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UNIFORM BANDS

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Abstract A semigroup B in which every element is an idempotent can be embedded into such a semigroup B', where all the local submonoids are isomorphic, and in such a way that B and B' satisfy the same equational identities. In view of the properties preserved under this embedding, a corresponding embedding theorem is obtained for regular semigroups whose idempotents form a subsemigroup.

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1. Introduction

We follow the notation and terminology of [**2**, **3**, **13**, **14**, **25**]. In the following, we recall the relevant facts.

A *band* B is a semigroup in which each element is an idempotent. The class of all bands forms a variety $\boldsymbol{B}$ of semigroups. A subvariety of $\boldsymbol{B}$ is also an equational class, that is, consists of all the bands that satisfy a given set of equational identities. The lattice $\mathcal{L}(\boldsymbol{B})$ of subvarieties of $\boldsymbol{B}$ is countable and completely distributive; we refer the reader to [**25**] for further references and an explicit description of the lower part of $\mathcal{L}(\boldsymbol{B})$. A semigroup S is said to be *regular* if for every $a \in S$ there exists $a' \in S$ such that $aa'a = a$, and a regular semigroup S is said to be an *orthodox semigroup* if the set $E(S)$ of idempotents of S forms a subsemigroup of S. An orthodox semigroup S is called *fundamental* if the equality on S is the only congruence on S that separates the elements of $E(S)$. An *inverse semigroup* S is an orthodox semigroup for which the set $E(S)$ of idempotents is a semilattice. A semigroup S is called *bisimple* if $S \times S$ is Green's $\mathcal{D}$-relation on S. In particular, a bisimple semigroup is *simple*, that is, has no non-trivial ideals.

If B is a band, then we define the so-called *natural partial order* $\leqslant$ on B, for $e, f \in B$, by $f \leqslant e$ if $ef = f = fe$. Given $e \in B$, the set $eBe = \{ege \mid g \in B\}$ forms a subsemigroup of B, and $eBe = \{f \in B \mid f \leqslant e\}$. Such a subsemigroup eBe of B is called a *local submonoid* of B, because e is the identity element of the band eBe. We define the equivalence relation $\mathcal{U}_B$ on B, for $e, g \in B$, by $e\mathcal{U}_B g$ if eBe and gBg are isomorphic (as bands). A *partial* *isomorphism* of B is an isomorphism $eBe \mapsto gBg$ for $e, g \in B$ with $e\mathcal{U}_B g$. The band B is said to be *uniform* if $\mathcal{U}_B = B \times B$, that is, if all local submonoids are isomorphic. If there exists an automorphism of B that maps $e \in B$ to $g \in B$, then the restriction to eBe of this automorphism is a partial isomorphism of eBe onto gBg. So, if the band B has a transitive automorphism group, it is, in particular, uniform. A band is uniform if and only if it is the band of idempotents of some bisimple orthodox semigroup (see [**8**–**11**] and [**23**, Chapter 6]).

Every semigroup can be embedded into a bisimple semigroup [**28**] (see also [**3**, § 8.6]), and every inverse semigroup can be embedded into a bisimple inverse semigroup [**29**]. In fact, every inverse semigroup can be embedded into a bisimple inverse semigroup that has no non-trivial congruences [**16**]. In particular, every semilattice can be embedded into a uniform semilattice, and every fundamental inverse semigroup can be embedded into a bisimple fundamental inverse semigroup. As the abstract states, we generalize the latter results for orthodox semigroups: we show that every orthodox semigroup S can be embedded into a bisimple orthodox semigroup S' such that the bands $E(S)$ and $E(S')$ generate the same band variety. Following the results of [**20**], we obtain an embedding of any semilattice into a uniform semilattice that preserves several structural properties. The technique of embedding presented there is an inspiration for the construction that follows.

2. An embedding of bands

Let B be a band. We denote by B^0 the band B with an *extra* zero adjoined: $0 \notin B$, and $a0 = 0a = 0$ for every $a \in B^0$. $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers and $\mathbb{Z}^+$ is the set of positive integers.

The *power* $(B^0)^{\mathbb{N}\times B}$ consists of all the mappings $\alpha\colon \mathbb{N} \times B \to B^0$ endowed with the pointwise multiplication, which we denote by '$\cdot$'; for any $\alpha_1, \alpha_2 \in (B^0)^{\mathbb{N}\times B}$, $\alpha_1 \cdot \alpha_2 \in (B^0)^{\mathbb{N}\times B}$ is such that, for any $(i, e) \in \mathbb{N} \times B$,

$$(i, e)(\alpha_1 \cdot \alpha_2) = ((i, e)\alpha_1)((i, e)\alpha_2)$$

is the product of $(i, e)\alpha_1$ and $(i, e)\alpha_2$ in B^0. We let B_1 be the set of all $\alpha \in (B^0)^{\mathbb{N}\times B}$ satisfying the following conditions:

(1) (i) $(0, e)\alpha = (0, g)\alpha$ for all $e, g \in B$,

(ii) $(i, e)\alpha \leqslant e$ in B^0 for all $e \in B$, $i \in \mathbb{Z}^+$,

(iii) $(i, e)\alpha \neq e$ for only finitely many $(i, e) \in \mathbb{Z}^+ \times B$.

It is easy to see that $(B^0)^{\mathbb{N}\times B}$ is a band, and that B_1 is a subband of $(B^0)^{\mathbb{N}\times B}$.

For every $e \in B^0$, we let $\epsilon_e \in B_1$ be defined by

$$\left.\begin{aligned} (0, g)\epsilon_e &= e \quad \text{for every } g \in B, \\ (i, g)\epsilon_e &= g \quad \text{for every } (i, g) \in \mathbb{Z}^+ \times B. \end{aligned}\right\} \tag{2.1}$$

Recall that a subset A of B_1 is said to be a *filter* of B_1 if

$$\alpha \in A,\ \beta \in B_1,\ \alpha \leqslant \beta \quad \text{in } (B_1, \leqslant) \quad \Rightarrow \quad \beta \in A.$$

Lemma 2.1.

(i) *The mapping*

$$\begin{aligned} \iota_1\colon B &\mapsto B_1 \\ e &\to \epsilon_e \end{aligned} \tag{2.2}$$

is an embedding of bands.

(ii) *For every $e \in B^0$, $\epsilon_e B_1 \epsilon_e$ consists of the $\alpha \in (B^0)^{\mathbb{N}\times B}$ such that*

(a) $(0,e)\alpha = (0,g)\alpha \leqslant e$ *in* B^0 *for every* $g \in B$,

(b) $(i,g)\alpha \leqslant g$ *for every* $(i,g) \in \mathbb{Z}^+ \times B$,

(c) $(i,g)\alpha \neq g$ *for only finitely many* $(i,g) \in \mathbb{Z}^+ \times B$,

(iii) $B\iota_1$ *is a filter of* B_1.

Proof. The proof follows a routine verification. We provide some details concerning (iii). Therefore, let $e \in B$, let $\alpha \in B_1$, and suppose that $\epsilon_e \leqslant \alpha$ in B_1. Let $(0,e)\alpha = (0,g)\alpha = f$ for all $g \in B$. Then, $e = (0,e)\epsilon_e \leqslant (0,e)\alpha = f$ in B^0, whence $f \in B$. Furthermore, for every $(i,g) \in \mathbb{Z}^+ \times B$, $g = (i,g)\epsilon_e \leqslant (i,g)\alpha$, whereas $D_{(i,g)\alpha} \leqslant D_g$ in $B^0/\mathcal{D}$. It follows that $(i,g)\alpha = g$ for every $(i,g) \in \mathbb{Z}^+ \times B$. Thus, $\alpha = \epsilon_f \in B\iota_1$. □

Lemma 2.2.

(i) *For every $e \in B$, let the mapping $\varphi_e\colon \epsilon_e B_1 \epsilon_e \to \epsilon_0 B_1 \epsilon_0$ be given, for $\alpha \in \epsilon_e B_1 \epsilon_e$, by*

$$\begin{aligned} (0,g)(\alpha\varphi_e) &= 0 && \text{for every } g \in B, \\ (i,e)(\alpha\varphi_e) &= (i-1,e)\alpha && \text{for every } i \in \mathbb{Z}^+, \\ (i,g)(\alpha\varphi_e) &= (i,g)\alpha && \text{for every } i \in \mathbb{Z}^+ \text{ and } g \neq e \text{ in } B. \end{aligned}$$

Then, φ_e is a partial isomorphism that maps $\epsilon_e B_1 \epsilon_e$ isomorphically onto $\epsilon_0 B_1 \epsilon_0$.

(ii) *Let $\theta\colon eBe \mapsto gBg$ be a partial isomorphism of B. The partial isomorphism $\iota_1^{-1}\theta\iota_1\colon \epsilon_e(B\iota_1)\epsilon_e \mapsto \epsilon_g(B\iota_1)\epsilon_g$ of $B\iota_1$ can then be extended to a partial isomorphism $\theta_1\colon \epsilon_e B_1 \epsilon_e \mapsto \epsilon_g B_1 \epsilon_g$.*

(iii) $(B\iota_1) \times (B\iota_1) \subseteq \mathcal{U}_{B_1}$.

Proof. (i) Using Lemma 2.1 (ii), one routinely verifies that for every $\alpha \in \epsilon_e B_1 \epsilon_e$ we have that $\alpha\varphi_e \in \epsilon_0 B_1 \epsilon_0$. We prove that φ_e is one-to-one. If $\alpha_1, \alpha_2 \in \epsilon_e B_1 \epsilon_e$ and $(i,e)\alpha_1 \neq (i,e)\alpha_2$ for some $i \in \mathbb{N}$, then $(i+1,e)(\alpha,\varphi_e) \neq (i+1,e)(\alpha_2\varphi_e)$, and if $(i,g)\alpha_1 \neq (i,g)\alpha_2$ for some $i \in \mathbb{Z}^+$ and $g \neq e$ in B, then $(i,g)(\alpha_1\varphi_e) \neq (i,g)(\alpha_2\varphi_e)$. We next prove that φ_e is onto. Therefore, let $\beta \in \epsilon_0 B_1 \epsilon_0$. Define $\alpha \in (B^0)^{\mathbb{N}\times B}$ by

$$\begin{aligned} (0,g)\alpha &= (1,e)\beta && \text{for every } g \in B, \\ (i,e)\alpha &= (i+1,e)\beta && \text{for every } i \in \mathbb{N}, \\ (i,g)\alpha &= (i,g)\beta && \text{for every } i \in \mathbb{Z}^+ \text{ and } g \neq e \text{ in } B. \end{aligned}$$

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One verifies that $\alpha \in \epsilon_e B_1 \epsilon_e$ and $\alpha\varphi_e = \beta$. We conclude that φ_e is a bijection of $\epsilon_e B_1 \epsilon_e$ onto $\epsilon_0 B_1 \epsilon_0$.

In order to prove (i) it suffices to prove that φ_e is a band homomorphism. Therefore, let $\alpha_1, \alpha_2 \in \epsilon_e B_1 \epsilon_e$ and calculate $(\alpha_1 \cdot \alpha_2)\varphi_e$ and $(\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)$: for any $g \in B$,

$$\begin{aligned}(0,g)((\alpha_1 \cdot \alpha_2)\varphi_e) &= 0\\ &= 00\\ &= (0,g)((\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e));\end{aligned}$$

for any $i \in \mathbb{Z}^+$,

$$\begin{aligned}(i,e)((\alpha_1 \cdot \alpha_2)\varphi_e) &= (i-1,e)(\alpha_1 \cdot \alpha_2)\\ &= (i,e)((\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e));\end{aligned}$$

and, for every $i \in \mathbb{Z}^+$ and $g \neq e$ in B,

$$\begin{aligned}(i,g)((\alpha_1 \cdot \alpha_2)\varphi_e) &= (i,g)(\alpha_1 \cdot \alpha_2)\\ &= (i,g)((\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)).\end{aligned}$$

Therefore, $(\alpha_1 \cdot \alpha_2)\varphi_e = (\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)$, and we conclude that φ_e is a partial isomorphism of B_1.

(ii) For the partial isomorphism $\theta\colon eBe \mapsto gBg$ of B, define $\theta_1\colon \epsilon_e B_1 \epsilon_e \mapsto \epsilon_g B_1 \epsilon_g$, for $\alpha \in \epsilon_e B_1 \epsilon_e$, as follows: $\alpha\theta_1$ is given by

$$\left.\begin{aligned}(0,g)(\alpha\theta_1) &= ((0,g)\alpha)\theta \qquad \text{for every } g \in B,\\ (i,g)(\alpha\theta_1) &= (i,g)\alpha \qquad\quad \text{for every } i \in \mathbb{Z}^+,\ g \in B.\end{aligned}\right\} \tag{2.3}$$

Using Lemma 2.1 (ii), one routinely verifies that θ_1 is a partial isomorphism of B_1. Furthermore, if $\epsilon_f \in \epsilon_e B_1 \epsilon_e$, that is, $f \in eBe$, then

$$\begin{aligned}(0,g)(\epsilon_f\theta_1) &= ((0,g)\epsilon_f)\theta = f\theta \quad \text{for every } g \in B,\\ (i,g)(\epsilon_f\theta_1) &= (i,g)\epsilon_f = g \qquad\quad \text{for every } i \in \mathbb{Z}^+,\ g \in B;\end{aligned}$$

thus, $\epsilon_f\theta_1 = \epsilon_{f\theta}$. Therefore, θ_1 extends $\iota_1^{-1}\theta\iota_1$.

(iii) From (i) it follows that $\epsilon_e \mathcal{U}_{B_1} \epsilon_0$ for every $e \in B$. Therefore, $(B\iota_1) \times (B\iota_1) \subseteq \mathcal{U}_{B_1}$. □

For any band B, we let $\underline{T}_B$ be the set of partial isomorphisms of B. We consider the sequence of bands

$$B = B_0, B_1, \dots, B_j, B_{j+1}, \dots, \quad j < \omega, \tag{2.4}$$

and the embeddings $\iota_{j+1}\colon B_j \to B_{j+1}$, where, for every $j < \omega$, B_{j+1} is obtained from B_j in the same way as B_1 was obtained from B, as in the foregoing discussion, and the embedding ι_{j+1} is defined along the same lines as $\iota_1\colon B \to B_1$, which was given by (2.2). We thus obtain a direct family of bands B_j, $j < \omega$, and we let B' be the direct limit of this direct family (in the sense of [**7**, § 21]). For notational convenience, we identify B_j with $B_j\iota_{j+1}$ for every $j < \omega$. When doing so, we have that $B' = \bigcup_{j<\omega} B_j$ is a band and the B_j, $j < \omega$, form a chain of subbands of B'. In the following we also consider the sequence of sets

$$\underline{T}_B = \underline{T}_{B_0}, \underline{T}_{B_1}, \dots, \underline{T}_{B_j}, \underline{T}_{B_{j+1}}, \dots, \quad j < \omega, \tag{2.5}$$

of partial isomorphisms of the respective bands in (2.4). For any $j < \omega$, $\theta_j \in \underline{T}_{B_j}$, we denote by $\theta_{j+1} \in \underline{T}_{B_{j+1}}$ the partial isomorphism obtained from θ_j in the same way as θ_1 was obtained from θ in (2.3). In view of the identification of B_j with $B_j\iota_{j+1}$ mentioned in the preceding paragraph, we have $\theta_j \subseteq \theta_{j+1}$ by Lemma 2.2 (ii).

If $\boldsymbol{K}$ is an algebraic class of bands that is closed under adding an extra zero, subdirect powers, direct limits (see [**7**, §§ 20,21]), and $B \in \boldsymbol{K}$, then the band B' constructed from B as described above also belongs to $\boldsymbol{K}$. This is, in particular, the case if $\boldsymbol{K}$ is a variety of bands that contains the variety of semilattices.

Theorem 2.3. *Every band B can be embedded into a uniform band B' such that B and B' generate the same band variety.*

Proof. If B is a rectangular band, we take $B = B'$ and the result follows. We henceforth assume that B is not a rectangular band. The variety $\boldsymbol{K}$ generated by B then contains the variety of all semilattices. Let B' be constructed from B as described in this section. Since B is a subband of B', it follows from the remark made in the preceding paragraph that B and B' generate the same band variety $\boldsymbol{K}$.

Let $e, g \in B'$. There exists $j < \omega$ such that $e, g \in B_{j-1}$. By Lemma 2.2 (iii) there exists a $\theta_j \in \underline{T}_{B_j}$ that maps eB_je isomorphically onto gB_jg. Consider the sequence of partial isomorphisms

$$\theta_j \subseteq \theta_{j+1} \subseteq \cdots \subseteq \theta_{j+k} \subseteq \theta_{j+k+1} \subseteq \cdots, \quad k < \omega, \tag{2.6}$$

where, for each $k < \omega$, $\theta_{j+k} \in \underline{T}_{B_{j+k}}$ and θ_{j+k+1} is obtained from θ_{j+k} as θ_1 was obtained from θ in (2.3). Set $\theta'_j = \bigcup_{k<\omega} \theta_{j+k}$. Then, $\theta'_j \colon eB'e \to gB'g$ is a partial isomorphism of B', whence $e\mathcal{U}_{B'}g$. We conclude that B' is uniform. □

We conclude this section with some additional properties that are satisfied by the embedding of the band B into the band B', as in Theorem 2.3.

Theorem 2.4. *Let B and B' be bands, as in Theorem 2.3. The following then hold.*

(i) *If B is not a rectangular band, then B' is countably infinite if B is finite, and, otherwise, B and B' have the same cardinality.*

(ii) *B is a filter of B'.*

(iii) *Every endomorphism γ of B can be extended to an endomorphism γ' of B' such that $\operatorname{End} B \mapsto \operatorname{End} B'$, $\gamma \to \gamma'$ is an embedding of endomorphism monoids that induces an embedding $\operatorname{Aut} B \mapsto \operatorname{Aut} B'$ of automorphism groups.*

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(iv) *Every congruence ρ on B is the restriction to B of a congruence ρ' on B' such that $\operatorname{Con} B \mapsto \operatorname{Con} B'$, $\rho \to \rho'$ embeds the congruence lattice of B as a complete sublattice of the congruence lattice of B'.*

Proof. (i) This property is guaranteed by § 2 (1) (iii).

(ii) This property follows from Lemma 2.1 (iii).

(iii) In the following we adopt the notation of Lemma 2.1. For $\gamma \in \operatorname{End} B$, let $\iota_1^{-1}\gamma\iota_1 \colon \epsilon_e \to \epsilon_{e\gamma}$ be the corresponding endomorphism in $B\iota_1$. This endomorphism of $B\iota_1$ can be extended to the endomorphism γ_1 of B_1, where, for every $\alpha \in B_1$, $\alpha\gamma_1$ is given by

$$\begin{aligned} (0,g)(\alpha\gamma_1) &= ((0,g)\alpha)\gamma \quad \text{for every } g \in B \text{ such that } (0,g)\alpha \neq 0,\\ (i,g)(\alpha\gamma_1) &= (i,g)\alpha \qquad\quad \text{otherwise.} \end{aligned}$$

It should be clear that $\operatorname{End} B \mapsto \operatorname{End} B_1$, $\gamma \to \gamma_1$ is an embedding of endomorphism monoids. If we adopt the convention that B is identified with its isomorphic image $B\iota_1$, then $\gamma \subseteq \gamma_1$ for every $\gamma \in \operatorname{End} B$. We note that if $\gamma \in \operatorname{Aut} B$, then $\gamma_1 \in \operatorname{Aut} B_1$; thus, $\operatorname{Aut} B \mapsto \operatorname{Aut} B_1$, $\gamma \to \gamma_1$ is an embedding of automorphism groups.

We now consider the sequence (2.4) of bands B_j, $j < \omega$, whose direct limit is B', and the corresponding sequence

$$\operatorname{End} B = \operatorname{End} B_0, \operatorname{End} B_1, \dots, \operatorname{End} B_j, \operatorname{End} B_{j+1}, \dots, \quad j < \omega,$$

of endomorphism monoids. For any $j < \omega$ and $\gamma \in \operatorname{End} B$, we construct the $\gamma_j \in \operatorname{End} B_j$, $j < \omega$, inductively using

$$\gamma_0 = \gamma,$$
$$\text{and, for any } j < \omega, \quad \gamma_{j+1} \text{ is constructed from } \gamma_j \text{ as } \gamma_1 \text{ is constructed from } \gamma.$$

We thus obtain a sequence of endomorphisms

$$\gamma = \gamma_0 \subseteq \gamma_1 \subseteq \cdots \subseteq \gamma_j \subseteq \gamma_{j+1} \subseteq \cdots, \quad j < \omega,$$

and we set $\gamma' = \bigcup_{j<\omega} \gamma_j$. One verifies that $\gamma' \in \operatorname{End} B'$, and $\operatorname{End} B \mapsto \operatorname{End} B'$ is an embedding of endomorphism monoids.

(iv) The proof of (iv) follows the same lines as the proof of (iii). We only indicate here how to construct $\rho_1 \in \operatorname{Con} B_1$ from a given $\rho \in \operatorname{Con} B$. For $\alpha_1, \alpha_2 \in B_1$ we set $(\alpha_1, \alpha_2) \in \rho_1$ if and only if

$$\begin{aligned} ((0,g)\alpha_1, (0,g)\alpha_2) \in \rho &\quad \text{for every } g \in B \text{ with } (0,g)\alpha_1 \neq 0 \neq (0,g)\alpha_2,\\ (i,g)\alpha_1 = (i,g)\alpha_2 &\quad \text{otherwise.} \end{aligned}$$

□

Following our procedure for constructing the uniform band B' from the band B, one can set up a faithful functor from the category of bands to the category of uniform bands in a straightforward way. We refrain from exploring this line of investigation here.

3. An embedding of orthodox semigroups

For any band B, we adopt the notation of § 2: B_1 is the band constructed from B as in § 2 (1), and we once more adopt the convention that in the sequence of bands (2.4) we have $B = B_0$ and $B_j \subseteq B_{j+1}$ for every $j < \omega$, and $B' = \bigcup_{j<\omega} B_j$. Corresponding to the sequence (2.4) is the sequence (2.5) of sets of partial isomorphisms of the respective bands of (2.4). For every $j < \omega$, $\theta_j \in \underline{T}_{B_j}$, let $\theta_{j+k} \in \underline{T}_{B_{j+k}}$, $k < \omega$, as in the sequence (2.6), and as in the proof of Theorem 2.3 we set $\theta'_j = \bigcup_{k<\omega} \theta_{j+k} \in \underline{T}_{B'}$, a partial isomorphism of $B' = \bigcup_{k<\omega} B_k$. In particular, any $\theta = \theta_0 \in \underline{T}_B = \underline{T}_{B_0}$ extends to a partial isomorphism $\theta' = \cup\theta_j \in \underline{T}_{B'}$.

We recall some facts of [**9**]. We prefer to use the notation and basic results of [**23**, Chapter 6]; in this paper we use the more conventional notation $\underline{T}_B$ instead of the notation $\underline{\Phi}_B$ that was used in [**23**].

For any $e, g \in B_j$, $j < \omega$,

$$\begin{aligned} \pi_j(e,g)\colon egeB_jege &\mapsto gegB_jgeg \\ d &\to gdg \end{aligned} \tag{3.1}$$

is a partial isomorphism of B_j.

Lemma 3.1. *Let $e, g \in B_j$ and $\pi_j(e,g) \in \underline{T}_{B_j}$ as in (3.1). For any $k < \omega$, define $\pi_{j,k}(e,g)$, $k < \omega$, inductively using that $\pi_{j,k+1}(e,g)$ is obtained from $\pi_{j.k}(e,g)$ as $\theta_1 \in \underline{T}_{B_1}$ is obtained from $\theta \in \underline{T}_B$ in (2.3). Then, $\pi'(e,g) = \bigcup_{k<\omega} \pi_{j,k}(e,g) \in \underline{T}_{B'}$, where*

$$\begin{aligned} \pi'(e,g)\colon egeB'ege &\mapsto gegB'geg \\ d &\to gdg. \end{aligned}$$

Proof. The proof easily follows from an inductive argument and the details of (2.3). □

Let $j < \omega$ and $\sigma_j, \theta_j \in \underline{T}_{B_j}$, where

$$\sigma_j\colon eB_je \mapsto fB_jf, \qquad \theta_j\colon gB_jg \mapsto hB_jh \tag{3.2}$$

for some $e, f, g, h \in B_j$. We defined a product $\circ$ on $\underline{T}_{B_j}$ by

$$\sigma_j \circ \theta_j = \sigma_j \pi'(f,g)\theta_j = \sigma_j \pi_j(f,g)\theta_j, \tag{3.3}$$

where in the right-hand side of (3.3) juxtaposition denotes a composition of partial one-to-one transformations.

With the notation introduced above, we have the following lemma.

Lemma 3.2. *For any $j, k < \omega$ and $\theta_j \in \underline{T}_{B_j}$, let $\theta_{j,k} \in \underline{T}_{B_{j+k}}$ be inductively defined using*

$$\theta_{j,0} = \theta_j,$$

$$\theta_{j,k+1} \in \underline{T}_{B_{j+k+1}} \text{ is obtained from } \theta_{j,k} \in \underline{T}_{B_{j+k}} \text{as } \theta_1 \text{ is obtained from } \theta \text{ as in (2.3).}$$

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Then,

$$\begin{aligned} \underline{\tau}_{j,k} \colon \underline{T}_{B_j} &\mapsto \underline{T}_{B_{j+k}} \\ \theta_j &\to \theta_{j,k} \end{aligned} \tag{3.4}$$

is an embedding of $(\underline{T}_{B_j}, \circ)$ *into* $(\underline{T}_{B_{j+k}}, \circ)$.

Proof. The proof follows from Lemma 3.1, the details of (2.3) and the definition (3.3). □

Lemma 3.3. *With the notation of Lemma 3.2, for* $j < \omega$, $\theta_j \in \underline{T}_{B_j}$, $\theta'_j = \bigcup_{k<\omega} \theta_{j,k}$. *Then,*

$$\begin{aligned} \underline{\tau}'_j \colon \underline{T}_{B_j} &\mapsto \underline{T}_{B'} \\ \theta_j &\to \theta'_j \end{aligned} \tag{3.5}$$

is an embedding of $(\underline{T}_{B_j}, \circ)$ *into* $(\underline{T}_{B'}, \circ)$, *where the product* $\circ$ *is defined on* $\underline{T}_{B'}$ *as follows: for* $\sigma', \theta' \in \underline{T}_{B'}$, *with*

$$\begin{gathered} \sigma' \colon eB'e \mapsto fB'f, \quad \theta' \colon gB'g \mapsto hB'h \quad \text{for some } e, f, g, h \in B', \\ \sigma' \circ \theta' = \sigma' \pi'(f, g) \theta'. \end{gathered} \tag{3.6}$$

Proof. The proof follows from Lemma 3.1 and a direct verification. □

We note here that the algebras $(\underline{T}_{B_{j+k}}, \circ)$ and $(\underline{T}_{B'}, \circ)$ mentioned in Lemmas 3.2 and 3.3 are in fact orthodox semigroups (see [**23**, Theorem 4.3]). From Lemmas 3.2 and 3.3 we then have the following corollary.

Corollary 3.4. *For any* $j < \omega$, *the direct limit of the direct system of orthodox semigroups* $(\underline{T}_{B_{j+k}}, \circ)$, $k < \omega$, *given by (3.4), is an orthodox subsemigroup of* $(\underline{T}_{B'}, \circ)$, *and the mapping (3.5) embeds each orthodox semigroup* $(\underline{T}_{B_j}, \circ)$ *isomorphically into the orthodox semigroup* $(\underline{T}_{B'}, \circ)$.

For any $j < \omega$ and $\sigma_j, \theta_j \in \underline{T}_{B_j}$ as in (3.2) we set

$$\sigma_j \kappa_j \theta_j \iff e\mathcal{R}g, f\mathcal{L}h \text{ in } B_j \quad \text{and} \quad \pi_j(e,g)\theta_j = \sigma_j \pi_j(f,h) \tag{3.7}$$

(see [**23**, (1.7)]). Similarly, for any $\sigma', \theta' \in \underline{T}_{B'}$ as in (3.3), we set

$$\sigma' \kappa' \theta' \iff e\mathcal{R}g, f\mathcal{L}h \text{ in } B' \quad \text{and} \quad \pi'(e,g)\theta' = \sigma' \pi'(f,h). \tag{3.8}$$

Again, juxtaposition in the right-hand sides of (3.7) and (3.8) denotes a composition of partial one-to-one transformations. By [**23**, Theorem 4.1], the κ_j, $j < \omega$, and κ' defined above are congruence relations whose idempotent classes form rectangular bands, and so the canonical homomorphisms

$$\left.\begin{aligned} \kappa_j^\natural \colon \underline{T}_{B_j} &\to \underline{T}_{B_j}/\kappa_j = T_{B_j}, \quad j < \omega, \\ \kappa'^\natural \colon \underline{T}_{B'} &\to \underline{T}_{B'}/\kappa' = T_{B'} \end{aligned}\right\} \tag{3.9}$$

are homomorphisms of orthodox semigroups. As in [**23**], for each $j < \omega$, we call $\underline{T}_{B_j}$ the *augmented hull* of B_j and $T_{B_j} = \underline{T}_{B_j}/\kappa_j$ the *hull* of B_j. Similarly, $\underline{T}_{B'}$ is the augmented hull of B' and $T_{B'} = \underline{T}_{B'}/\kappa'$ is the hull of B'.

For $j < \omega$ and $\sigma_j \in \underline{T}_{B_j}$, use the notation $\bar{\sigma}_j = \sigma_j \kappa_j^{\natural}$, and for $\sigma' \in T_{B'}$ use $\bar{\sigma}' = \sigma' \kappa'^{\natural}$. From [**23**, Theorem 1.5] it then follows that, for $j < \omega$,

$$B_j \mapsto E(T_{B_j}), \qquad e \to \overline{\pi_j(e, e)}, \tag{3.10}$$

and also

$$B' \mapsto E(T_{B'}), \qquad e \to \overline{\pi'(e, e)} \tag{3.11}$$

are isomorphisms of bands. Here, we use the conventional notation, where $E(S)$ denotes the set of idempotents of the semigroup S. Note that, for any $j < \omega$, and $e \in B_j$, $\pi_j(e, e)$ is the identity transformation on eB_je, whereas, for every $e \in B'$, $\pi'(e, e)$ is the identity transformation on $eB'e$.

Using Lemmas 3.1, 3.2, 3.3 and the notation used therein, and the definitions of κ_j, $j < \omega$, and κ' in (3.7), (3.8), we obtain, in sequence, the following.

Lemma 3.5. *For any $j, k < \omega$,*

(i) *for every $\sigma_j, \theta_j \in \underline{T}_{B_j}$,*

$$\sigma_{j,k} \kappa_{j+k} \theta_{j,k} \iff \sigma_{j,k+1} \kappa_{j+k+1} \theta_{j,k+1},$$

(ii) *for every $\sigma_j, \theta_j \in \underline{T}_{B_j}$,*

$$\sigma_j \kappa_j \theta_j \iff \sigma'_j \kappa' \theta'_j,$$

(iii)

$$\begin{array}{ccc} \underline{T}_{B_j} & \xrightarrow{\underline{\tau}_{j,k}} & \underline{T}_{B_{j+k}} \\ {\scriptstyle \kappa_j^{\natural}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{j+k}^{\natural}} \\ T_{B_j} & \xrightarrow[\tau_{j+k}]{} & T_{B_{j+k}} \end{array} \qquad \begin{array}{ccc} \theta_j & \xrightarrow{\underline{\tau}_{j,k}} & \theta_{j,k} \\ {\scriptstyle \kappa_j^{\natural}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{j+k}^{\natural}} \\ \bar{\theta}_j & \xrightarrow[\tau_{j,k}]{} & \bar{\theta}_{j,k} \end{array} \tag{3.12}$$

and

$$\begin{array}{ccc} \underline{T}_{B_j} & \xrightarrow{\underline{\tau}'_j} & \underline{T}_{B'} \\ {\scriptstyle \kappa_j^{\natural}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa'^{\natural}} \\ T_{B_j} & \xrightarrow[\tau'_j]{} & T_{B'} \end{array} \qquad \begin{array}{ccc} \theta_j & \xrightarrow{\underline{\tau}'_j} & \theta'_j \\ {\scriptstyle \kappa_j^{\natural}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa'^{\natural}} \\ \bar{\theta}_j & \xrightarrow[\tau'_j]{} & \bar{\theta}'_j \end{array} \tag{3.13}$$

are commuting diagrams.

We therefore have the following corollary.

Corollary 3.6. *For any $j < \omega$, the direct system of orthodox semigroups $T_{B_{j+k}}$, $k < \omega$, given by (3.12) is an orthodox subsemigroup of $T_{B'}$, and the mapping τ'_j given by (3.13) embeds T_{B_j} isomorphically into $T_{B'}$.*

We mention the following intermediate result for clarity.

Proposition 3.7. *Let B be a band that is not a rectangular band and let S be any fundamental orthodox semigroup such that $E(S) = B$ is the band of idempotents of S. Let B' be the band constructed from B as in Theorem 2.3. Then, S can be embedded into the orthodox semigroup $T_{B'}$ that is bisimple and fundamental, where B and $B' \cong E(T_{B'})$ generate the same band variety.*

Proof. We set $B = B_0$ as in (2.4). Following Corollary 3.6, with $j = 0$, T_B can be embedded into $T_{B'}$. Since $B = B_0 \cong E(T_B)$ and $B' \cong E(T_{B'})$ via (3.10) and (3.11), we have that $E(T_B)$ and $E(T_{B'})$ generate the same band variety by Theorem 2.3. By [**23**, Theorem 1.5], there exists an idempotent separating homomorphism of S into T_B that induces the isomorphism (3.10) of bands (for $j = 0$). This homomorphism is one-to-one, since S is assumed to be fundamental. Thus, S embeds isomorphically into $T_{B'}$. The orthodox semigroup $T_{B'}$ is bisimple and fundamental by [**23**, Lemmas 1.8 and 6.4]. □

The proof of the following theorem refers to the primary references, but it may be useful to instead consult the survey paper [**21**], or [**24**].

Theorem 3.8. *Let S be an orthodox semigroup. Then, S can be embedded into an orthodox semigroup S' that is bisimple and such that the bands $E(S)$ and $E(S')$ of idempotents of S and S' generate the same band variety. Moreover, if S is not a rectangular group, then S' can be chosen to be fundamental.*

Proof. If S is a rectangular group, then take $S' = S$. We henceforth assume that S is not a rectangular group, that is, the variety of bands generated by $E(S)$ contains the variety of all semilattices. By Proposition 3.7 it suffices to embed the given orthodox semigroup S into a fundamental orthodox semigroup S_0 whose band $B = E(S_0)$ generates the same band variety as $E(S)$. The following device will do.

We let $\mathcal{Y}$ be the least inverse congruence on the orthodox semigroup S as described in [**14**, § 6.2]. We next embed $S/\mathcal{Y}$ into a fundamental inverse semigroup I: this can, for instance, be done using the Vagner–Preston representation that embeds $S/\mathcal{Y}$ isomorphically into an appropriate symmetric inverse semigroup (see [**14**, Chapter 5, Theorem 5.1.7 and Exercise 22]). We may as well assume that S and I are disjoint, and we let $\varphi\colon S \to I$ be the homomorphism that results from the composition of the canonical homomorphism $\mathcal{Y}^\natural$ and the embedding of $S/\mathcal{Y}$ into I. We let S_0 be the *strong composition* (Płonka sum) of S and I using φ (see [**25**, § I.8.7] and [**26**, **27**]): $S_0 = S \cup I$ and the multiplication on S_0 extends those given on S and I, and, for $s \in S$, $i \in I$, $si = (s\varphi)i$ and $is = i(s\varphi)$ as in I. The band $B = E(S_0)$ of idempotents of S_0 is the strong composition of the band $E(S)$ and the semilattice $E(I)$ using the band homomorphism $\varphi|_{E(S)}$. Since we assumed that the variety generated by $E(S)$ contains the variety of all semilattices, it follows from [**25**, Lemma I.8.8] or [**26**] that $E(S)$ and $B = E(S_0)$ generate the same band variety.

It remains to show that S_0 is a fundamental orthodox semigroup. Let μ be the greatest idempotent separating congruence on S_0. By [**4**] it suffices to show that if $a \in S_0$, $a \neq a^2$, belongs to a maximal subgroup of S_0 that has $e \in E(S_0)$ as its identity element, then

a cannot be μ-related to e. This is surely the case if $e \in E(I)$, since I is fundamental. Otherwise, $a, e \in S$, and we let $a^{-1} \in S$ be the inverse of a in the maximal subgroup of S that contains a and e, and $a \neq e$. From the description of $\mathcal{Y}$ in [**14**, § 6.2] it follows that φ isomorphically embeds the maximal subgroup of S containing a, a^{-1} and $e = aa^{-1} = a^{-1}a$ into the maximal subgroup of I that has identity element $e\varphi$. In particular, $e\varphi \neq a\varphi$, where $a\varphi$ and $a^{-1}\varphi = (a\varphi)^{-1}$ are mutually inverse elements in the maximal subgroup containing $e\varphi$ as its identity element. Since I is fundamental, there exists $f \leqslant e\varphi \leqslant e$ in $B = E(S_0)$ such that $f \neq (a^{-1}\varphi)f(a\varphi) = a^{-1}fa$, and, therefore, a is not μ-related to e, as required (see the description of μ in [**12**, § 4] or [**23**, (1.58)]). □

4. Final remarks

Many of the results obtained so far find their analogues in other settings. We only give an outline of the required proofs. Let S be a regular semigroup, let $E(S)$ be its set of idempotents, and define a partial operation $\circ$ on $E(S)$ as follows: for $e, f \in E(S)$, $f \circ e$ is defined if and only if $\{e, f\} \cap \{ef, fe\} \neq \emptyset$, and if this the case, then $f \circ e = fe$. Here, the products ef and fe are as in the given regular semigroup S. As in [**17**] we call $(E(S), \circ)$ the *(regular) biordered set* of S.

There are at least two natural settings where we can extend the partial operation $\circ$ on $E(S)$ to a binary operation $\wedge$ on $E(S)$ as follows.

(1) If S is a regular semigroup whose idempotents generate a completely regular semigroup, then, for $e, f \in E(S)$, $f \wedge e = (fe)^0$ is the identity of the maximal subgroup of S that contains fe.

(2) If, for every $e \in E(S)$, eSe is an inverse semigroup, then $f \wedge e$ is the unique inverse of ef that belongs to fSe (see [**1**, § 2], [**22**, § 5], [**23**, § 4.1] and [**18**]).

In (1) we call S a *solid regular semigroup* and $(E(S), \wedge)$ a *regular solid idempotent algebra*, and in (2) we call S a *locally inverse semigroup* and $(E(S), \wedge)$ a *pseudo-semilattice*. It was shown that the classes of regular solid idempotent algebras and of pseudo-semilattices each form a variety [**1**, **18**]. The lattice of varieties of pseudo-semilattices was investigated extensively in [**19**]. The variety of bands is a subvariety of the variety of regular solid idempotent algebras; unlike the variety of all bands, the latter variety has a lattice of subvarieties that is of the power of the continuum (see the final remarks of [**22**, § 5]).

If $(B, \wedge)$ is a binary algebra as in (1) or (2), and S is a regular semigroup that has B as its biordered set, then, for every $e \in B$,

$$\begin{aligned}(e \wedge B) \wedge e &= \{(e \wedge f) \wedge e \mid f \in B\}\\ &= e \wedge (B \wedge e)\\ &= \{e \wedge (f \wedge e) \mid f \in B\}\\ &= E(eSe)\\ &= \{f \in B \mid f \leqslant e\}\\ &= \{f \in B \mid e \wedge f = f = f \wedge e\}\end{aligned}$$

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is a subalgebra of $(B, \wedge)$, which we denote by $e \wedge B \wedge e$. We call $(B, \wedge)$ *uniform* if, for every $e, g \in B$, we have that $e \wedge B \wedge e \cong g \wedge B \wedge g$. Following the procedure of § 2 step by step, we obtain the following analogue of Theorem 2.3.

Theorem 4.1. *Every regular solid idempotent algebra (pseudo-semilattice) $(B, \wedge)$ can be embedded into a regular solid idempotent algebra (pseudo-semilattice) $(B', \wedge)$ that is uniform, and such that B and B' generate the same variety.*

The analogue of Theorem 2.4 also holds true.

For the binary idempotent algebras $(B, \wedge)$ considered above, again let $\underline{T}_B$ be the set of partial isomorphisms of B, that is, isomorphisms of the form $e \wedge B \wedge e \to g \wedge B \wedge g$ for $e, g \in B$, and, in analogy with (3.3), define a product $\circ$ on $\underline{T}_B$ as follows: for $\sigma, \theta \in \underline{T}_B$, where $\sigma\colon e \wedge B \wedge e \to f \wedge B \wedge f$ and $\theta\colon g \wedge B \wedge g \to h \wedge B \wedge h$,

$$\sigma \circ \theta = \sigma\pi(f \wedge (g \wedge f), g \wedge f)\pi(g \wedge f, (g \wedge f) \wedge g)\theta. \tag{4.1}$$

In analogy with (3.7), then define the (congruence) relation κ on $\underline{T}_B$ and set $T_B = \underline{T}_B/\kappa$. Then, T_B is a fundamental regular semigroup and the analogue of (3.10) yields an isomorphism of binary algebras: T_B is a solid regular (locally inverse) semigroup if and only if B is a regular solid idempotent algebra (pseudo-semilattice); moreover, T_B is bisimple if and only if B is uniform (use [**17**, Proposition 3.6 and Theorems 4.12, 5.2], and [**23**, § 4.1], or [**12**]).

Following the same reasoning that leads to Proposition 3.7, we can then prove the following.

Proposition 4.2. *Every solid regular (locally inverse) semigroup S that is fundamental can be embedded into a solid regular (locally inverse) semigroup S' that is fundamental and bisimple, such that $(E(S), \wedge)$ and $(E(S'), \wedge)$ generate the same variety.*

Thus, for instance, from [**5**, **6**], [**15**] or [**30**] it follows that the free completely regular semigroup on a countably infinite set of generators is fundamental and can therefore be embedded into a solid regular semigroup that is bisimple and fundamental. In order to prove the analogue of Theorem 3.8, one needs to prove that every solid regular (locally inverse) semigroup S can be embedded into such a fundamental regular semigroup S' such that $(E(S), \wedge)$ and $(E(S'), \wedge)$ generate the same band variety. In other words, the device used in the proof of Theorem 3.8 needs to be modified. We do not elaborate any further on this here.

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