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CARLESON MEASURES THEOREMS FOR GENERALIZED BERGMAN SPACES ON THE UNIT POLYDISK*

ARTUR MICHALAK

Wydział Matematyki i Informatyki, A. Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland
e-mail: michalak@amu.edu.pl

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Abstract. We provide sufficient conditions on a positive finite rotation invariant Borel measure on $\mathbb{D}^n$ which guarantee that the analogue of the Carleson measure theorem remains valid for Bergman spaces of holomorphic and n-harmonic functions on $\mathbb{D}^n$ generated by the measure.

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Theorems characterizing for a given space H of holomorphic or harmonic functions measures ω for which H is naturally embedded into the space $L^p(\omega)$, are usually called Carleson measures theorems. They go back to Carleson (see [**1**], [**6**], [**7**]), who described measures ω on $\mathbb{D}$ for which the Hardy space $H^p(\mathbb{D})$ is naturally embedded in $L^p(\omega)$. The description is given in terms of values of ω on "Carleson squares". This result was generalized by L. Hörmander for Hardy spaces on strictly pseudoconvex domains in $\mathbb{C}^n$ (see [**9**]). W. Hastings [**8**] proved a Carleson type theorem for Bergman spaces of holomorphic and harmonic functions on the disk and polydisk. A version for weighted Bergman spaces of holomorphic functions on the disk appears in a paper of D. A. Stegenga [**23**] (see also [**12**], [**18**], [**17**], [**24**]). J. A. Cima and W. R. Wogen [**3**] gave a Carleson type theorem for weighted Bergman spaces on the ball in $\mathbb{C}^n$. D. H. Luecking in [**11**] showed a general method to characterize Carleson measures on generalized Bergman spaces. This method was used by J. A. Cima and P. R. Mercer [**2**] to prove a Carleson measure theorem for weighted Bergman spaces on strictly pseudoconvex domains in $\mathbb{C}^n$. The analogue of the Carleson theorem for $H^p(\mathbb{D}^n)$ for $n > 1$ is not true.

In the paper we consider Bergman type spaces that are generalizations of the weighted Bergman spaces. Roughly speaking, a weighted Bergman space on the disk $\mathbb{D}$ is generated by the measure of the form $(1-r^2)^\alpha r dr \times \lambda$ in the polar coordinates for some $\alpha > -1$, where dr and λ are the Lebesgue measures on $[0,1)$ and $\mathbb{T}$, respectively. The spaces $b^p(\mu)$ and $B^p(\mu)$ of n-harmonic and holomorphic functions on $\mathbb{D}^n$ are generated by the measure $\mu \otimes \lambda_n = \Phi_*(\mu \times \lambda)$ on $\mathbb{D}^n$, where μ is a positive finite Borel measure on $[0,1)^n$ that does not vanish near $(1,\dots,1)$, λ_n is the Haar measure on $\mathbb{T}^n$ and the function $\Phi : [0,1)^n \times \mathbb{T}^n \to \mathbb{D}^n$ is given by $\Phi((r,t)) = rt$. Properties of these spaces were considered in [**22**], [**13**], [**14**]. One can show that each positive finite rotation invariant Borel measure on $\mathbb{D}^n$ has the form $\mu \otimes \lambda_n$ for some μ.

The aim of the paper is to prove Carleson type theorems for spaces $b^p(\mu)$ for $1 \leqslant p < \infty$ and $B^p(\mu)$ for $0 < p < \infty$. The paper is divided into three sections. The

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terminology and basic facts are explained in the first. In the second section we provide conditions on μ which guarantee that the spaces $b^p(\mu)$ for $1 \leqslant p < \infty$ and $B^p(\mu)$ for $0 < p < \infty$ are naturally embedded in $L^q(\omega)$ for $p \leqslant q < \infty$, if values of ω on "Carleson squares" are suitable for μ (see Theorem 3). Furthermore, we provide less restrictive conditions for μ which guarantee that the spaces $b^p(\mu)$ and $B^p(\mu)$ are naturally embedded in $L^q(\omega)$, if values of ω on a sequence of "suitable squares" inside the polydisk are appropriate for μ; (see Theorem 4). A crucial role in proofs of these results is played by Lemma 2 which describes a natural property of the Poisson kernels. For $n = 1$ (and spaces $b^p(\mu)$ for $1 < p < \infty$ and $B^p(\mu)$ for $0 < p < \infty$) the role of the lemma can be played by the Carleson theorem. Our method provides in the case of Bergman spaces generated by radial weights more precise results than the D. H. Luecking method. For Bergman spaces generated by "regular" radial weights they yield similar results. In the third section we give conditions for μ which guarantee that if $B^p(\mu)$ is naturally embedded in $L^q(\omega)$, then values of ω on "Carleson squares" are suitable for μ; (see Theorem 8). The proof of Theorem 8 applies some ideas that are standard in such considerations (see [**15**]). The conditions on μ in Theorems 3 and 4 are expressed in terms of values of μ on some sequence of parallelepipeds in $[0, 1)^n$. In Theorem 8 they look similar to those introduced by Shields and Williams in [**21**]. Propositions 6 and 10 show that in both cases they are quite easy to verify for wide families of measures. Theorems 3 and 8 together show that the classical Carleson theorem can be adapted for $b^p(\mu)$ and $B^p(\mu)$ spaces for a quite wide class of measures μ.

1. Basic properties of generalized Bergman spaces on the unit polydisk. We start by explaining basic notation used in this paper. As usual, $\mathbb{N}$, $\mathbb{D}$ and $\mathbb{T}$ will stand for the set of all positive integers, for the open unit disk and for the unit circle in the complex plane $\mathbb{C}$, respectively. Throughout the paper, n will be a positive integer and μ a positive finite Borel measure on $[0, 1)^n$ with $\mathbf{1} = (1, \ldots, 1) \in \mathrm{supp}(\mu)$; (the support of μ briefly denoted by $\mathrm{supp}(\mu)$ is the smallest closed set $C \subset [0, 1]^n$ such that $\mu(C) = \mu([0, 1)^n)$). The normalized Lebesgue (Haar) measures on $[0, 1)^n$, $\mathbb{T}$ and $\mathbb{T}^n$ will be denoted by η_n, λ and λ_n, respectively. For $r = (r_1, \ldots, r_n) \in [0, 1)^n$ and $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$ the element $rt = (r_1 t_1, \ldots, r_n t_n)$ is a member of $\mathbb{D}^n$. We denote by $\mu \otimes \lambda_n$ the Borel measure on $\mathbb{D}^n$ given by $\mu \otimes \lambda_n(A) = \mu \times \lambda_n(\Phi^{-1}(A))$, where $\Phi : [0, 1)^n \times \mathbb{T}^n \to \mathbb{D}^n$ is given by $\Phi((r, t)) = rt$. We shall use the following convention: if q is a member of an n-fold product X^n, then q_l is the l-coordinate of q for $l = 1, \ldots n$. The integer part of $x \in \mathbb{R}$ will be denoted by $[x]$.

For a positive finite Borel measure μ on $[0, 1)^n$ with $\mathbf{1} \in \mathrm{supp}(\mu)$ and $0 < p < \infty$, we denote by $b^p(\mu)$ the space of all n-harmonic functions (continuous and harmonic in each variable separately (see [**19**])) $f : \mathbb{D}^n \to \mathbb{C}$ such that

$$\left(\int_{\mathbb{D}^n} |f|^p \, d\mu \otimes \lambda_n\right)^{\frac{1}{p}} < \infty,$$

equipped with the norm (when $1 \leqslant p < \infty$) and quasi-norm (when $0 < p < 1$) defined by

$$\|f\| = \left(\int_{\mathbb{D}^n} |f|^p \, d\mu \otimes \lambda_n\right)^{\frac{1}{p}}.$$

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Its subspace consisting of all holomorphic functions will be denoted by $B^p(\mu)$. This definition covers many classical examples of Bergman type spaces on $\mathbb{D}^n$. For example: the space $B^p(r_1 \dots r_n \eta_n)$ is the classical Bergman space of holomorphic functions on $\mathbb{D}^n$, the space $B^p\big(\big(\prod_{l=1}^n (1-r_l^2)_l^{\alpha} r_l\big)\eta_n\big)$ for $\alpha_l > -1$, $l = 1, \dots, n$ are analogues of the classical weighted Bergman spaces of holomorphic functions; (see [**5**]). We concentrate only on properties of $b^p(\mu)$ spaces for $1 \leqslant p < \infty$ and $B^p(\mu)$ spaces for $0 < p < \infty$.

For $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{D}^n$ with $|z_l| < |w_l|$, for every $l = 1, \dots, n$ let

$$\mathbb{P}_z(w) = \prod_{l=1}^n \frac{|w_l|^2 - |z_l|^2}{|w_l - z_l|^2}.$$

For every $h = (h_1, \dots, h_n) \in (0, 1]^n$ let

$$P_h = \{(u_1, \dots, u_n) \in [0, 1)^n : 1 - h_l \leqslant u_l < 1,\ l = 1, \dots, n\}.$$

Proposition 1. *Let μ be a positive finite Borel measure on $[0,1)^n$ such that $\mathbf{1} \in \operatorname{supp}(\mu)$. For every $f \in b^p(\mu)$ with $1 \leqslant p < \infty$, and $f \in B^p(\mu)$ with $0 < p < \infty$, $z = (z_1, \dots, z_n) \in \mathbb{D}^n$, we have*

$$|f(z)| \leqslant \left(\frac{4^n}{\mu([\frac{1+|z_1|}{2}, 1) \times \dots \times [\frac{1+|z_n|}{2}, 1)) \prod_{l=1}^n (1 - |z_l|)}\right)^{\frac{1}{p}} \|f\|.$$

Proof. Applying the fact that $|f|^p$ is a subharmonic function in each variable separately we get

$$|f(z)|^p \leqslant \int_{\mathbb{T}^n} \mathbb{P}_z(Rt)\, |f(Rt)|^p \, d\lambda_n(t) \leqslant \left(\prod_{l=1}^n \frac{R_l + |z_l|}{R_l - |z_l|}\right) \int_{\mathbb{T}^n} |f(Rt)|^p \, d\lambda_n(t),$$

for every $R = (R_1, \dots, R_n)$ with $|z_l| < R_l < 1$ for $l = 1, \dots, n$. Therefore,

$$\begin{aligned} |f(z)|^p &\leqslant \frac{4^n}{\mu(P_{(\frac{1-|z_1|}{2}, \dots, \frac{1-|z_n|}{2})})\big(\prod_{l=1}^n (1-|z_l|)\big)} \int_{P_{(\frac{1-|z_1|}{2}, \dots, \frac{1-|z_n|}{2})}} \int_{\mathbb{T}^n} |f(rt)|^p \, d\lambda_n(t) d\mu(r) \\ &\leqslant \frac{4^n \, \|f\|^p}{\mu(P_{(\frac{1-|z_1|}{2}, \dots, \frac{1-|z_n|}{2})})\big(\prod_{l=1}^n (1-|z_l|)\big)}. \end{aligned}$$

From the proposition above, it follows that the topology of $b^p(\mu)$, if $1 \leqslant p < \infty$, and $B^p(\mu)$, if $0 < p < \infty$, is stronger than the topology of uniform convergence on compact subsets of $\mathbb{D}^n$. Then the spaces $b^p(\mu)$ and $B^p(\mu)$ are Banach when $1 \leqslant p < \infty$ and p-Banach when $0 < p < 1$. Therefore $b^p(\mu)$ and $B^p(\mu)$ are closed subspaces of $L^p(\mu \otimes \lambda)$. The space $L^p(\mu \otimes \lambda)$ is separable, for every $0 < p < \infty$; this is a straightforward consequence of the Lusin theorem and the fact that the space $C(\overline{\mathbb{D}}^n)$ is separable. It follows that the spaces $b^p(\mu)$ and $B^p(\mu)$ are separable. For every $t \in \mathbb{T}^n$ and $f \in b^p(\mu)$, $\|f\| = \|f_t\|$, where $f_t(z) = f(zt)$. Furthermore, the norm (quasi-norm) of $b^p(\mu)$ and $B^p(\mu)$ is lower semicontinuous in the topology of uniform convergence

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on compact subsets of $\mathbb{D}^n$. Consequently, harmonic polynomials are dense in $b^p(\mu)$ spaces, and polynomials are dense in $B^p(\mu)$, for every μ. (See **[10]** and **[16]**.) If $B^p(\mu)$ or $b^p(\mu)$ is naturally embedded in $L^q(\omega)$ for some positive finite Borel measure ω on $\mathbb{D}^n$ and $0 < q < \infty$, then the embedding is continuous. Moreover if the embedding is compact, then it maps bounded pointwise null sequences into norm null sequences. The first fact follows from the closed graph theorem and the fact that the natural embedding is continuous when $b^p(\mu)$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}^n$ and $L^q(\omega)$ with the topology of convergence in measure. The second follows from the fact that the closed unit balls of $b^p(\mu)$ and $B^p(\mu)$ are compact in the topology of uniform convergence on compact subsets of $\mathbb{D}^n$.

2. Sufficient conditions for Carleson measures on $B^p(\mu)$ and $b^p(\mu)$. For every $h = (h_1, \dots, h_n) \in (0, 1]^n$ and $t = (e^{iu_1}, \dots, e^{iu_n}) \in \mathbb{T}^n$ let

$$S_{h,t} = \{(r_1e^{is_1}, \dots, r_ne^{is_n}) \in \mathbb{D}^n : 1 - h_l \leqslant r_l < 1,\ u_l - h_l \leqslant s_l \leqslant u_l,\ l = 1, \dots, n\}.$$

For each Borel measure μ on $[0, 1)^n$ we have

$$\mu \otimes \lambda_n(S_{h,t}) = \mu(P_h) \prod_{l=1}^{n} \frac{h_l}{2\pi},$$

for every $h = (h_1, \dots, h_n) \in (0, 1]^n$ and $t \in \mathbb{T}^n$. The crucial role in our estimations will be played by the following result.

Lemma 2. *Let $0 \leqslant r_l < R_l < 1$ and $0 < \varepsilon_l \leqslant R_l - r_l$, for $l = 1, \dots, n$. Moreover, put $N_l = \left[\frac{2\pi}{\varepsilon_l}\right]$ and $J = \{0, 1, \dots, N_1 - 1\} \times \cdots \times \{0, 1, \dots, N_n - 1\}$. Finally, let $U_j = \{(e^{is_1}, \dots, e^{is_n}) \in \mathbb{T}^n : \frac{2\pi j_l}{N_l} \leqslant s_l \leqslant \frac{2\pi(j_l+1)}{N_l}, l = 1, \dots, n\}$, for $j = (j_1, \dots, j_n) \in J$. Then for every $t \in \mathbb{T}^n$ and $u_j \in U_j$, $j \in J$, we have*

$$\sum_{j \in J} \mathbb{P}_{ru_j}(Rt) \leqslant 3^n N_1 \dots N_n,$$

where $r = (r_1, \dots, r_n)$ and $R = (R_1, \dots, R_n)$.

The proof of the lemma is very technical but the idea is quite simple: we replace the sum by the lower Riemann integral sum of $\mathbb{P}_r$ on $\mathbb{T}^n$.

Proof. Let us take any $t = (t_1, \dots, t_n) \in \mathbb{T}^n$. Let $\bar{t} = (\bar{t}_1, \dots, \bar{t}_n)$. Let $\mathbf{e} : \mathbb{R} \to \mathbb{T}$ be given by $\mathbf{e}(s) = e^{2\pi is}$. For every $j \in J$ let $\tilde{U}_j = \mathbf{e}\big(\big[\frac{j_1}{N_1}, \frac{j_1+1}{N_1}\big)\big) \times \dots \times \mathbf{e}\big(\big[\frac{j_n}{N_n}, \frac{j_n+1}{N_n}\big)\big)$. The sets $\tilde{U}_j$ are pairwise disjoint and $\mathbb{T}^n = \bigcup_{j \in J} \bar{t}\tilde{U}_j$. Let $w_j = \big(e^{i\frac{2\pi j_1}{N_1}}, \dots, e^{i\frac{2\pi j_n}{N_n}}\big)$. Each w_j belongs to exactly one of the sets $\{\bar{t}\tilde{U}_m\}_{m \in J}$ and each of the sets contains exactly one element w_j. Denote by $\varphi : J \to J$ the bijection defined in this way:

$$\varphi(j) = m \qquad \text{if } w_m \in \bar{t}\tilde{U}_j.$$

Let $v_j = (v_{j,1}, \dots, v_{j,n})$, where

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$$v_{j,l} = \begin{cases} 1 & \text{if } \varphi(j)_l = 0, \\ \mathbf{e}\left(\frac{\varphi(j)_l - 1}{N_l}\right) & \text{if } 0 < \frac{\varphi(j)_l}{N_l} < \frac{1}{2}, \\ \mathbf{e}\left(\frac{\varphi(j)_l + 1}{N_l}\right) & \text{if } \frac{\varphi(j)_l}{N_l} \geqslant \frac{1}{2}. \end{cases}$$

It is clear that $\bar{t}U_j \subset \mathbf{e}(W_{j,1}) \times \ldots \times \mathbf{e}(W_{j,n})$, where

$$W_{j,l} = \begin{cases} \left[\frac{\varphi(j)_l - 1}{N_l}, \frac{\varphi(j)_l + 1}{N_l}\right] & \text{if } \varphi(j)_l \geqslant 1, \\ \left[0, \frac{1}{N_l}\right] \cup \left[\frac{N_l - 1}{N_l}, 1\right] & \text{if } \varphi(j)_l = 0. \end{cases}$$

Since the function $s \to \frac{R_l^2 - r_l^2}{|R_l e^{is} - r_l|^2}$ is increasing on $(\pi, 2\pi)$ and decreasing on $(0, \pi)$, for each $l = 1, \ldots, n$,

$$\mathbb{P}_{ru_j}(Rt) = \mathbb{P}_r(Rt\bar{u}_j) = \mathbb{P}_r(Ru_j\bar{t}) \leqslant \mathbb{P}_r(Rv_j).$$

For every $l = 1, \ldots, n$, we have

$$\frac{R_l^2 - r_l^2}{|R_l v_{j,l} - r_l|^2} \leqslant \frac{R_l + r_l}{R_l - r_l} \leqslant \frac{2}{\varepsilon_l}.$$

Let $V_j = \mathbf{e}(T_{j,1}) \times \ldots \times \mathbf{e}(T_{j,n})$, where

$$T_{j,l} = \begin{cases} \left[\frac{\max\{0, \varphi(j)_l - 2\}}{N_l}, \frac{\max\{1, \varphi(j)_l - 1\}}{N_l}\right) & \text{if } \frac{\varphi(j)_l}{N_l} < \frac{1}{2}, \\ \left[\frac{\min\{N_l - 1, \varphi(j)_l + 1\}}{N_l}, \frac{\min\{N_l, \varphi(j)_l + 2\}}{N_l}\right) & \text{if } \frac{\varphi(j)_l}{N_l} \geqslant \frac{1}{2}. \end{cases}$$

For every $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{0, 1, 2\}^n$ let $C_\sigma = B_1^{\sigma_1} \times \ldots \times B_n^{\sigma_n}$, where $B_l^0 = \{2, 3, \ldots, N_l - 3\}$, $B_l^1 = \{0, N_l - 1\}$ and $B_l^2 = \{1, N_l - 2\}$. Let $E_\sigma = D_1^{\sigma_1} \times \ldots \times D_n^{\sigma_n}$ where $D_l^0 = \mathbb{T}$ and $D_l^1 = D_l^2 = \mathbf{e}([-\frac{1}{N_l}, \frac{1}{N_l}])$. Applying the following facts

$$\bigcup_{\{j : \varphi(j) \in C_\sigma\}} V_j \subset E_\sigma \quad \text{and} \quad \int_{\mathbb{T}} \frac{R_l^2 - r_l^2}{|R_l x - r_l|^2} \, d\lambda(x) = 1$$

for every $\sigma \in \{0, 1, 2\}^n$ and $l = 1, \ldots, n$, we get

$$\begin{aligned}
\frac{1}{N_1 \ldots N_n} \sum_{j \in J} \mathbb{P}_{ru_j}(Rt) &\leqslant \frac{1}{N_1 \ldots N_n} \sum_{j \in J} \mathbb{P}_r(Rv_j) \\
&\leqslant \sum_{\sigma \in \{0,1,2\}^n} \sum_{\{j : \varphi(j) \in C_\sigma\}} \mathbb{P}_r(Rv_j) \lambda_n(V_j) \\
&\leqslant \sum_{\sigma \in \{0,1,2\}^n} \sum_{\{j : \varphi(j) \in C_\sigma\}} \Big(\prod_{\{l : \sigma_l \neq 0\}} \frac{2}{\varepsilon_l} \Big) \Big(\prod_{\{l : \sigma_l = 0\}} \frac{R_l^2 - r_l^2}{|R_l v_{j,l} - r_l|^2} \Big) \lambda_n(V_j) \\
&\leqslant \sum_{\sigma \in \{0,1,2\}^n} \int_{E_\sigma} \Big(\prod_{\{l : \sigma_l \neq 0\}} \frac{2}{\varepsilon_l} \Big) \Big(\prod_{\{l : \sigma_l = 0\}} \frac{R_l^2 - r_l^2}{|R_l x_l - r_l|^2} \Big) \, d\lambda_n(x) \\
&\leqslant \sum_{\sigma \in \{0,1,2\}^n} \prod_{\{l : \sigma_l \neq 0\}} \frac{2}{N_l} \frac{2}{\varepsilon_l} \leqslant \prod_{l=1}^n \Big(1 + \frac{8}{N_l \varepsilon_l} \Big) \leqslant \Big(1 + \frac{8}{2\pi - 1} \Big)^n.
\end{aligned}$$

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Now we are ready to state the Carleson theorem for $b^p(\mu)$ and $B^p(\mu)$ spaces.

THEOREM 3. *Let μ be a finite positive Borel measure on $[0,1)^n$ with $\mathbf{1} \in \operatorname{supp}(\mu)$. If there exist $C_1 > 0$, $C_2 > 0$, strictly decreasing sequences $(a_{m,1}), \dots, (a_{m,n}) \subset (0,1)$ converging to zero, increasing sequences $(b_{m,1}), \dots, (b_{m,n}) \subset \mathbb{N}$ and $d \in \mathbb{N}$ such that*

(i) $\dfrac{a_{m-1,l}}{a_{m,l} - a_{m+b_{m,l},l}} \leqslant C_1$, *for every $m \in \mathbb{N}$ and $l = 1, \dots, n$,*

(ii) $\dfrac{\mu(P_{(a_{k_1-1,1},\dots,a_{k_n-1,n})})}{\mu(B_k)} \leqslant C_2$, *for every $k = (k_1, \dots, k_n) \in \mathbb{N}^n \setminus \{\mathbf{1}\}$, where* $B_k = [1 - a_{k_1+b_{k_1,1},1}, 1 - a_{k_1+d+b_{k_1+d,1},1}) \times \dots \times [1 - a_{k_n+b_{k_n,n},n}, 1 - a_{k_n+d+b_{k_n+d,n},n})$ *and* $a_{0,1} = \dots = a_{0,n} = 1$,

then

(a) *for every finite positive Borel measure ω on $\mathbb{D}^n$ such that*

$$\omega(S_{h,t}) \leqslant C_3 \Big(\mu(P_h) \prod_{l=1}^{n} h_l\Big)^{\frac{q}{p}},$$

for some $C_3 > 0$, every $h = (h_1, \dots, h_n) \in (0,1]^n$ and every $t \in \mathbb{T}^n$, the space $b^p(\mu)$ is naturally embedded in $L^q(\omega)$, for every $1 \leqslant p \leqslant q < \infty$, and the space $B^p(\mu)$ is naturally embedded in $L^q(\omega)$, for every $0 < p \leqslant q < \infty$;

(b) *for every finite positive Borel measure ω on $\mathbb{D}$ such that*

$$\lim_{\delta \to 0^+} \sup \Big\{ \frac{\omega(S_{h,t})}{(\mu(P_h) \prod_{l=1}^{n} h_l)^{\frac{q}{p}}} : h = (h_1, \dots, h_n) \in (0,1]^n,\ \min_l \{h_l\} \leqslant \delta,\ t \in \mathbb{T}^n \Big\} = 0,$$

the embedding $I : b^p(\mu) \to L^q(\omega)$ is compact, for every $1 \leqslant p \leqslant q < \infty$, and the embedding $I : B^p(\mu) \to L^q(\omega)$ is compact, for every $0 < p \leqslant q < \infty$.

Proof. For every $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $l = 1, \dots, n$ let

$$A_k = [1 - a_{k_1-1,1}, 1 - a_{k_1,1}) \times \dots \times [1 - a_{k_n-1,n}, 1 - a_{k_n,n})$$

and $N_{k,l} = \Big[\frac{2\pi}{a_{k_l,l} - a_{k_l+b_{k_l,l},l}}\Big]$. Let

$$J_k = \{0, 1, \dots, N_{k,1} - 1\} \times \dots \times \{0, 1, \dots, N_{k,n} - 1\}.$$

For every $j = (j_1, \dots, j_n) \in J_k$ let

$$U_{k,j} = \{(e^{is_1}, \dots, e^{is_n}) \in \mathbb{T}^n : \frac{2\pi j_l}{N_{k,l}} \leqslant s_l \leqslant \frac{2\pi(j_l+1)}{N_{k,l}}, l = 1, \dots, n\}$$

and $A_{k,j} = \Phi(A_k \times U_{k,j})$, where $\Phi : [0,1)^n \times \mathbb{T}^n \to \mathbb{D}^n$ is given by $\Phi((r,t)) = rt$. Then

$$A_{k,j} \subset \bigcup_{\sigma \in \{0,1\}^n} S_{(a_{k_1-1,1},\dots,a_{k_n-1,n}),\Big(e^{i(\frac{2\pi(j_1+1)}{N_{k,l}} - \sigma_1 a_{k-1,1})},\dots,e^{i(\frac{2\pi(j_n+1)}{N_{k,l}} - \sigma_n a_{k-1,n})}\Big)}.$$

Hence

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$$\omega(A_{k,j}) \leqslant 2^n C_3\big(\mu(P_{(a_{k_1-1,1},\ldots,a_{k_n-1,n})})\prod_{l=1}^{n} a_{k_l-1,l}\big)^{\frac{q}{p}}.$$

Let us take any $f \in b^p(\mu)$, for $1 \leqslant p < \infty$, or $f \in B^p(\mu)$, for $0 < p < \infty$. Let $|f|$ attain its maximum on $\overline{A_{k,j}}$ at $w_{k,j} = (|w_{k,j,1}|e^{is_{k,j,1}}, \ldots, |w_{k,j,n}|e^{is_{k,j,n}}) \in \overline{A_{k,j}}$; $|f(w_{k,j})| = \sup\{|f(z)| : z \in A_{k,j}\}$. Let $u_{k,j} = (e^{is_{k,j,1}}, \ldots, e^{is_{k,j,n}})$. Applying the fact that $|f|^p$ is a n-subharmonic function we get

$$|f(w_{k,j})|^p \leqslant \int_{\mathbb{T}^n} \mathbb{P}_{w_{k,j}}(Rt)|f(Rt)|^p \, d\lambda_n(t)$$

for every $R = (R_1, \ldots, R_n) \in [0, 1)^n$ with $|w_{k,j,l}| < R_l < 1$, for $l = 1, \ldots, n$. For every $R = (R_1, \ldots, R_n) \in B_k$,

$$\begin{aligned}\sup_{s\in\mathbb{R}} \frac{\frac{R_l^2-|w_{k,j,l}|^2}{|R_le^{is}-w_{k,j,l}|^2}}{\frac{R_l^2-|1-a_{k_l,l}|^2}{|R_le^{is}-(1-a_{k_l,l})u_{k,j,l}|^2}} &\leqslant \sup_{s\in\mathbb{R}} \frac{(R_l-|w_{k,j,l}|)\big((R_l-1+a_{k_l,l})^2+4R_l(1-a_{k_l,l})\sin^2\big(\frac{s-s_{k,j,l}}{2}\big)\big)}{(R_l-1+a_{k_l,l})\big((R_l-|w_{k,j,l}|)^2+4R_l|w_{k,j,l}|\sin^2\big(\frac{s-s_{k,j,l}}{2}\big)\big)} \\ &\leqslant \Big(1+\frac{a_{k_l-1,l}-a_{k_l,l}}{a_{k_l,l}-a_{k_l+b_{k_l,l},l}}\Big)\frac{(R_l+1-a_{k_l,l})^2}{(R_l+|w_{k,j,l}|)^2} \leqslant 4C_1,\end{aligned}$$

for every $l = 1, \ldots, n$. The second inequality follows from the fact that the linear fractional mapping

$$\rho \to \frac{(R_l-1+a_{k_l,l})^2+4R_l(1-a_{k_l,l})\rho}{(R_l-|w_{k,j,l}|)^2+4R_l|w_{k,j,l}|\rho}$$

attains its maximum on $[0, 1]$ at 0 or 1; (in this case at 1). We now set $r_k = (1 - a_{k_1,1}, \ldots, 1 - a_{k_n,n})$. Applying Lemma 2 we get

$$\begin{aligned}\sum_{j\in J_k} |f(w_{k,j})|^p &\leqslant \int_{\mathbb{T}^n} \sum_{j\in J_k} \mathbb{P}_{w_{k,j}}(Rt)|f(Rt)|^p \, d\lambda_n(t) \\ &\leqslant \int_{\mathbb{T}^n} \Big(\sum_{j\in J_k} 4^n C_1^n \mathbb{P}_{r_k u_{k,j}}(Rt)\Big)|f(Rt)|^p \, d\lambda_n(t) \\ &\leqslant 12^n C_1^n \big(\prod_{l=1}^{n} N_{k,l}\big) \int_{\mathbb{T}^n} |f(Rt)|^p \, d\lambda_n(t).\end{aligned}$$

Since the estimation above holds for every $R \in B_k$, it follows that

$$\sum_{j\in J_k} |f(w_{k,j})|^p \leqslant 12^n C_1^n \big(\prod_{l=1}^{n} N_{k,l}\big) \frac{1}{\mu(B_k)} \int_{B_k} \int_{\mathbb{T}^n} |f(rt)|^p \, d\lambda_n(t) d\mu(r).$$

Applying the fact that the Banach space l^1 is contained in $l^{\frac{q}{p}}$ we get

$$\Big(\sum_{j\in J_k} |f(w_{k,j})|^q\Big)^{\frac{p}{q}} \leqslant 12^n C_1^n \big(\prod_{l=1}^{n} N_{k,l}\big) \frac{1}{\mu(B_k)} \int_{B_k} \int_{\mathbb{T}^n} |f(rt)|^p \, d\lambda_n(t) d\mu(r).$$

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Since the sets $[1-a_{m_1+b_{m_1,l},l}, 1-a_{m_1+d+b_{m_1+d,l},l})$ and $[1-a_{m_2+b_{m_2,l},l}, 1-a_{m_2+d+b_{m_2+d,l},l})$ are disjoint if $|m_1-m_2|\geqslant d$, each element of $\mathbb{D}^n$ belongs to at most d^n distinct B_k sets. Let $E=(1-a_{1,1})\mathbb{D}\times\ldots\times(1-a_{1,n})\mathbb{D}$. Then

$$\begin{aligned}\int_{\mathbb{D}^n\setminus E}|f|^q\,d\omega &\leqslant \sum_{k\in\mathbb{N}^n\setminus\{\mathbf{1}\}}\sum_{j\in J_k}|f(w_{k,j})|^q\omega(A_{k,j})\\ &\leqslant \sum_{k\in\mathbb{N}^n\setminus\{\mathbf{1}\}}\sum_{j\in J_k}|f(w_{k,j})|^q 2^n\,C_3\big(\mu(P_{(a_{k_1-1,1},\ldots,a_{k_n-1,n})})\prod_{l=1}^n a_{k_l-1,l}\big)^{\frac{q}{p}}\\ &\leqslant \sum_{k\in\mathbb{N}^n\setminus\{\mathbf{1}\}}2^nC_3\Big(\Big(\prod_{l=1}^n\frac{24\pi C_1\,a_{k_l-1,l}}{a_{k_l,l}-a_{k_l+b_{k_l,l},l}}\Big)\frac{\mu(P_{(a_{k_1-1,1},\ldots,a_{k_n-1,n})})}{\mu(B_k)}\int_{B_k}\int_{\mathbb{T}^n}|f(rt)|^p\,d\lambda_n(t)d\mu(r)\Big)^{\frac{q}{p}}\\ &\leqslant 2^nC_3\big(C_2(24\pi dC_1^2)^n\|f\|^p\big)^{\frac{q}{p}}.\end{aligned}$$

The last inequality also follows from the fact that the Banach space l^1 is naturally embedded into $l^{\frac{q}{p}}$. By Proposition 1,

$$\int_E|f|^q\,d\omega\leqslant\omega(\mathbb{D}^n)\Big(\frac{4^n\|f\|^p}{\mu(P_{(\frac{a_{1,1}}{2},\ldots,\frac{a_{1,n}}{2})})\prod_{l=1}^n a_{1,l}}\Big)^{\frac{q}{p}}$$

which completes the proof in the case (a).

(b) It is enough to show that I maps bounded pointwise null sequences to norm (quasi-norm) null sequences. Let (f_m) be a bounded pointwise null sequence either in $b^p(\mu)$ for $1\leqslant p<\infty$ or in $B^p(\mu)$ for $0<p<\infty$. By Proposition 1, (f_m) converges uniformly to zero on compact subsets of $\mathbb{D}^n$. For every $\varepsilon>0$ we can find $M=(M_1,\ldots,M_n)\in\mathbb{N}^n$ such that, applying the estimation above for sequences $a'_{m,l}=a_{M_l+m-1,l}$, we get

$$\limsup_m\int_{\mathbb{D}^n}|f_m|^q d\omega\leqslant\limsup_m\int_{E_M}|f_m|^q d\omega+C\|f_m\|^q\sup\frac{\omega(S_{h,t})}{(\mu(P_h)\prod_{l=1}^n h_l)^{\frac{q}{p}}}\leqslant\varepsilon,$$

where $E_M=(1-a_{M_1,1})\mathbb{D}\times\ldots\times(1-a_{M_n,n})\mathbb{D}$, the supremum is taken over all $h=(h_1,\ldots,h_n)$ with $\min_l\{h_l\}\leqslant\max_l\{a_{M_l,l}\}$ and $t\in\mathbb{T}^n$, $C=2^n\big(C_2(24\pi dC_1^2)^n\big)^{\frac{q}{p}}$.

REMARKS. (1) We shall apply in the paper the theorem above (and the next theorem) only with $d=b_{k,l}=1$ for every $k\in\mathbb{N}$, and $l=1,\ldots,n$. Then $B_k=A_{k+(2,\ldots,2)}$.

The condition (i) needs some explanation. Easy estimations show that

$$\frac{a_{m+1,l}}{a_{m,l}}\geqslant\frac{1}{C_1}\quad\text{and}\quad\frac{a_{m+b_{m,l},l}}{a_{m,l}}\leqslant\frac{C_1-1}{C_1}$$

for every $l=1,\ldots,n$. It is clear that both inequalities together with independent constants from the interval $(0,1)$ on the right hand sides are equivalent to (i). A natural test of verification whether or not a measure μ satisfies the conditions of Theorem 3 in the case $n=1$ is to choose $c>1$, find a sequence $a_{m,1}\subset(0,1)$ such that $\mu([1-a_{m,1},1))=c^{-m}\mu([0,1))$ and verify whether or not the sequence $(a_{m,1})$ satisfies the condition (i).

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(2) For $n = 1$ and spaces $b^p(\mu)$, for $1 < p < \infty$, and $B^p(\mu)$, for $0 < p < \infty$, we can apply the classical Carleson theorem for Hardy spaces on $\mathbb{D}$ instead of Lemma 2. We give a sketch of the idea. We apply the Carleson theorem for the measure $\omega|_{\Phi([1-a_{k-1,1},1-a_{k,1})\times\mathbb{T})}$ and the disk $(1 - a_{k+b_{k,1},1})\mathbb{D}$. Then

$$\int_{\Phi([1-a_{k-1,1},1-a_{k,1})\times\mathbb{T})} |f|^q d\omega \leqslant CC_3\Big((1 - a_{k+b_{k,1},1})\frac{2a_{k-1}\mu([1 - a_{k-1,1}, 1))}{a_{k,1} - a_{k+b_{k,1},1}}\int_{\mathbb{T}} |f(rt)|^p d\lambda(t)\Big)^{\frac{q}{p}},$$

for every $1 > r \geqslant 1 - a_{k+b_{k,1},1}$, where C is either the constant from the Carleson theorem for the Hardy space $h^p(\mathbb{D})$ of harmonic functions on $\mathbb{D}$ if $f \in b^p(\mu)$ or the constant from the Carleson theorem for the Hardy space $H^p(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ if $f \in B^p(\mu)$; (see **[6]**, **[7]**). Thus we save two pages of technicalities. The rest of the consideration proceeds similarly as in the proof above.

From the estimations in the proof above, it follows that the "Carleson squares" $S_{h,t}$ are not the most suitable tools for Carleson type theorems for some measures μ.

THEOREM 4. *Let μ be a finite positive Borel measure on $[0, 1)^n$ such that $\mathbf{1} \in \operatorname{supp}(\mu)$. Suppose that there exist $C_1' > 0$, $C_4 > 0$, strictly decreasing sequences $(a_{m,1}), \ldots, (a_{m,n})$, with each $a_{m,r} \in (0, 1)$, converging to zero, increasing sequences $(b_{m,1}), \ldots, (b_{m,n}) \subset \mathbb{N}$ and $d \in \mathbb{N}$ such that*

(i) $\dfrac{a_{m-1,l} - a_{m,l}}{a_{m,l} - a_{m+b_{m,l},l}} \leqslant C_1'$, *for every $m \in \mathbb{N}$ and $l = 1, \ldots, n$,*

(ii) $\dfrac{\mu(A_k)}{\mu(B_k)} \leqslant C_4$, *for every $k \in \mathbb{N}^n \setminus \{\mathbf{1}\}$,*

where $a_{0,1} = \ldots = a_{0,n} = 1$,

$$A_k = [1 - a_{k_1-1,1}, 1 - a_{k_1,1}) \times \ldots \times [1 - a_{k_n-1,n}, 1 - a_{k_n,n})$$

and

$$B_k = [1 - a_{k_1+b_{k_1,1},1}, 1 - a_{k_1+d+b_{k_1+d,1},1}) \times \ldots \times [1 - a_{k_n+b_{k_n,n},n}, 1 - a_{k_n+d+b_{k_n+d,n},n}).$$

Moreover, put $|k| = k_1 + \ldots + k_n$, $N_{k,l} = [\frac{2\pi}{a_{k_l,l} - a_{k_l+b_{k,l},l}}]$ and

$$J_k = \{0, 1, \ldots, N_{k,1} - 1\} \times \ldots \times \{0, 1, \ldots, N_{k,n} - 1\},$$

for every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$. Finally, let

$$U_{k,j} = \{(e^{is_1}, \ldots, e^{is_n}) \in \mathbb{T}^n : \frac{2\pi j_l}{N_{k,l}} \leqslant s_l \leqslant \frac{2\pi(j_l + 1)}{N_{k,l}}, l = 1, \ldots, n\}$$

and $A_{k,j} = \Phi(A_k \times U_{k,j})$ for every $j = (j_1, \ldots, j_n) \in J_k$. ($\Phi : [0, 1)^n \times \mathbb{T}^n \to \mathbb{D}^n$ is given by $\Phi((r, t)) = rt$.) Then

(a) *for every finite positive Borel measure ω on $\mathbb{D}^n$ such that*

$$\omega(A_{k,j}) \leqslant C_5\big(\mu(A_k)\prod_{l=1}^{n}(a_{k_l-1,l} - a_{k_l,l})\big)^{\frac{q}{p}}$$

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for some $C_5 > 0$, *every* $k \in \mathbb{N}^n \setminus \{\mathbf{1}\}$ *and every* $j \in J_k$, *the space* $b^p(\mu)$ *is naturally embedded in* $L^q(\omega)$, *for every* $1 \leqslant p \leqslant q < \infty$, *and the space* $B^p(\mu)$ *is naturally embedded in* $L^q(\omega)$, *for every* $0 < p \leqslant q < \infty$;

(b) *for every finite positive Borel measure* ω *on* $\mathbb{D}$ *such that*

$$\lim_{|k|} \sup_{j \in J_k} \left\{ \frac{\omega(A_{k,j})}{(\mu \otimes \lambda_n(A_{k,j}))^{\frac{q}{p}}} \right\} = 0,$$

the embedding $I : b^p(\mu) \to L^q(\omega)$ *is compact, for every* $1 \leqslant p \leqslant q < \infty$, *and the embedding* $I : B^p(\mu) \to L^q(\omega)$ *is compact, for every* $0 < p \leqslant q < \infty$.

Proof. We use the notation and the estimations from the proof of Theorem 3. Then for every $f \in b^p(\mu)$, for $1 \leqslant p < \infty$, or $f \in B^p(\mu)$, for $0 < p < \infty$, we have

$$\begin{aligned}
\int_{\mathbb{D}^n \setminus E} |f|^q \, d\omega &\leqslant \sum_{k \in \mathbb{N}^n \setminus \{\mathbf{1}\}} \sum_{j \in J_k} |f(w_{k,j})|^q \omega(A_{k,j}) \\
&\leqslant \sum_{k \in \mathbb{N}^n \setminus \{\mathbf{1}\}} \sum_{j \in J_k} |f(w_{k,j})|^q C_5 \Big(\mu(A_k) \prod_{l=1}^n (a_{k_l - 1, l} - a_{k_l, l}) \Big)^{\frac{q}{p}} \\
&\leqslant \sum_{k \in \mathbb{N}^n \setminus \{\mathbf{1}\}} C_5 \bigg(\Big(\prod_{l=1}^n 24\pi (C_1' + 1) \frac{a_{k_l - 1, l} - a_{k_l, l}}{a_{k_l, l} - a_{k_l + b_{k_l, l}, l}} \Big) \frac{\mu(A_k)}{\mu(B_k)} \int_{B_k} \int_{\mathbb{T}^n} |f(rt)|^p d\lambda_n(t) d\mu(r) \bigg)^{\frac{q}{p}} \\
&\leqslant C_5 \big(C_4 (24\pi d (C_1' + 1) C_1')^n \|f\|^p \big)^{\frac{q}{p}},
\end{aligned}$$

where $E = (1 - a_{1,1})\mathbb{D} \times \ldots \times (1 - a_{1,n})\mathbb{D}$. The rest of the proof of part (a) and the proof of (b) proceeds similarly as in the proof of Theorem 3.

Corollary 5. *Suppose that* μ *fulfills the assumptions of Theorem 4.*

(a) *If* ν *is a finite positive Borel measure on* $[0, 1)^n$ *with* $\mathbf{1} \in \operatorname{supp}(\nu)$ *such that*

$$\limsup_{|k|} \frac{\nu(A_k)}{\mu(A_k)} < \infty,$$

then $b^p(\mu)$ *is naturally embedded in* $b^p(\nu)$, *for every* $1 \leqslant p < \infty$ *and* $B^p(\mu)$ *is naturally embedded in* $B^p(\nu)$, *for every* $0 < p < \infty$.

(b) *If* ν *is a finite positive Borel measure on* $[0, 1)^n$ *with* $\mathbf{1} \in \operatorname{supp}(\nu)$ *such that*

$$\limsup_{|k|} \frac{\nu(A_k)}{\mu(A_k)} = 0,$$

then the embedding $I : b^p(\mu) \to b^p(\nu)$ *is compact, for every* $1 \leqslant p < \infty$, *and the embedding* $I : B^p(\mu) \to B^p(\nu)$ *is compact, for every* $0 < p < \infty$.

The corollary shows that if a measure μ fulfills the assumptions of Theorem 4, then its values on sets A_k can be modified (within some limits, naturally) and still it generates the same spaces $b^p(\mu)$ and $B^p(\mu)$. In particular $b^p(\mu)$ and $B^p(\mu)$ do not depend on values of μ on any compact subset of $[0, 1)^n$. The next result shows that a measure $\mu = f(r_1, \ldots, r_n) \prod_{l=1}^n (1 - r_l)^{\alpha_l} \eta_n$ for $\alpha_l > -1$, satisfies conditions of

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Theorem 3 if the function $f: [0,1)^n \to \mathbb{R}_+$ increases or decreases not too quickly in each variable separately.

PROPOSITION 6. *Let* $f: [0,1)^n \to \mathbb{R}_+$, $\alpha = (\alpha_1, \dots, \alpha_n) \in (-1, \infty)^n$. *Let* $f_{l,r}: (0,1) \to \mathbb{R}_+$ *be given by* $f_{l,r}(x) = f(r_1, \dots, r_l - 1, x, r_l + 1, \dots, r_n)$, *for every* $l = 1, \dots, n$ *and* $r \in [0,1)^n$. *If for each* $l = 1, \dots, n$ *the function* $f_{l,r}$ *is either increasing, for every* $r \in [0,1)^n$, *and*

$$\limsup_k \frac{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k+1}\right)}{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k}\right)} = C_l^{\alpha_l+1} < m_l^{\alpha_l+1}, \qquad \text{for some } m_l > 1 \text{ and every } r \in [0,1)^n,$$

or decreasing, for every $r \in [0,1)^n$, *and*

$$\limsup_k \frac{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k}\right)}{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k+1}\right)} \leqslant C, \qquad \text{for some } m_l > 1 \text{ and every } r \in [0,1)^n,$$

and the measure $\mu = f(r_1, \dots, r_n) \prod_{l=1}^n (1 - r_l)^{\alpha_l} \eta_n$ *is finite, then it fulfills the assumptions of Theorem 3.*

Proof. Let $K_l \in \mathbb{N}$ be such that

$$\sup_{k > K_l} \frac{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k+1}\right)}{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k}\right)} < \left(\frac{C_l + m_l}{2}\right)^{\alpha_l+1} \qquad \text{if } f_{l,r} \text{ is increasing,}$$

or

$$\sup_{k > K_l} \frac{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k}\right)}{f_{l,r}\left(1 - \left(\frac{1}{m_l}\right)^{k+1}\right)} < C + 1 \qquad \text{if } f_{l,r} \text{ is decreasing.}$$

It is clear that the sequences $a_{k,l} = \frac{1}{m_l^{K_l k}}$ for $l = 1, \dots, n$ fulfill the condition (i) of Theorem 3. Let $A_k = [1 - a_{k_1-1,1}, 1 - a_{k_1,1}) \times \dots \times [1 - a_{k_n-1,n}, 1 - a_{k_n,n})$ for every $k = (k_1, \dots, k_n) \in \mathbb{N}^n$. Then

$$\int_{A_k} \prod_{l=1}^n (1 - r_l)^{\alpha_l} \, d\eta_n(r) = \prod_{l=1}^n \frac{1 - \left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l}}{(\alpha_l + 1) m_l^{(\alpha_l+1)(K_l(k_l-1))}}.$$

Let $L = \{l : f_{l,r} \text{ is increasing for every } r \in [0,1)^n\}$ and $L' = \{1, \dots, n\} \setminus L$. Let $\mathbf{2} = (2, \dots, 2)$. For every $k \in \mathbb{N}^n \setminus \{\mathbf{1}\}$ we have

$$f(Q_k) \prod_{l=1}^n \frac{1 - \left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l}}{(\alpha_l + 1) m_l^{(\alpha_l+1)(K_l(k_l-1))}} \leqslant \mu(A_k) \leqslant f(R_k) \prod_{l=1}^n \frac{1 - \left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l}}{(\alpha_l + 1) m_l^{(\alpha_l+1)(K_l(k_l-1))}},$$

where $R_k = (R_{k,1}, \dots, R_{k,n})$ and $Q_k = (Q_{k,1}, \dots, Q_{k,n})$ for

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$$R_{k,l}=\begin{cases}1-\left(\frac{1}{m_l}\right)^{K_lk_l} & \text{if } l\in L,\\ 1-\left(\frac{1}{m_l}\right)^{K_l(k_l-1)} & \text{if } l\in L',\end{cases}\qquad Q_{k,l}=\begin{cases}1-\left(\frac{1}{m_l}\right)^{K_l(k_l-1)} & \text{if } l\in L,\\ 1-\left(\frac{1}{m_l}\right)^{K_lk_l} & \text{if } l\in L'.\end{cases}$$

Hence, for every $p=(p_1,\dots,p_n)\in\mathbb{N}^n\setminus\{\mathbf{1}\}$, we have

$$\frac{\mu(P_{(a_{p_1-1,1},\dots,a_{p_n-1,n})})}{\mu(A_{p+\mathbf{2}})}\leqslant\frac{\sum_{k\in\mathbb{N}^n,k\geqslant p}f(R_k)\prod_{l=1}^n\left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l(k_l-1)}}{f(Q_{p+\mathbf{2}})\prod_{l=1}^n\left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l(p_l+1)}}$$
$$=\sum_{k\in(\mathbb{N}\cup\{0,-1,-2\})^n}\frac{f(R_{p+\mathbf{2}+k})}{f(Q_{p+\mathbf{2}})}\prod_{l=1}^n\left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_lk_l}.$$

For every $\sigma=(\sigma_1,\dots,\sigma_n)\in\{0,1\}^n$, let $D_\sigma=B_1^{\sigma_1}\times\cdots\times B_n^{\sigma_n}$, where $B_l^0=\{-1,-2\}$ and $B_l^1=\mathbb{N}\cup\{0\}$. Let $L'_{\sigma,0}=\{l:\sigma_l=0\}\cap L'$, $L'_{\sigma,1}=\{l:\sigma_l=1\}\cap L'$, $L_{\sigma,0}=\{l:\sigma_l=0\}\cap L$ and $L_{\sigma,1}=\{l:\sigma_l=1\}\cap L$. For every $k\in D_\sigma$, we have

$$\frac{f(R_{p+\mathbf{2}+k})}{f(Q_{p+\mathbf{2}})}\leqslant\prod_{l\in L'_{\sigma,0}}(C+1)^{K_l(-k_l+1)}\prod_{l\in\{l:k_l=0\}\cap L'_{\sigma,1}}(C+1)^{K_l}\prod_{l\in L_{\sigma,1}}\left(\frac{C_l+m_l}{2}\right)^{(\alpha_l+1)K_l(k_l+1)}.$$

Hence

$$\frac{\mu(P_{(a_{p_1-1,1},\dots,a_{p_n-1,n})})}{\mu(A_{p+\mathbf{2}})}\leqslant\sum_{\sigma\in\{0,1\}^n}\sum_{k\in D_\sigma}\prod_{l\in L'_{\sigma,0}}m_l^{2(\alpha_l+1)K_l}(C+1)^{3K_l}\prod_{l\in L'_{\sigma,1}}\frac{(C+1)^{K_l}}{m_l^{(\alpha_l+1)K_lk_l}}$$
$$\cdot\prod_{l\in L_{\sigma,0}}m_l^{2(\alpha_l+1)K_l}\prod_{l\in L_{\sigma,1}}\left(\frac{C_l+m_l}{2}\right)^{(\alpha_l+1)K_l}\left(\frac{C_l+m_l}{2m_l}\right)^{(\alpha_l+1)K_lk_l}$$
$$\leqslant\sum_{\sigma\in\{0,1\}^n}2^n\prod_{l=1}^n(C+1)^{3K_l}m_l^{2(\alpha_l+1)K_l}\prod_{l\in L'_{\sigma,1}}\frac{1}{1-\left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l}}\prod_{l\in L_{\sigma,1}}\frac{\left(\frac{C_l+m_l}{2}\right)^{(\alpha_l+1)K_l}}{1-\left(\frac{C_l+m_l}{2m_l}\right)^{(\alpha_l+1)K_l}}$$
$$\leqslant 4^n\prod_{l=1}^n(C+1)^{3K_l}\prod_{l\in L'}\frac{m_l^{2(\alpha_l+1)K_l}}{1-\left(\frac{1}{m_l}\right)^{(\alpha_l+1)K_l}}\prod_{l\in L}\frac{(m_l^2(C_l+m_l))^{(\alpha_l+1)K_l}}{1-\left(\frac{C_l+m_l}{2m_l}\right)^{(\alpha_l+1)K_l}}.$$

Thus we showed that μ fulfills the conditions of Theorem 3 for $b_{m,l}=d=1$, for every $m\in\mathbb{N}$ and $l=1,\dots,n$.

For $n=1$, by Proposition 6 and the remark after Corollary 5, every measure of the form $(1-r^2)^\alpha r\eta_1$, $\chi_{[\frac{1}{2},1]}|\ln(1-r)|^\gamma(1-r)^\alpha\eta_1$, $\chi_{[\frac{1}{2},1]}\ln^\gamma(|\ln(1-r)|)(1-r)^\alpha\eta_1$ for $\alpha>-1$ and $\gamma\in\mathbb{R}$ fulfills the conditions of Theorem 3. It is clear that every measure which fulfills the conditions of Theorem 3 also fulfills the conditions of Theorem 4. The measures $\chi_{[\frac{1}{2},1]}|\ln(1-r)|^\gamma(1-r)^{-1}\eta_1$ for $\gamma<-1$, $\sum_{n=1}^\infty\frac{1}{n(n+1)}\delta_{1-\frac{1}{2^n}}$, (where δ_r is the Dirac measure concentrated at r), and $\exp\left(-\frac{1}{1-r}\right)(1-r)^{-2-k}\eta_1$ for $k\in\mathbb{N}$, fulfill the assumptions of Theorem 4 (in the first and second example for $a_{m,1}=\frac{1}{2^n}$, $d=b_{m,1}=1$ and in the third for $a_n=\frac{1}{n}$, $d=b_{m,1}=1$) but does not fulfill the conditions of Theorem 3. W. Lusky **[13]** considered the following conditions:

(∗) $\sup_n\frac{\mu([1-2^{-n},1])}{\mu([1-2^{-(n+1)},1])}<\infty$,

(∗∗) $\inf_k\limsup_n\frac{\mu([1-2^{-(n+k)},1])}{\mu([1-2^{-n},1])}<1$, for positive Borel measures on [0, 1) with $1 \in \operatorname{supp}(\mu)$, to get the Banach description of the generalized Bergman spaces on the disk. The last example does not satisfy the condition $(*)$. The measure $\sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{1-\frac{1}{n!}}$ fulfills the condition $(*)$ but does not fulfill the conditions of Theorem 4. The measure $\sum_{n=1}^{\infty} \frac{1}{n!}\delta_{1-\frac{1}{2^n}}$ satisfies the condition $(**)$ but does not satisfy the conditions of Theorem 3. On the other hand, if μ fulfills the conditions $(*)$ and $(**)$, then it fulfills the assumptions of Theorem 3. The case $n > 1$ is much more complicated. It is clear that the measures $\prod_{l=1}^{n}(1-r_l^2)^{\alpha_l} r_l \eta_n$ and $\prod_{l=1}^{n} |\ln(1-r_l)|^{\gamma_l}(1-r_l)^{\alpha_l} r_l \eta_n$ for $\alpha_l > -1$ and $\gamma_l > -1$ fulfill the conditions of Propositions 6 and consequently the conditions of Theorem 3. We show in Example 11 that also the measure $(1-\max_l^2\{r_l\})^{\alpha}\left(\prod_{l=1}^{n} r_l\right)\eta_n$ for $\alpha > -1$ satisfies the conditions of Theorem 3.

3. Necessary conditions for Carleson measures on $B^p(\mu)$ and $b^p(\mu)$. In the sequel we shall need the following fact; (see [**4**, p. 27] or [**20**, Prop 1.4.10]).

FACT 7. *For every $\beta > 1$ there exists $C_\beta > 0$ such that*

$$\int_{\mathbb{T}} \frac{d\lambda(t)}{|R-rt|^{\beta}} \leqslant \frac{C_\beta}{R(R-r)^{\beta-1}}, \qquad \text{for every } R > r > 0.$$

THEOREM 8. *Let $0 < p \leqslant q < \infty$. Let μ be a positive finite Borel measure on $[0,1)^n$ such that $\mathbf{1} \in \operatorname{supp}(\mu)$. If there exist $\beta = (\beta_1, \dots, \beta_n) \in (1,\infty)^n$ and $C_1 > 0$ such that, for every $\sigma = (\sigma_1, \dots, \sigma_n) \in \{0,1\}^n$ and $h = (h_1, \dots h_n) \in (0,1]^n$, we have*

$$\int_{[0,1)^n} \frac{\mu(P_{(r_1^{\sigma_1},\dots,r_n^{\sigma_n})})}{\prod_{l=1}^{n}(h_l+r_l)^{\sigma_l\beta_l}}\, d\eta_n(r) \leqslant C_1 \mu(P_h) \prod_{l=1}^{n} \frac{1}{h_l^{\beta_l-1}},$$

then

(a) *for every positive finite Borel measure ω on $\mathbb{D}^n$ such that $B^p(\mu)$ is naturally embedded in $L^q(\omega)$ there exists $C > 0$ such that*

$$\omega(S_{h,t}) \leqslant C\Big(\mu(P_h)\prod_{l=1}^{n} h_l\Big)^{\frac{q}{p}},$$

for every $h = (h_1, \dots h_n) \in (0,1]^n$ and every $t \in \mathbb{T}^n$;

(b) *for every positive finite Borel measure ω on $\mathbb{D}$ such that the natural embedding $I: B^p(\mu) \to L^q(\omega)$ is compact, we have*

$$\lim_{\rho\to 0^+} \sup\left\{\frac{\omega(S_{h,t})}{(\mu(P_h)\prod_{l=1}^{n} h_l)^{\frac{q}{p}}} : h = (h_1, \dots h_n) \in (0,1]^n,\ \min_l\{h_l\} \leqslant \rho,\ t \in \mathbb{T}^n\right\} = 0.$$

Proof. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$, $h = (h_1, \dots h_n) \in (0,1]^n$ and $t = (t_1, \dots, t_n) \in \mathbb{T}^n$ let $g_{h,t,\alpha} : \mathbb{D}^n \to \mathbb{C}$ be given by

$$g_{h,t,\alpha}(z_1, \dots, z_n) = \prod_{l=1}^{n} \frac{1}{(1+h_l - z_l\bar{t}_l)^{\frac{\alpha_l}{p}}}.$$

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Then, for every $z \in S_{h,t}$, we have

$$|g_{h,t,\beta}(z)|^p \geqslant \prod_{l=1}^{n} \frac{1}{|1+h_l-(1-h_l)e^{-ih_l}|^{\beta_l}} \geqslant \prod_{l=1}^{n} \frac{1}{(3h_l)^{\beta_l}}.$$

Let C_2 be the norm of the embedding $I : B^p(\mu) \to L^q(\omega)$. For any $0 \neq g \in B^p(\mu)$

$$1 = \int_{\mathbb{D}^n} \frac{|g|^p}{\|g\|^p}\, d\mu \otimes \lambda \geqslant \frac{1}{C_2^p}\Big(\int_{\mathbb{D}^n} \frac{|g|^q}{\|g\|^q}\, d\omega\Big)^{\frac{p}{q}} \geqslant \frac{1}{C_2^p\|g\|^p} \inf_{z\in S_{h,t}} |g(z)|^p \big(\omega(S_{h,t})\big)^{\frac{p}{q}}.$$

Applying Fact 7 we get

$$\begin{aligned}
\|g_{h,t,\beta}\|^p &= \int_{[0,1)^n}\int_{\mathbb{T}^n} \prod_{l=1}^{n} \frac{1}{|1+h_l-r_l s_l|^{\beta_l}}\, d\lambda_n(s) d\mu(r) \leqslant \int_{[0,1)^n} \prod_{l=1}^{n} \frac{C_{\beta_l}}{(1+h_l-r_l)^{\beta_l-1}}\, d\mu(r) \\
&\leqslant \int_{[0,1)^n} \Big(\prod_{l=1}^{n} C_{\beta_l}\Big(\int_0^{r_l} \frac{\beta_l-1\, dx}{(1+h_l-x)^{\beta_l}} + \frac{1}{(1+h_l)^{\beta_l-1}}\Big)\Big)\, d\mu(r) \\
&\leqslant \Big(\prod_{l=1}^{n} C_{\beta_l}\Big) \int_{[0,1)^n} \sum_{\sigma\in\{0,1\}^n} \prod_{l=1}^{n}\Big(\Big(\frac{1}{(1+h_l)^{\beta_l-1}}\Big)^{1-\sigma_l}\Big(\int_{1-r_l}^{1} \frac{\beta_l-1\, dx}{(h_l+x)^{\beta_l}}\Big)^{\sigma_l}\Big)\, d\mu(r) \\
&\leqslant \Big(\prod_{l=1}^{n} C_{\beta_l}\Big) \sum_{\sigma\in\{0,1\}^n} \int_{[0,1)^n}\int_{P_{(r_1^{\sigma_1},\ldots,r_n^{\sigma_n})}} \prod_{l=1}^{n}\Big(\frac{\beta_l-1}{(h_l+x_l)^{\beta_l}}\Big)^{\sigma_l} d\eta_n(x) d\mu(r) \\
&\leqslant \Big(\prod_{l=1}^{n} C_{\beta_l}\Big) \sum_{\sigma\in\{0,1\}^n} \int_{[0,1)^n} \mu(P_{(r_1^{\sigma_1},\ldots,r_n^{\sigma_n})}) \prod_{l=1}^{n}\Big(\frac{\beta_l-1}{(h_l+r_l)^{\beta_l}}\Big)^{\sigma_l} d\eta_n(r) \\
&\leqslant \Big(\prod_{l=1}^{n} C_{\beta_l}\Big) \sum_{\sigma\in\{0,1\}^n} C_1\mu(P_h) \prod_{l=1}^{n}\Big(\frac{(\beta_l-1)^{\sigma_l}}{h_l^{\beta_l-1}}\Big) \leqslant C_1\, \mu(P_h) \prod_{l=1}^{n} \frac{\beta_l C_{\beta_l}}{h_l^{\beta_l-1}},
\end{aligned}$$

for every $h \in (0, 1]^n$. Hence

$$\omega(S_{h,t}) \leqslant \Big(C_2^p C_1\, \mu(P_h) \prod_{l=1}^{n} 3^{\beta_l}\beta_l C_{\beta_l} h_l\Big)^{\frac{q}{p}},$$

for every $h \in (0, 1]^n$, which completes the proof in the case (a).

(b) Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}_+^n$. For every $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{0, 1\}^n$ and $h = (h_1, \ldots, h_n) \in (0, 1]^n$, we have

$$\int_{[0,1)^n} \frac{\mu(P_{(r_1^{\sigma_1},\ldots,r_n^{\sigma_n})})}{\prod_{l=1}^{n}(h_l+r_l)^{\sigma_l(\beta_l+\gamma_l)}}\, d\eta_n(r) \leqslant C_1\mu(P_h) \prod_{l=1}^{n} \frac{1}{h_l^{\beta_l+\gamma_l-1}}.$$

The sets $\{(\frac{\prod_{l=1}^{n} h_l^{\beta_l-1}}{\mu(P_h)})^{\frac{1}{p}} g_{h,t,\beta} : h \in (0, 1]^n, t \in \mathbb{T}^n\}$ and $\{(\frac{\prod_{l=1}^{n} h_l^{\beta_l+\gamma_l-1}}{\mu(P_h)})^{\frac{1}{p}} g_{h,t,\beta+\gamma} : h \in (0, 1]^n$, $t \in \mathbb{T}^n\}$ are bounded in $B^p(\mu)$, in view of estimations above. The functions $(\prod_{l=1}^{n} h_l^{\frac{\gamma_l}{p}}) g_{h,t,\gamma}$ converge pointwise to zero if $\prod_{l=1}^{n} h_l$ converges to zero. Therefore the sequence $((\frac{\prod_{l=1}^{n} h_{k,l}^{\beta_l+\gamma_l-1}}{\mu(P_h)})^{\frac{1}{p}} g_{h_k,t_k,\beta+\gamma})$ is bounded and converges pointwise to zero, for any sequences $(h_k) = ((h_{k,1}, \ldots, h_{k,n})) \subset (0, 1]^n$ and $(t_k) \subset \mathbb{T}^n$ such that $\lim_k \min_l\{h_{k,l}\} = 0$. Since I is compact,

$$\lim_{\rho\to 0^+}\sup\Big\{\Big\| I\Big(\Big(\prod_{l=1}^{n}\frac{h_l^{\beta_l+\gamma_l-1}}{\mu(P_h)}\Big)^{\frac{1}{p}}g_{h,t,\beta+\gamma}\Big)\Big\| : h=(h_1,\dots,h_n)\in(0,1]^n,\ \min_l\{h_l\}\leqslant\rho\Big\}=0.$$

Hence, for every $\varepsilon>0$ there exists $\delta\in(0,1)$ such that

$$\|g_{h,t,\beta+\gamma}\|_{L^q(\omega)}\leqslant\varepsilon\Big(\mu(P_h)\prod_{l=1}^{n}\frac{1}{h_l^{\beta_l+\gamma_l-1}}\Big)^{\frac{1}{p}},\qquad \text{for every } h\in(0,1]^n \text{ with } \min_l\{h_l\}\leqslant\delta.$$

For every $z\in S_{h,t}$, we have

$$|g_{h,t,\beta+\gamma}(z)|^p\geqslant\prod_{l=1}^{n}\frac{1}{|1+h_l-(1-h_l)e^{-ih_l}|^{\beta_l+\gamma_l}}\geqslant\prod_{l=1}^{n}\frac{1}{(3h_l)^{\beta_l+\gamma_l}}.$$

For every $0\neq g\in L^q(\omega)$, we have

$$1=\int_{\mathbb{D}^n}\frac{|g|^q}{\|g\|^q_{L^q(\omega)}}\,d\omega\geqslant\Big(\frac{\inf_{z\in S_{h,t}}|g(z)|}{\|g\|_{L^q(\omega)}}\Big)^q\omega(S_{h,t}).$$

Applying estimations above we get

$$\sup\Big\{\frac{\omega(S_{h,t})}{(\mu(P_h)\prod_{l=1}^{n}h_l)^{\frac{q}{p}}} : h\in(0,1]^n,\ \min_l\{h_l\}\leqslant\delta,\ t\in\mathbb{T}^n\Big\}\leqslant\varepsilon^q\prod_{l=1}^{n}3^{\frac{(\beta_l+\gamma_l)q}{p}}.$$

Remark. Since $B^p(\mu)$ is a subspace of $b^p(\mu)$ the theorem above remains valid also for spaces $b^p(\mu)$, for $1\leqslant p<\infty$.

The next result shows that a measure $\mu=f(r_1,\dots,r_n)\prod_{l=1}^{n}(1-r_l)^{\alpha_l}\eta_n$, for $\alpha_l>-1$, satisfies the conditions of Theorem 8 if the function $f:[0,1)^n\to\mathbb{R}_+$ is increasing in each variable separately. For this purpose we shall need the following result.

Fact 9. *Let $\alpha>-1$. If $g:[0,1)\to\mathbb{R}_+$ is an increasing function, then*

$$\frac{1}{s^{\alpha+1}}\int_{1-s}^{1}g(x)(1-x)^\alpha\,dx\leqslant\frac{1}{(\gamma s)^{\alpha+1}}\int_{1-\gamma s}^{1}g(x)(1-x)^\alpha\,dx,$$

for every $\gamma\in(0,1]$ and $s\in(0,1]$.

Proof.

$$\begin{aligned}\frac{1}{s^{\alpha+1}}\int_{1-s}^{1}g(x)(1-x)^\alpha\,dx&=\frac{1}{s^{\alpha+1}}\Big(\int_{1-s}^{1-\gamma s}+\int_{1-\gamma s}^{1}\Big)g(x)(1-x)^\alpha\,dx\\&\leqslant\Big(\frac{1-\gamma^{\alpha+1}}{\alpha+1}\Big)g(1-\gamma s)+\frac{\gamma^{\alpha+1}}{(\gamma s)^{\alpha+1}}\int_{1-\gamma s}^{1}g(x)(1-x)^\alpha\,dx\\&\leqslant\Big(\frac{1-\gamma^{\alpha+1}}{(\gamma s)^{\alpha+1}}+\frac{\gamma^{\alpha+1}}{(\gamma s)^{\alpha+1}}\Big)\int_{1-\gamma s}^{1}g(x)(1-x)^\alpha\,dx\\&\leqslant\frac{1}{(\gamma s)^{\alpha+1}}\int_{1-\gamma s}^{1}g(x)(1-x)^\alpha\,dx.\end{aligned}$$

Proposition 10. *Let* $f:[0,1)^n \to \mathbb{R}_+$ *and* $\alpha=(\alpha_1,\dots,\alpha_n)\in(-1,\infty)^n$. *If the function* $f_{l,r}(x)=f(r_1,\dots,r_l-1,x,r_l+1,\dots,r_n)$ *on* $(0,1)$ *is increasing, for every* $l=1,\dots,n$, *and* $r\in[0,1)^n$, *and the measure* $\mu=f(r_1,\dots,r_n)\prod_{l=1}^n(1-r_l)^{\alpha_l}\eta_n$ *is finite, then it fulfills the assumptions of Theorem 8.*

Proof. Let $\beta=\alpha+(3,\dots,3)$. Applying Fact 9 for each variable separately we get

$$\frac{1}{\prod_{l=1}^n r_l^{\alpha_l+1}}\int_{P_{(r_1,\dots,r_n)}} f\,d\nu_\alpha \leqslant \frac{1}{\prod_{l=1}^n(\gamma_l r_l)^{\alpha_l+1}}\int_{P_{(\gamma_1 r_1,\dots,\gamma_n r_n)}} f\,d\nu_\alpha \qquad (*)$$

for every $\gamma=(\gamma_1,\dots,\gamma_n)\in(0,1]^n$ and $r=(r_1,\dots,r_n)\in(0,1]^n$, where $\nu_\alpha=\left(\prod_{l=1}^n(1-r_l)^{\alpha_l}\right)\eta_n$. Therefore the function $(r_1,\dots,r_n)\to\frac{\mu(P_{(r_1,\dots,r_n)})}{\prod_{l=1}^n r_l^{\alpha_l+1}}$ on $(0,1]^n$ is decreasing in each variable separately. Let $h=(h_1,\dots,h_n)\in(0,1]^n$ and $\sigma=(\sigma_1,\dots,\sigma_n)\in\{0,1\}^n$. For every $\tau=(\tau_1,\dots,\tau_n)\in\{0,1\}^n$ with $\tau\leqslant\sigma$ let $D_\tau=B_1^{\tau_1}\times\dots\times B_n^{\tau_n}$, where $B_l^0=[h_l\sigma_l,1)$ and $B_l^1=[0,h_l)$. Let $L_\tau=\{l:\tau_l=0,\ \sigma_l=1\}$ and $M_\tau=\{l:\tau_l=1,\sigma_l=1\}$. Then for every $(r_1,\dots,r_n)\in D_\tau$, we have

$$\frac{\mu(P_{(r_1^{\sigma_1},\dots,r_n^{\sigma_n})})}{\prod_{l\in L_\tau} r_l^{\alpha_l+1}}\leqslant\frac{\mu(P_{(R_1,\dots,R_n)})}{\prod_{l\in L_\tau} h_l^{\alpha_l+1}}\leqslant\frac{\mu(P_{(h_1^{\sigma_1},\dots,h_n^{\sigma_n})})}{\prod_{l\in L_\tau} h_l^{\alpha_l+1}},$$

where

$$R_l=\begin{cases} r_l & \text{if } \sigma_l=\tau_l=1,\\ h_l & \text{if } \sigma_l=1,\ \tau_l=0,\\ 1 & \text{if } \sigma_l=0.\end{cases}$$

Hence

$$\begin{aligned}
&\int_{[0,1)^n}\frac{\mu(P_{(r_1^{\sigma_1},\dots,r_n^{\sigma_n})})}{\prod_{l=1}^n(h_l+r_l)^{\sigma_l(\alpha_l+3)}}\,d\eta_n(r)\\
&\quad=\sum_{\tau\in\{0,1\}^n,\tau\leqslant\sigma}\int_{D_\tau}\frac{\mu(P_{(r_1^{\sigma_1},\dots,r_n^{\sigma_n})})\prod_{l\in L_\tau} r_l^{\alpha_l+1}}{\prod_{l\in L_\tau} r_l^{\alpha_l+1}\prod_{l=1}^n(h_l+r_l)^{\sigma_l(\alpha_l+3)}}\,d\eta_n(r)\\
&\quad\leqslant\sum_{\tau\in\{0,1\}^n,\tau\leqslant\sigma}\frac{\mu(P_{(h_1^{\sigma_1},\dots,h_n^{\sigma_n})})}{\prod_{l\in L_\tau} h_l^{\alpha_l+1}}\int_{D_\tau}\frac{d\eta_n(r)}{\prod_{l\in L_\tau}(h_l+r_l)^2\prod_{l\in M_\tau}(h_l+r_l)^{\alpha_l+3}}\\
&\quad\leqslant\sum_{\tau\in\{0,1\}^n,\tau\leqslant\sigma}\frac{\mu(P_{(h_1^{\sigma_1},\dots,h_n^{\sigma_n})})}{\prod_{l\in L_\tau} h_l^{\alpha_l+1}}\prod_{l\in L_\tau}\int_{h_l}^1\frac{dx}{(h_l+x)^2}\prod_{l\in M_\tau}\int_0^{h_l}\frac{dx}{(h_l+x)^{\alpha_l+3}}\\
&\quad\leqslant\sum_{\tau\in\{0,1\}^n\tau\leqslant\sigma}\frac{\mu(P_{(h_1^{\sigma_1},\dots,h_n^{\sigma_n})})}{\prod_{l\in L_\tau} h_l^{\alpha_l+1+1}\prod_{l\in M_\tau} h_l^{\alpha_l+2}}.
\end{aligned}$$

Applying $(*)$ once again we get

$$\mu(P_{(h_1^{\sigma_1},\dots,h_n^{\sigma_n})})\leqslant\frac{\prod_{l=1}^n h_l^{\sigma_l(\alpha_l+1)}}{\prod_{l=1}^n h_l^{\alpha_l+1}}\mu(P_h).$$

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Hence

$$\int_{[0,1)^n} \frac{\mu(P_{(r_1^{\sigma_1},\dots,r_n^{\sigma_n})})}{\prod_{l=1}^n (h_l+r_l)^{\sigma_l(\alpha_l+3)}}\,d\eta_n(r) \leqslant \sum_{\tau\in\{0,1\}^n,\tau\leqslant\sigma} \frac{\mu(P_h)}{\prod_{l\in L_\tau\cup M_\tau} h_l \prod_{l=1}^n h_l^{\alpha_l+1}}$$
$$\leqslant 2^n \mu(P_h)\prod_{l=1}^n \frac{1}{h_l^{\alpha_l+2}}.$$

It is clear that the values of a measure μ on any compact subset of $[0,1)^n$ do not have any influence on this whether or not μ fulfills the assumptions of Theorem 8. For $n=1$, by Proposition 10 every measure of the form $(1-r^2)^\alpha r\eta_1$, $\chi_{[\frac12,1)}|\ln(1-r)|^\gamma(1-r)^\alpha\eta_1$, $\chi_{[\frac12,1)}\ln^\gamma(|\ln(1-r)|)(1-r)^\alpha\eta_1$ for $\alpha>-1$ and $\gamma\in\mathbb{R}$, $\chi_{[\frac12,1)}|\ln(1-r)|^\gamma(1-r)^{-1}\eta_1$ for $\gamma<-1$ fulfills the conditions of Theorem 8. The measure $\exp\big(-\frac{1}{1-r}\big)(1-r)^{-2}\eta_1$ does not satisfy the condition for $\sigma=0$ of Theorem 8. For $n>1$ according to Proposition 10 the measures $\prod_{l=1}^n(1-r_l^2)^{\alpha_l}r_l\eta_n$ and $\prod_{l=1}^n|\ln(1-r_l)|^{\gamma_l}(1-r_l)^{\alpha_l}r_l\eta_n$, for $\alpha_l>-1$ and $\gamma_l>0$, fulfill the conditions of Theorem 8. We show below that also the measure $(1-\max_l^2\{r_l\})^\alpha\big(\prod_{l=1}^n r_l\big)\eta_n$, for $\alpha>-1$, satisfies the conditions of Theorem 8.

Example 11. The measure $\mu=(1-\max_l^2\{r_l\})^\alpha\big(\prod_{l=1}^n r_l\big)\eta_n$ for $\alpha>-1$ satisfies the conditions of Theorems 3 and 8. In order to show this we apply the properties of the measure $\mu_1=(1-\max_l^2\{r_l\})^\alpha\eta_n$. It is clear that μ_1 is finite. Let $f:[0,1)^n\to\mathbb{R}_+$ and $f_{l,r}:(0,1)\to\mathbb{R}$ be given by $f(r_1,\dots,r_n)=(1-\max_l^2\{r_l\})^\alpha$ and $f_{l,r}(x)=f(r_1,\dots,r_{l-1},x,r_{l+1},\dots,r_n)$, respectively. Then, for every $r\in[0,1)^n$ and $l=1,\dots,n$, we have

$$f_{l,r}(x)=\begin{cases}(1-\max^2\{r_j:j\neq l\})^\alpha & \text{for } x\leqslant\max\{r_j:j\neq l\},\\ (1-x^2)^\alpha & \text{for } x\geqslant\max\{r_j:j\neq l\}.\end{cases}$$

If $-1<\alpha\leqslant 0$, the functions $f_{l,r}$ are increasing and

$$\limsup_k \frac{f_{l,r}\big(1-\big(\frac12\big)^{k+1}\big)}{f_{l,r}\big(1-\big(\frac12\big)^{k}\big)}=\limsup_k \frac{\big(1-\big(1-\big(\frac12\big)^{k+1}\big)^2\big)^\alpha}{\big(1-\big(1-\big(\frac12\big)^{k}\big)^2\big)^\alpha}=2^{-\alpha}.$$

If $\alpha\geqslant 0$, the functions $f_{r,l}$ are decreasing and

$$\limsup_k \frac{f_{l,r}\big(1-\big(\frac12\big)^{k}\big)}{f_{l,r}\big(1-\big(\frac12\big)^{k+1}\big)}=\limsup_k \frac{\big(1-\big(1-\big(\frac12\big)^{k}\big)^2\big)^\alpha}{\big(1-\big(1-\big(\frac12\big)^{k+1}\big)^2\big)^\alpha}=2^{\alpha}.$$

According to Propositions 6 and 10 the measure μ_1 satisfies the conditions of Theorems 3 and 8 for $\alpha\leqslant 0$ and the conditions of Theorem 3 for $\alpha\geqslant 0$. Let $g:[0,1)^n\to\mathbb{R}_+$ be given by $g(r_1,\dots,r_n)=(1-\max_l^2\{r_l\})^\alpha\prod_{l=1}^n(1-r_l)^{-\alpha}$. For every $r\in[0,1)^n$, $l=1,\dots,n$ and $\alpha\geqslant 0$ the function

$$g_{l,r}(x)=\begin{cases}(1-x)^{-\alpha}(1-\max^2\{r_j:j\neq l\})^\alpha\prod_{j\neq l}(1-r_j)^{-\alpha} & \text{if } x\leqslant\max\{r_j:j\neq l\},\\ (1+x)^\alpha\prod_{j\neq l}(1-r_j)^{-\alpha} & \text{if } x\geqslant\max\{r_j:j\neq l\},\end{cases}$$

is increasing. According to Proposition 10 the measure $\mu_1=g(r_1,\dots,r_n)\prod_{l=1}^n(1-r_l)^\alpha\eta_n$ satisfies the conditions of Theorem 8 for $\alpha\geqslant 0$. It is easy to check that

$$1 \geqslant \frac{\mu(A_k)}{\mu_1(A_k)} \geqslant \frac{1}{2^{\alpha+n}},$$

for every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n \setminus \{\mathbf{1}\}$, where $A_k = [1 - 2^{1-k_1}, 1 - 2^{-k_1}) \times \ldots \times [1 - 2^{1-k_n}, 1 - 2^{-k_n})$. Hence there exists $C > 0$ such that $\mu_1(P_h) \geqslant \mu(P_h) \geqslant C\mu_1(P_h)$, for every $h \in (0, 1]^n$. Therefore μ satisfies the conditions of Theorems 3 and 8 for $\alpha > -1$.

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