \iota_S$ if and only if $E\omega \cap E\zeta = E$.*

Proof. By Theorems 2.5 and 3.1, we have that $(x, y) \in \sigma \cap \mu$ if and only if $x^{-1}x = y^{-1}y$ and $xy^{-1} \in E\omega \cap E\zeta$. Suppose first that $E\omega \cap E\zeta = E$ and that $(x, y) \in \sigma \cap \mu$. The equalities (6), (7) and (8) then follow exactly as in the proof of Theorem 2.7. Thus $x = y$, and so $\sigma \cap \mu = \iota_S$ as required.

Conversely, suppose that $\sigma \cap \mu = \iota_S$, and let $z \in E\omega \cap E\zeta$. If we denote $z^{-1}z$ by e, it is clear that $ze^{-1}(= ze = z)$ belongs to $E\omega$; hence $(z, e) \in \sigma$. Also, $ze^{-1} = z \in E\zeta$ and $z^{-1}z = e^{-1}e(= e)$; hence $(z, e) \in \mu$ (Theorem 2.5). Since by assumption $\sigma \cap \mu = \iota_S$, we must therefore have that $z = e \in E$. Hence $E\omega \cap E\zeta = E$ as required. This completes the proof.

We require some preliminaries before investigating the nature of $\sigma \vee \mu$. A subsemigroup H of an inverse semigroup S is called an *inverse* subsemigroup if x^{-1} belongs to H whenever x does. An inverse subsemigroup H of S will be called *self-conjugate* if zxz^{-1} belongs to H for any z whenever x belongs to H.

The next two lemmas are implicit in Šaĭn's paper (**9**).

Lemma 3.3. *Let K be a closed, self-conjugate inverse subsemigroup of an inverse semigroup S. Suppose further that $K \supseteq E$, the semilattice of idempotents of S. Then the relation ρ_K defined by the rule that $(x, y) \in \rho_K$ if and only if $xy^{-1} \in K$ is a congruence on S.*

Proof. Since $xx^{-1} \in E \subseteq K$, we have that ρ_K is reflexive. It is symmetric since $yx^{-1} = (xy^{-1})^{-1}$ belongs to K whenever xy^{-1} does. Suppose now that $xy^{-1}, yz^{-1} \in K$. Then $xy^{-1}yz^{-1} \in K$ since K is a subsemigroup. But

$$xz^{-1} \geqq xy^{-1}yz^{-1}$$

by (3), and so $xz^{-1} \in K$ since K is closed. Thus ρ_K is transitive.

Now suppose that $xy^{-1} \in K$ and that z is an arbitrary element of S. Then $(zx)(zy)^{-1} = zxy^{-1}z^{-1} \in K$ since K is self-conjugate. Also,

$$(xz)(yz)^{-1} = xzz^{-1}y^{-1} = xy^{-1} \,.\, yzz^{-1}y^{-1} \in K \,.\, E \subseteq K.$$

Thus ρ_K is a congruence.

Lemma 3.4. *If H is a self-conjugate inverse subsemigroup of an inverse semigroup S, then so is $H\omega$.*

Proof. Let x and y be elements of $H\omega$, and let h and k be the elements of H such that $x \geqq h$ and $y \geqq k$. From the compatibility of the order relation it now follows that $xy \geqq hk \in H$; hence $xy \in H\omega$. By (2), we have that

$$x^{-1} \geqq h^{-1} \in H;$$

hence $x^{-1} \in H\omega$. Thus $H\omega$ is an inverse subsemigroup. Finally, if z is an arbitrary element of S, it follows, again from the compatibility of the order relation, that $zxz^{-1} \geqq zhz^{-1} \in H$; hence $zxz^{-1} \in H\omega$.

We also have

Lemma 3.5. *Let S be an inverse semigroup with semilattice of idempotents E. Then the centraliser $E\zeta$ of E in S is a self-conjugate inverse subsemigroup of S.*

Proof. It is clear that xy belongs to $E\zeta$ if x and y do. If $x \in E\zeta$, then $xe = ex$ for every e in E. Taking inverses, we find that $ex^{-1} = x^{-1}e$ for every e in E; hence $x^{-1} \in E\zeta$. Now let $x \in E\zeta$ and let z be an arbitrary element of S. Then

$$\begin{aligned} zxz^{-1}e = zxz^{-1}zz^{-1}e = zxz^{-1}ezz^{-1} &= z(x \,.\, z^{-1}ez)z^{-1} \\ &= z(z^{-1}ez \,.\, x)z^{-1} = zz^{-1}ezxz^{-1} = ezz^{-1}zxz^{-1} = ezxz^{-1} \end{aligned}$$

for every e in E; hence $zxz^{-1} \in E\zeta$.

As an immediate consequence of the last two lemmas, we have

Lemma 3.6. *If S is an inverse semigroup with semilattice of idempotents E, then $(E\zeta)\omega$ is a closed, self-conjugate inverse subsemigroup of S.*

Since $(E\zeta)\omega$ certainly contains E, it now follows from Lemma 3.3 that the relation $\rho_{(E\zeta)\omega}$, which from now on we shall denote simply by ρ, is a congruence on S.

The following theorem characterises $\sigma \vee \mu$.

Theorem 3.7. *Let σ be the minimum group congruence and μ the maximum idempotent-separating congruence on an inverse semigroup S with semilattice of idempotents E. Then the relation ρ, defined by the rule that $(x, y) \in \rho$ if and only if $xy^{-1} \in (E\zeta)\omega$ is equal to $\sigma \vee \mu$.*

Proof. We have already remarked that ρ is a congruence on S. Moreover, it follows immediately from Theorems 3.1 and 2.5 that $\sigma \subseteq \rho$ and $\mu \subseteq \rho$; hence $\sigma \vee \mu \subseteq \rho$. It remains to prove that $\rho \subseteq \sigma \vee \mu$. We prove in fact that $\rho \subseteq \sigma \circ \mu \circ \sigma$, which is clearly sufficient.

Suppose, then, that $(x, y) \in \rho$. Then there exists $z \in E\zeta$ such that $xy^{-1} \geqq z$. Let

$$u = zy \text{ and } v = z^{-1}zy.$$

Then $xu^{-1} = xy^{-1}z^{-1} \geqq zz^{-1} \in E$ and so $xu^{-1} \in E\omega$. Thus $(x, u) \in \sigma$. Also,

$$v^{-1}v = y^{-1}z^{-1}zz^{-1}zy = y^{-1}z^{-1}zy = u^{-1}u$$

and, for every e in E,

$$uv^{-1}e = zyy^{-1}z^{-1}ze = zeyy^{-1}z^{-1}z \quad (\text{since } yy^{-1}z^{-1}z \in E)$$
$$= ezyy^{-1}z^{-1}z = euv^{-1} \quad (\text{since } z \in E\zeta).$$

Thus $uv^{-1} \in E\zeta$ and so, by Theorem 2.5, we have that $(u, v) \in \mu$. Finally, $vy^{-1} = z^{-1}zyy^{-1} \in E \subseteq E\omega$, and so $(v, y) \in \sigma$. Summarising, we have that

$$(x, u) \in \sigma, \quad (u, v) \in \mu, \quad (v, y) \in \sigma,$$

and so $(x, y) \in \sigma \circ \mu \circ \sigma$ as required. This completes the proof.

An obvious consequence of the theorem is

Corollary 3.8. *The smallest congruence $\sigma \vee \mu$ containing σ and μ is the universal congruence if and only if $(E\zeta)\omega = S$.*

Proof. It is clear that $\sigma \vee \mu = \omega_S$ if $(E\zeta)\omega = S$. Conversely, if $\sigma \vee \mu = \omega_S$, then $xy^{-1} \in (E\zeta)\omega$ for all x, y in S. In particular, for all x in S,

$$x(x^{-1}x)^{-1} = xx^{-1}x = x \in (E\zeta)\omega.$$

Thus $(E\zeta)\omega = S$.

Note. In the proof of Theorem 3.7 it emerged incidentally that

$$\sigma \vee \mu = \sigma \circ \mu \circ \sigma.$$

This remains true if μ is replaced by any congruence whatever on S:

Theorem 3.9. *Let σ be the minimum group congruence on an inverse semigroup S, and let ξ be an arbitrary congruence on S. Then $\sigma \vee \xi = \sigma \circ \xi \circ \sigma$.*

Proof. Clearly $\sigma \circ \xi \circ \sigma \subseteq \sigma \vee \xi$. To show the opposite inclusion it suffices to prove that $\sigma \circ \xi \circ \sigma$ is transitive, for it is then a congruence containing σ and ξ (and therefore containing $\sigma \vee \xi$). Suppose, then, that (x, y) and (y, z) belong to $\sigma \circ \xi \circ \sigma$. Then there exist a, b, c, d in S such that

$$(x, a) \in \sigma, \quad (a, b) \in \xi, \quad (b, y) \in \sigma,$$
$$(y, c) \in \sigma, \quad (c, d) \in \xi, \quad (d, z) \in \sigma.$$

Now, by the transitivity of σ, we have immediately that $(b, c) \in \sigma$, and so there exists an idempotent e such that $eb = ec$. By the left-compatibility of ξ, we have that

$$(ea, eb) \in \xi, \quad (ec, ed) \in \xi,$$

and so $(ea, ed) \in \xi$. Moreover, $ea = e \, . \, ea$, and so $(a, ea) \in \sigma$; hence, by transitivity, $(x, ea) \in \sigma$. A similar argument shows that $(ed, z) \in \sigma$. Hence, summarising, we have that

$$(x, ea) \in \sigma, \quad (ea, ed) \in \xi, \quad (ed, z) \in \sigma,$$

and so $(x, z) \in \sigma \circ \xi \circ \sigma$ as required.

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