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LANDAU'S THEOREM FOR SOLUTIONS OF THE $\overline{\partial}$-EQUATION IN DIRICHLET-TYPE SPACES

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Abstract

The main aim of this article is to establish analogues of Landau's theorem for solutions to the $\overline{\partial}$-equation in Dirichlet-type spaces.

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1. Introduction and main results

Let $\mathbb{C} \cong \mathbb{R}^2$ be the complex plane. For $a \in \mathbb{C}$ and $r > 0$, set $\mathbb{D}(a, r) := \{z : |z - a| < r\}$ and $\mathbb{D}_r := \mathbb{D}(0, r)$ so that $\mathbb{D} := \mathbb{D}_1$ is the open unit disk in $\mathbb{C}$. For a real 2×2 matrix A, we use the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. Set $z = x + iy \in \mathbb{C}$. The formal derivative, namely, the Jacobian matrix, D_f, of the complex-valued function $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x \; u_y \\ v_x \; v_y \end{pmatrix},$$

so that $\|D_f\| = |f_z| + |f_{\overline{z}}|$ and $l(D_f) = ||f_z| - |f_{\overline{z}}||$, where

$$f_z = \frac{\partial f}{\partial z} = \partial_z f = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \overline{\partial}_z f = \frac{1}{2}(f_x + i f_y)$$

are the usual partial derivatives. We write

$$J_f := \det D_f = |f_z|^2 - |f_{\overline{z}}|^2$$

to denote the Jacobian of f and $\Delta f = 4 f_{z\overline{z}}$ for the Laplacian of a C^2-function f.

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A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is nonincreasing for $t > 0$ (compare with [15]). Given a subset Ω of $\mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that, for all $z, w \in \Omega$,

$$|f(z) - f(w)| \le C\omega(|z - w|).$$

Throughout the article, we denote by $C^m(\mathbb{D})$ the set of all complex-valued m-times continuously differentiable functions from $\mathbb{D}$ into $\mathbb{C}$, where $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, $C(\mathbb{D}) := C^0(\mathbb{D})$ is the set of all continuous functions on $\mathbb{D}$.

We are primarily interested in functions $f \in C^1(\mathbb{D})$ which satisfy the $\overline{\partial}$*-equation*

$$f_{\overline{z}} = u \tag{1.1}$$

for some $u \in C(\mathbb{D})$. *Hörmander's* solutions to (1.1) in $\mathbb{C}$ are discussed in [11, 12]. In particular, if the function $u \in C(\mathbb{D})$ in (1.1) is real valued, then we call f a *u-gradient mapping*. If the function u in (1.1) is real valued and the solution f is sense preserving, then we call f a *sense-preserving u-gradient mapping* (compare with [1]).

We use $\mathcal{F}$ to denote the set of all analytic functions f defined in $\mathbb{D}$ satisfying the standard normalisation $f(0) = f'(0) - 1 = 0$. In [13], Landau proved that there is a constant $\rho > 0$, independent of $f \in \mathcal{F}$, such that $f(\mathbb{D})$ contains a disk of radius ρ. For $f \in \mathcal{F}$, let L_f be the supremum of the set of positive numbers r such that $f(\mathbb{D})$ contains a disk of radius r. Then we call $\inf_{f\in\mathcal{F}} L_f$ the Landau–Bloch constant. One of the long-standing open problems in geometric function theory is to determine the precise value of the Landau–Bloch constant. It has attracted much attention (see, for example, [2, 14, 18] and the references therein). The Landau theorem is an important tool in geometric function theory of one complex variable (compare with [3, 20]). Unfortunately, there is no analogue of Landau's theorem for general classes of functions (see [4, 18]). In order to obtain analogues of Landau's theorem for more general classes of functions, it is necessary to restrict the class of functions considered (compare with [4–7, 18]). The first aim of this paper is to extend the classical Landau theorem to the solutions of (1.1).

Let $\mathcal{F}_u^M$ denote the class of all complex-valued functions f satisfying (1.1) with $f(0) = J_f(0) - 1 = 0$, $u \in C(\mathbb{D})$ and $\sup_{z\in\mathbb{D}} |f(z)| < M$, where M is a positive constant.

Theorem 1.1. *For a given $u \in C(\mathbb{D})$, let $f \in \mathcal{F}_u^M$. Then there is a positive constant r_0 depending only on M and u such that $\mathbb{D}_{r_0} \subset f(\mathbb{D})$.*

Remark 1.2. Since the proof of Theorem 1.1 is based on convergence considerations of function families, it is not possible to give an explicit form of the constant r_0. Although Theorem 1.1 provides the existence of the Landau–Bloch-type constant for functions $f \in \mathcal{F}_u^M$, an explicit estimate for $\inf_{f\in\mathcal{F}_u^M} L_f(u)$ remains an open problem.

Let $\mathcal{B}_t$ be the Bloch space consisting of all complex-valued functions $f \in C^1(\mathbb{D})$ with

$$\sup_{z\in\mathbb{D}}\{(1 - |z|^2)(|f_z(z)| + |f_{\overline{z}}(z)|)\} < +\infty.$$

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Corollary 1.3. *For a given function $u \in C(\mathbb{D})$, let $f \in \mathcal{B}_t$ be a solution to* (1.1) *with $f(0) = J_f(0) - 1 = 0$. Then there is a positive constant s_0, depending only on u, such that $\mathbb{D}_{s_0} \subset f(\mathbb{D})$.*

Proof. For $z \in \mathbb{D}$, let $F(z) = \sqrt{2} f(\sqrt{2}z/2)$. Since F is bounded in $\mathbb{D}$, the result follows from Theorem 1.1. □

We use $\mathcal{D}^{\omega}_{\gamma_1,\gamma_2}(\mathbb{D})$ to denote the *weighted Dirichlet-type space* consisting of all $f \in C^1(\mathbb{D})$ with the norm

$$\|f\|_{\mathcal{D}^{\omega}_{\gamma_1,\gamma_2}} = |f(0)| + \int_{\mathbb{D}} \omega((d(z))^{\gamma_1})\|D_f(z)\|^{\gamma_2}\, d\sigma(z) < \infty,$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, ω is a majorant and $d\sigma$ is the normalised area measure in $\mathbb{D}$.

This Dirichlet-type energy integral has been investigated by Yamashita [19] and Chen *et al.* in a series of papers (see [8, 9]). As an analogue of [9, Theorem 4], we prove the following result.

Theorem 1.4. *Let $\beta > 0$ and suppose $2 \le \eta < (2 + \beta)(1 + \alpha) - 2\beta$ if $\alpha \in [0, 2)$, and $2 \le \eta < 2 + \beta$ if $\alpha \in [2, 4]$. For $\mu \in \mathbb{R}$ and $p \ge 2$, if $g \in \mathcal{D}^{\omega}_{\beta,\eta}(\mathbb{D}) \cap C^3(\mathbb{D})$ and g_z is a sense-preserving u-gradient mapping, then*

$$\int_{\mathbb{D}} (d(z))^{(p-1)(\nu-1)+\alpha\nu}\Delta(|g(z)|^p)\, d\sigma(z) < +\infty, \tag{1.2}$$

where $u = \mu|g_z|^{\alpha}$, $\nu = (2 + \beta)/\eta$ and ω is a majorant.

For $p \in (0, \infty]$, the *generalised Hardy space $H^p_{\mathcal{G}}(\mathbb{D})$* consists of all those functions $f : \mathbb{D} \to \mathbb{C}$ such that each f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and $\|f\|_p < \infty$, where

$$M^p_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \quad \text{and} \quad \|f\|_p = \begin{cases} \sup_{0<r<1} M_p(r, f) & \text{if } p \in (0, \infty), \\ \sup_{z\in\mathbb{D}} |f(z)| & \text{if } p = \infty. \end{cases}$$

The following result easily follows from Theorem 1.4 and [8, Theorem 1].

Corollary 1.5. *Suppose $\beta > 0$ and $2 \le \eta < (2 + \beta)(1 + \alpha) - 2\beta$ with $\alpha \in [0, 1)$. For $\mu \in \mathbb{R}$ and $p = (1 - \alpha)\nu/(\nu - 1) \ge 2$, if $g \in \mathcal{D}^{\omega}_{\beta,\eta}(\mathbb{D}) \cap C^3(\mathbb{D})$ and g_z is a sense-preserving u-gradient mapping, then $g \in H^p_{\mathcal{G}}(\mathbb{D})$, where $u = \mu|g_z|^{\alpha}$, $\nu = (2 + \beta)/\eta$ and ω is a majorant.*

The proofs of Theorems 1.1 and 1.4 will be presented in Section 2.

2. The proofs of the main results

Let $\mathbb{R}^n$ $(n \ge 2)$ denote the n-dimensional Euclidean space, where $n \in \{2, 3, \ldots\}$. Let $f : \overline{\Omega} \to \mathbb{R}^n$ be a differentiable mapping and let x be a regular value of f, where

$x \notin f(\partial\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. The degree $\deg(f, \Omega, x)$ is defined by the formula (compare with [16, 17])

$$\deg(f, \Omega, x) := \sum_{y \in f^{-1}(x) \cap \Omega} \operatorname{sign}(\det J_f(y)).$$

Lemma 2.1 [16, pages 125–129]. *The* $\deg(f, \Omega, x)$ *has the following properties.*

(I) *If* $x \in \mathbb{R}^n \backslash f(\partial\Omega)$ *and* $\deg(f, \overline{\Omega}, x) \neq 0$*, then there exists a point* $w \in \Omega$ *such that* $f(w) = x$.

(II) *If* D *is a domain with* $\overline{D} \subset \Omega$ *and* $x \in \mathbb{R}^n \backslash f(\partial D)$*, then* $\deg(f, D, \cdot)$ *is constant on each component of* $\mathbb{R}^n \backslash f(\partial D)$.

Lemma 2.2 [10, Theorem 11]. *Suppose* $v : \Omega \to \mathbb{C}$ *is continuous and the partial derivatives* v_x*,* v_y *exist at every point on* Ω *except for countably many. If* $v_{\bar{z}} = 0$ *almost everywhere in* Ω*, then* v *is analytic on* Ω.

Proof of Theorem 1.1. Suppose that the conclusion of Theorem 1.1 fails for $f \in \mathcal{F}_u^M$. Then there is a sequence $\{b_k\}$ and a sequence of functions $\{f_k\} \subset \mathcal{F}_u^M$ such that $\lim_{k\to+\infty} b_k = 0$ and $b_k \notin f_k(\mathbb{D})$ for $k \in \mathbb{N}$. For $k \in \mathbb{N}$, set $F_k = f_k - f_0$, where f_0 is a particular solution to (1.1). By Lemma 2.2, for $k \in \mathbb{N}$, we see that F_k is analytic on $\mathbb{D}$. By the assumption, F_k is uniformly bounded on $\mathbb{D}$. Hence, by Montel's theorem, on any closed subset Ω of $\mathbb{D}$, there is a subsequence of $\{F_k\}$ which converges uniformly on Ω. Without loss of generality, we assume that the subsequence $\{F_{n_k}\}$ of $\{F_k\}$ converges uniformly on $\overline{\mathbb{D}}_{1/2}$ to $f^* - f_0$. Hence $\lim_{k\to+\infty} f_{n_k}(0) = f^*(0)$ and $\lim_{k\to+\infty} J_{f_{n_k}}(0) = J_{f^*}(0)$, which implies that $f^* \in \mathcal{F}_u^M$. Since $f^*(0) = J_{f^*}(0) - 1 = 0$, there are $r_1 \in (0, 1/2)$ and $r_2 > 0$ such that $J_{f^*} > 0$ on $\overline{\mathbb{D}}_{r_1}$, $\mathbb{D}_{r_2} \subset f^*(\mathbb{D}_{r_1})$ and $|f^*(z)| \geq r_2$ for $z \in \partial\mathbb{D}_{r_1}$. From the properties of the limits, there is a positive integer k_0 such that, $|f_{n_k}| \geq r_2/2$ on $\partial\mathbb{D}_{r_1}$ and $J_{f_{n_k}} > 0$ on $\overline{\mathbb{D}}_{r_1}$ for $k \geq k_0$. Since $\det(f_{n_k}, \mathbb{D}_{r_1}, 0) \geq 1$, by Lemma 2.1, we conclude that $\det(f_{n_k}, \mathbb{D}_{r_1}, w) \geq 1$ for $k \geq k_0$ and $w \in \mathbb{D}_{r_2/2}$. Therefore, $\mathbb{D}_{r_2/2} \subset f_{n_k}(\mathbb{D}_{r_1})$ for $k \geq k_0$, which leads to a contradiction. The proof of the theorem is complete. □

Lemma 2.3. *For a given* $\mu \in \mathbb{R}$ *and* $\alpha \in [0, 4]$*, let* $g \in C^3(\mathbb{D})$ *and let* g_z *be a u-gradient mapping, where* $u = \mu|g_z|^\alpha$*. Then, for* $p \in [2, +\infty)$*,* $|g_z|^p$ *is subharmonic in* $\mathbb{D}$.

Proof. Let $\mathcal{Z}_{g_z} = \{w \in \mathbb{D} : g_z(w) = 0\}$. Then $\mathcal{Z}_{g_z}$ is a closed set and so $\mathbb{D}\backslash\mathcal{Z}_{g_z}$ is an open set. Set $f = g_z$. By computation, for $z \in \mathbb{D}\backslash\mathcal{Z}_{g_z}$,

$$2\mathrm{Re}(\overline{f} f_{z\bar{z}}) = \mu\alpha|f|^{\alpha-2}\mathrm{Re}(f_z \overline{f}^2) + \alpha\mu^2|f|^{2\alpha},$$

which gives

$$\begin{aligned}\Delta(|f|^p) &= p(p-2)|f|^{p-4}|f_{\bar{z}}\overline{f} + f\overline{f_z}|^2 + 2p|f|^{p-2} \\ &\qquad \times [|f_z|^2 + |f_{\bar{z}}|^2 + \alpha\mu^2|f|^{2\alpha} + \alpha\mu|f|^{\alpha-2}\mathrm{Re}(f_z\overline{f}^2)] \\ &= p(p-2)|f|^{p-4}|f_{\bar{z}}\overline{f} + f\overline{f_z}|^2 + 2p|f|^{p-2} \\ &\qquad \times \bigg[\Big(1 - \frac{\alpha}{4}\Big)|f_z|^2 + |f_{\bar{z}}|^2 + \alpha\Big|\mu|f|^{\alpha-2}f^2 + \frac{1}{2}f_z\Big|^2\bigg] \\ &\geq 0.\end{aligned}$$

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Hence $|g_z|^p$ is subharmonic in $\mathbb{D}\backslash\mathcal{Z}_{g_z}$. For each point of $\mathcal{Z}_{g_z}$, the mean value inequality trivially holds. Therefore $|g_z|^p$ is subharmonic in $\mathbb{D}$. □

Proof of Theorem 1.4. The assumptions on α, β, η show that $\eta \in [2, 2+\beta)$. Hence, by Lemma 2.3, $|g_z|^\eta$ is subharmonic in $\mathbb{D}$. Then, for $z \in \mathbb{D}$ and $\rho \in [0, d(z))$,

$$|g_z(z)|^\eta \le \frac{1}{2\pi}\int_0^{2\pi} |g_z(z+\rho e^{i\theta})|^\eta \, d\theta. \tag{2.1}$$

Multiplying both sides of the inequality (2.1) by ρ and integrating from 0 to $d(z)/2$,

$$\begin{aligned}
\frac{(d(z))^2|g_z(z)|^\eta}{2} &\le \int_0^{d(z)/2} \rho \int_0^{2\pi} |g_z(z+\rho e^{i\theta})|^\eta \, \frac{d\theta\, d\rho}{\pi} \\
&= \int_{\mathbb{D}(z,d(z)/2)} |g_z(\zeta)|^\eta \, d\sigma(\zeta) \\
&\le 2^\beta (d(z))^{-\beta} \int_{\mathbb{D}(z,d(z)/2)} |g_z(\zeta)|^\eta (d(\zeta))^\beta \, d\sigma(\zeta) \\
&= 2^\beta (d(z))^{-\beta} \int_{\mathbb{D}(z,d(z)/2)} \omega((d(\zeta))^\beta)|g_z(\zeta)|^\eta \frac{(d(\zeta))^\beta}{\omega((d(\zeta))^\beta)} \, d\sigma(\zeta) \\
&\le \frac{2^\beta}{\omega(1)(d(z))^\beta} \int_{\mathbb{D}(z,d(z)/2)} \omega((d(\zeta))^\beta)|g_z(\zeta)|^\eta \, d\sigma(\zeta) \\
&\le \frac{2^\beta \|f\|_{\mathcal{D}^\omega_{\beta,\eta}}}{\omega(1)(d(z))^\beta},
\end{aligned}$$

which gives

$$|g_z(z)| \le \frac{C_1}{(d(z))^{(2+\beta)/\eta}} \quad \text{and} \quad C_1 = \frac{2^{(1+\beta)/\eta}\|f\|^{1/\eta}_{\mathcal{D}^\omega_{\beta,\eta}}}{(\omega(1))^{1/\eta}}. \tag{2.2}$$

By the assumption and (2.2), we also see that

$$\|D_g(z)\| \le 2|g_z(z)| \le \frac{2C_1}{(d(z))^{(2+\beta)/\eta}}. \tag{2.3}$$

By (2.3),

$$\begin{aligned}
|g(z)| &\le |g(0)| + \left|\int_{[0,z]} dg(\zeta)\right| \\
&\le |g(0)| + \int_{[0,z]} \|D_g(\zeta)\| \, |d\zeta| \\
&\le |g(0)| + \frac{C_2}{(d(z))^{((2+\beta)/\eta)-1}},
\end{aligned} \tag{2.4}$$

where $C_2 = 2C_1\eta/(2+\beta-\eta)$ and $[0, z]$ denotes the line segment from 0 to z. Applying (2.2),

$$|g_z(z)|^\alpha \le \frac{C_1^\alpha}{(d(z))^{(2+\beta)\alpha/\eta}}.$$

Now recall the well-known inequality

$$(a+b)^q \leq 2^{\max\{q-1,0\}}(a^q+b^q), \tag{2.5}$$

where $a, b \geq 0$ and $q > 0$.

For $p \geq 2$, by (2.4) and (2.5),

$$|g(z)|^{p-1} \leq \Big[|g(0)| + \frac{C_2}{d(z)^{((2+\beta)/\eta-1)}}\Big]^{p-1} \leq 2^{p-2}\Big[|g(0)|^{p-1} + \frac{C_2^{p-1}}{(d(z))^{((2+\beta)/\eta-1)(p-1)}}\Big]$$

and

$$\begin{aligned}|g(z)|^{p-2} &\leq 2^{\max\{p-3,0\}}\Big[|g(0)|^{p-2} + \frac{C_2^{p-2}}{(d(z))^{((2+\beta)/\eta-1)(p-2)}}\Big]\\ &\leq 2^{p-2}\Big[|g(0)|^{p-2} + \frac{C_2^{p-2}}{(d(z))^{((2+\beta)/\eta-1)(p-2)}}\Big].\end{aligned}$$

For the case $p \in [4, \infty)$, by computation and the fact that $g_{\bar z} = \mu |g_z|^\alpha$,

$$\begin{aligned}\Delta(|g|^p) &= p(p-2)|g|^{p-4}|g\overline{g_z} + g_{\bar z}\bar g|^2 + 2p|g|^{p-2}(|g_z|^2 + |g_{\bar z}|^2) + p|g|^{p-2}\mathrm{Re}\,(\bar g \Delta g)\\ &\leq p^2|g|^{p-2}\|D_g\|^2 + 4p\mu|g|^{p-1}|g_z|^\alpha,\end{aligned}$$

which implies that

$$\begin{aligned}&(d(z))^{(p-1)(\nu-1)+\alpha\nu}\Delta(|g|^p)\\ &\quad\leq\ p^2(d(z))^{(p-1)(\nu-1)+\alpha\nu}|g|^{p-2}\|D_g\|^2 + 4p\mu|g|^{p-1}|g_z|^\alpha(d(z))^{(p-1)(\nu-1)+\alpha\nu}\\ &\quad= p^2(d(z))^{(p-2)(\nu-1)}|g|^{p-2}\|D_g\|^2(d(z))^{2\beta/\eta}\\ &\qquad\times(d(z))^{\nu(1+\alpha)-1-2\beta/\eta} + 4p\mu|g|^{p-1}|g_z|^\alpha(d(z))^{(p-1)(\nu-1)+\alpha\nu}\\ &\quad\leq\ p^2C_3\|D_g\|^2(d(z))^{2\beta/\eta} + C_4,\end{aligned} \tag{2.6}$$

where $\nu = (2+\beta)/\eta$, $C_3 = 2^{p-2}(C_2^{p-2} + |g(0)|^{p-2})$ and $C_4 = 2^pC_1^\alpha(C_2^{p-1} + |g(0)|^{p-1})p\mu$. By Hölder's inequality and (2.6), we conclude that

$$\begin{aligned}&\int_{\mathbb{D}}(d(z))^{(p-1)(\nu-1)+\alpha\nu}\Delta(|g(z)|^p)\,d\sigma(z)\\ &\qquad\leq\ C_3p^2\int_{\mathbb{D}}\|D_g(z)\|^2(d(z))^{2\beta/\eta}\,d\sigma(z) + C_4\\ &\qquad\leq\ C_3p^2\Big(\int_{\mathbb{D}}\|D_g(z)\|^\eta(d(z))^\beta\,d\sigma(z)\Big)^{2/\eta}\Big(\int_{\mathbb{D}}d\sigma(z)\Big)^{1-2/\eta} + C_4\\ &\qquad= C_3p^2\Big(\int_{\mathbb{D}}\omega((d(z))^\beta)\|D_g(z)\|^\eta\frac{(d(z))^\beta}{\omega((d(z))^\beta)}\,d\sigma(z)\Big)^{2/\eta} + C_4\\ &\qquad\leq\ \frac{C_3p^2}{\omega(1)}(\|g\|_{\mathcal{D}^\omega_{\beta,\eta}})^{2/\eta} + C_4 < +\infty.\end{aligned} \tag{2.7}$$

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In the case $p \in [2, 4)$, let $G_n^p = (|g|^2 + 1/n)^{p/2}$ for $n \in \mathbb{N}$. Then $\Delta(G_n^p)$ is integrable in $\mathbb{D}_r$ for $r \in [0, 1)$. By (2.6), (2.7) and Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to+\infty} \int_{\mathbb{D}_r} & (d(z))^{(p-1)(\nu-1)+\alpha\nu} \Delta(G_n^p(z))\, d\sigma(z) \\
&= \int_{\mathbb{D}_r} (d(z))^{(p-1)(\nu-1)+\alpha\nu} \lim_{n\to+\infty} \Delta(G_n^p(z))\, d\sigma(z) \\
&\leq C_3 p^2 \int_{\mathbb{D}_r} \|D_g(z)\|^2 (d(z))^{2\beta/\eta}\, d\sigma(z) + C_4 \\
&\leq C_3 p^2 \Big(\int_{\mathbb{D}} \|D_g(z)\|^\eta (d(z))^\beta\, d\sigma(z) \Big)^{2/\eta} \Big(\int_{\mathbb{D}_r} d\sigma(z) \Big)^{1-2/\eta} + C_4 \\
&< +\infty
\end{aligned}$$

by the same argument as in (2.7). The desired conclusion (1.2) follows from the two cases. □

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