ON MAITLAND'S GENERALISED BESSEL FUNCTION

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1. Introduction. Maitland's generalised Bessel function [4] is defined by the equation

$$(1.1)\qquad J_v^{(u)}(x) \overset{\text{def}}{=} \sum_{m=0}^{\infty} \frac{(-x)^m}{m!\,\Gamma(v+1+um)}$$

where u is real and positive and v is any number real or complex. If $u = 1$, then (1.1) reduces to the form

$$(1.2)\qquad J_v^{(1)}(x) = x^{-\frac{1}{2}v} J_v(2\sqrt{x}) \quad .$$

The object of this paper is to obtain an inversion formula of the convolution transform involving the generalised Maitland Bessel function. In the following paragraph the classical Laplace transform as defined by the equation

$$(1.3)\qquad F(p) = p\int_0^{\infty} e^{-pt} f(t)dt \qquad\qquad \text{Re}\,(p) > 0$$

will be symbolically denoted by

$$(1.4)\qquad F(p) \doteqdot f(t) \quad .$$

It can be verified that

$$(1.5)\qquad t^v\, J_v^{(u)}(at^u) \doteqdot p^{-v} \exp[-ap^{-u}]$$

where $u > 0$, $\text{Re}\,(v) > -1$ and $\text{Re}\,(p) > 0$.

2. Bushman [1] and Widder [3] have obtained the inversion integrals of the integral transforms which have as kernels the Legendre and Laguerre polynomials respectively. In this article we will consider the similar problem involving the generalised Maitland Bessel function as its kernel. The technique used here is that of Widder [3].

https://doi.org/10.4153/CMB-1968-091-5 Published online by Cambridge University Press

THEOREM. Let v be a positive integer and suppose that $g(x)$ and its first v derivatives with respect to x are continuous in $0 \leq x < \infty$ and vanish at $x = 0$. If $f(x)$ is defined by

$$(2.1) \qquad f(x) \overset{\text{def}}{=} \int_0^x (x-t)^v J_v^{(u)} [a(x-t)^u] \, g(t) \, dt \qquad u > 0$$

then $f(x)$ and its first $(2v+1)$ derivatives are continuous in $0 \leq x < \infty$ and vanish at $x = 0$ and $g(x)$ is given by

$$(2.2) \qquad g(x) = \int_0^x (x-t)^v J_v^{(u)} [-a(x-t)^u] \{ D^{2v+2} f(t) \} \, dt .$$

Proof. Since the lowest degree of $(x-t)$ in $(x-t)^v J_v^{(u)}[a(x-t)^u]$ is v hence its first $(v-1)$ derivatives with respect to x are continuous in $0 \leq x < \infty$ and vanish at $x = t$. Thus differentiating (2.1) v times with respect to x we obtain

$$(2.3) \qquad D^v f(x) = \int_0^x D^v [(x-t)^v J_v^{(u)} [a(x-t)^u] \, g(t) \, dt .$$

Now since $g(x)$ and its first v derivatives are continuous in $0 \leq x < \infty$ and vanish at $x = 0$, it is obvious that on differentiating (2.2) further, $f(x)$ and all its first $(2v+1)$ derivatives are also continuous in $0 \leq x < \infty$ and vanish at $x = 0$.

If $F(p) \doteq f(t)$ and $G(p) \doteq g(t)$ then from (2.1), the result [2, pp. 131, (20)] and (1.5) we obtain

$$F(p) = p^{-(v+1)} \exp[-ap^{-u}] \, G(p)$$

or

$$(2.4) \qquad G(p) = p^{-(v+1)} \exp[ap^{-u}] \, . \, p^{2v+2} \, F(p) .$$

Now inverting (2.4) and making use of (1.5) and the result [2, pp. 129 (8)] we obtain (2.2).

Applications. Let

$$g(t) = t^s \qquad s > v$$

then from (2.1) we have

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$$(2.5) \qquad f(x) = \int_0^x (x-t)^v \, J_v^{(u)} [a(x-t)^u] \, t^s \, dt \, .$$

Using the above theorem it can be shown that

$$\int_0^x (x-t)^v \, J_v^{(u)} [-a(x-t)^u] \, D^{2v} [t^{v+s+1} \, J_{v+s+1}^{(u)} (at^u)] \, dt$$

$$= x^s / \Gamma(s+1) \, .$$

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https://doi.org/10.4153/CMB-1968-091-5 Published online by Cambridge University Press