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Closed Ideals in Some Algebras of Analytic Functions

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Abstract. We obtain a complete description of closed ideals of the algebra $\mathcal{D}\cap \mathrm{lip}_\alpha$, $0 < \alpha \le \frac{1}{2}$, where $\mathcal{D}$ is the Dirichlet space and lip_α is the algebra of analytic functions satisfying the Lipschitz condition of order α.

1 Introduction

The Dirichlet space $\mathcal{D}$ consists of the complex-valued analytic functions f on the unit disk $\mathbb{D}$ with finite Dirichlet integral

$$D(f) := \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty,$$

where $dA(z) = \frac{1}{\pi} r dr dt$ denotes the normalized area measure on $\mathbb{D}$. Equipped with the pointwise algebraic operations and the norm

$$\|f\|_{\mathcal{D}}^2 := \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt + D(f) = \sum_{n=0}^{\infty}(1+n)|\hat{f}(n)|^2,$$

$\mathcal{D}$ becomes a Hilbert space. For $0 < \alpha \le 1$, let lip_α be the algebra of analytic functions f on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$ satisfing the Lipschitz condition of order α on $\overline{\mathbb{D}}$:

$$|f(z) - f(w)| = o(|z-w|^\alpha) \qquad (\,|z-w| \to 0).$$

Note that this condition is equivalent to

$$|f'(z)| = o((1-|z|)^{\alpha-1}) \qquad (|z| \to 1^-).$$

Then, lip_α is a Banach algebra when equipped with the norm

$$\|f\|_\alpha := \|f\|_\infty + \sup\{(1-|z|)^{1-\alpha}|f'(z)| : z \in \mathbb{D}\}.$$

Here $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$. Unlike the case when $0 < \alpha \le 1/2$, the inclusion $\mathrm{lip}_\alpha \subset \mathcal{D}$ always holds provided that $1/2 < \alpha \le 1$. In what follows, let $0 < \alpha \le 1/2$

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and define $\mathcal{A}_\alpha := \mathcal{D} \cap \mathrm{lip}_\alpha$. It is easy to check that $\mathcal{A}_\alpha$ is a commutative Banach algebra when it is endowed with the pointwise algebraic operations and the norm

$$\|f\|_{\mathcal{A}_\alpha} := \|f\|_\alpha + D^{1/2}(f) \qquad (f \in \mathcal{A}_\alpha).$$

In order to describe the closed ideals in subalgebras of the disc algebra $A(\mathbb{D})$, it is natural to make use of Nevanlinna's factorization theory. For $f \in A(\mathbb{D})$ there is a canonical factorization $f = C_f U_f O_f$, where C_f is a constant, U_f an inner function that is $|U_f| = 1$ *a.e* on $\mathbb{T}$ and O_f the outer function given by

$$O_f(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|d\theta\right\}.$$

Denote by $\mathcal{H}^\infty(\mathbb{D})$ the algebra of bounded analytic functions. Note that $\mathcal{A}_\alpha$ has the so-called F-property: if $f \in \mathcal{A}_\alpha$ and U is an inner function such that $f/U \in \mathcal{H}^\infty(\mathbb{D})$, then $f/U \in \mathcal{A}_\alpha$ and $\|f/U\|_{\mathcal{A}_\alpha} \leq C_\alpha \|f\|_{\mathcal{A}_\alpha}$, where C_α is independent of f (see [1, 9]). Korenblum [6] has described the closed ideals of the algebra H_1^2 of analytic functions f such that $f' \in H^2$, where H^2 is the Hardy space. This result has been extended to some other Banach algebras of analytic functions, by Matheson for lip_α [7] and by Shamoyan for the algebra $\lambda_\omega^{(n)}$ of analytic functions f on $\mathbb{D}$ such that

$$|f^{(n)}(\zeta_1) - f^{(n)}(\zeta_2)| = o(\omega(|\zeta_1 - \zeta_2|)) \ \textit{as}\ |\zeta_1 - \zeta_2| \to 0,$$

where n is a nonnegative integer and ω is an arbitrary nonnegative nondecreasing subadditive function on $(0, +\infty)$ [8]. Shirokov [9, 10] has given a complete description of closed ideals for Besov algebras $AB_{p,q}^s$ of analytic functions and particularly for the case $s > 1/2$ and $p = q = 2$

$$AB_{2,2}^s = \left\{ f \in A(\mathbb{D}) : \sum_{n\geq 0} |\widehat{f}(n)|^2(1+n)^{2s} < \infty \right\}.$$

Note that in the case of $AB_{2,2}^{1/2} = A(\mathbb{D}) \cap \mathcal{D}$ the problem of description of closed ideals appears to be much more difficult (see [2,4]). The purpose of this paper is to describe the structure of the closed ideals of the Banach algebras $\mathcal{A}_\alpha$. More precisely we prove that these ideals are standard in the sense of the Beurling–Rudin characterization of the closed ideals in the disc algebra [5].

Theorem 1.1 *If $\mathcal{I}$ is a closed ideal of $\mathcal{A}_\alpha$, then*

$$\mathcal{I} = \left\{ f \in \mathcal{A}_\alpha : f_{|E_{\mathcal{I}}} = 0 \textit{ and } f/U_{\mathcal{I}} \in \mathcal{H}^\infty(\mathbb{D}) \right\},$$

where $E_{\mathcal{I}} := \{z \in \mathbb{T} : f(z) = 0, \ \forall f \in \mathcal{I}\}$ and $U_{\mathcal{I}}$ is the greatest common divisor of the inner parts of the non-zero functions in $\mathcal{I}$.

Such characterization of closed ideals can be reduced further to a problem of approximation of outer functions using the Beurling–Carleman–Domar resolvent method. Define $d(\xi, E)$ to be the distance from $\xi \in \mathbb{T}$ to the set $E \subset \mathbb{T}$. Suppose that $\mathcal{I}$ is a closed ideal in $\mathcal{A}_\alpha$ such that $U_\mathcal{I} = 1$. We have $Z_\mathcal{I} = E_\mathcal{I}$, where

$$Z_\mathcal{I} := \{z \in \overline{\mathbb{D}} : f(z) = 0,\ \forall f \in \mathcal{I}\}.$$

Next, for $f \in \mathcal{A}_\alpha$ such that

$$|f(\xi)| \le C d(\xi, E_\mathcal{I})^{M_\alpha} \qquad (\xi \in \mathbb{T}),$$

where M_α is a positive constant depending only on $\mathcal{A}_\alpha$, we have $f \in \mathcal{I}$ (see Section 3 for more precisions). Now, to prove Theorem 1.1 we need Theorem 1.2 below, which states that every function in $\mathcal{A}_\alpha \setminus \{0\}$ can be approximated in $\mathcal{A}_\alpha$ by functions with boundary zeros of arbitrary high order.

Theorem 1.2 *Let f be a function in $\mathcal{A}_\alpha \setminus \{0\}$ and let $M > 0$. There exists a sequence of functions $\{g_n\}_{n=1}^{\infty} \subset A(\mathbb{D})$ such that:*

(i) *For all $n \in \mathbb{N}$, we have $f_n = f g_n \in \mathcal{A}_\alpha$ and $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{A}_\alpha} = 0$.*

(ii) $|g_n(\xi)| \le C_n d^M(\xi, E_f) \quad (\xi \in \mathbb{T})$, *where $E_f := \{\xi \in \mathbb{T} : f(\xi) = 0\}$.*

To prove this theorem, we give a refinement of the classical Korenblum approximation theory [6–10].

2 Main Result on Approximation of Functions in $\mathcal{A}_\alpha$

We begin by fixing some notations. Let $f \in \mathcal{A}_\alpha$ and let $\{\gamma_n := (a_n, b_n)\}_{n \ge 0}$ be the countable collection of the (disjoint open) arcs of $\mathbb{T} \setminus E_f$. Without loss of the generality, we can suppose that the arc lengths of γ_n are less than $1/2$. In what follows, we denote by Γ the union of a family of arcs γ_n. Define

$$f_\Gamma(z) := \exp\left\{ \frac{1}{2\pi} \int_\Gamma \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})| d\theta \right\}.$$

The difficult part in the proof of Theorem 1.2 is to establish the following.

Theorem 2.1 *Let $f \in \mathcal{A}_\alpha \setminus \{0\}$ be an outer function such that $\|f\|_{\mathcal{A}_\alpha} \le 1$, and let $N \ge 1$ and $\rho > 1$. Then we have $f^\rho f_\Gamma^N \in \mathcal{A}_\alpha$ and*

$$\sup_\Gamma \|f^\rho f_\Gamma^N\|_{\mathcal{A}_\alpha} \le C_{N,\rho}, \tag{2.1}$$

where $C_{N,\rho}$ is a positive constant independent of Γ.

Remark 2.2 For a set $S \subset A(\mathbb{D})$, we denote by $co(S)$ the convex hull of S consisting of the intersection of all convex sets that contain S. Set $\Gamma_n = \cup_{m \ge n} \gamma_m$, and let f be as in Theorem 2.1. It is clear that the sequence $(f^\rho f_{\Gamma_n}^N)_n$ converges uniformly on compact subsets of $\mathbb{D}$ to f^ρ. We use (2.1) to deduce, by the Hilbertian structure of $\mathcal{D}$, that there is a sequence $h_n \in co(\{f^\rho f_{\Gamma_m}^N\}_{m=n}^{\infty})$ converging to f^ρ in $\mathcal{D}$. Also, by [7, Section 4], we obtain that h_n converges to f^ρ in lip_α for sufficiently large N. (In fact, we can prove that this result remains true for every $N \ge 1$.) Therefore, $\|h_n - f^\rho\|_{\mathcal{A}_\alpha} \to 0$, as $n \to \infty$.

Define $\mathcal{J}(F)$ to be the closed ideal of all functions in $\mathcal{A}_\alpha$ that vanish on $F \subset \overline{\mathbb{D}}$. In the proof of Theorem 1.2, we need the following classical lemma. (See for instance [7, Lemma 4] and [6, Lemma 24]).

Lemma 2.3 *Let $f \in \mathcal{A}_\alpha$ and E' be a finite subset of $\mathbb{T}$ such that $f_{|E'} = 0$. Let $M > 0$ be given. For every $\varepsilon > 0$ there is an outer function F in $\mathcal{J}(E')$ such that*

(i) $\|Ff - f\|_{\mathcal{A}_\alpha} \leq \varepsilon$,
(ii) $|F(\xi)| \leq C d^M(\xi, E') \qquad (\xi \in \mathbb{T})$.

Proof of Theorem 1.2 Now, we can deduce the proof of Theorem 1.2 by using Theorem 2.1 and Lemma 2.3. Indeed, let f be a function in $\mathcal{A}_\alpha \setminus \{0\}$ such that $\|f\|_{\mathcal{A}_\alpha} \leq 1$, and let $\epsilon > 0$. For $m \geq 1$ we have

$$(fO_f^{\frac{1}{m}} - f)' = (O_f^{\frac{1}{m}} - 1)f' + \frac{1}{m} U_f O_f^{\frac{1}{m}} O_f'.$$

The F-property of $\mathcal{A}_\alpha$ implies that $O_f \in \mathcal{A}_\alpha$. Then, there exists $\eta_0 \in \mathbb{N}$ such that

$$\|fO_f^{\frac{1}{m}} - f\|_{\mathcal{A}_\alpha} < \epsilon/3 \qquad (m \geq \eta_0).$$

Set $\Gamma_n = \bigcup_{p \geq n} \gamma_p$ and $N \geq M/\alpha$ for a given $M > 0$. By Remark 2.2 applied to O_f (with $\rho = 1 + \frac{1}{m}$), there is a sequence $k_{n,m} \in co(\{f_{\Gamma_p}^N\}_{p=n}^\infty)$ such that

$$\|O_f^{1+\frac{1}{m}} k_{n,m} - O_f^{1+\frac{1}{m}}\|_{\mathcal{A}_\alpha} < \frac{1}{m} \qquad (n \in \mathbb{N},\ m \geq 1).$$

It is clear that

$$\|O_f^{\frac{1}{m}} f_{\Gamma_n}^N - O_f^{\frac{1}{m}}\|_\infty \longrightarrow 0 \qquad (n \longrightarrow +\infty).$$

Then for every $m \geq 1$ we get

$$\|O_f^{\frac{1}{m}} k_{n,m} - O_f^{\frac{1}{m}}\|_\infty \longrightarrow 0 \qquad (n \longrightarrow +\infty).$$

So, there is a sequence $k_m \in co(\{f_{\Gamma_p}^N\}_{p=m}^\infty)$ such that

$$\begin{cases} \|O_f^{1+\frac{1}{m}} k_m - O_f^{1+\frac{1}{m}}\|_{\mathcal{A}_\alpha} \leq \frac{1}{m} & (m \geq 1), \\ \|O_f^{\frac{1}{m}} k_m - O_f^{\frac{1}{m}}\|_\infty \leq \frac{1}{m} & (m \geq 1). \end{cases}$$

We have

$$(fO_f^{\frac{1}{m}} k_m - fO_f^{\frac{1}{m}})' = (f' - U_f O_f')(O_f^{\frac{1}{m}} k_m - O_f^{\frac{1}{m}}) + U_f (O_f^{1+\frac{1}{m}} k_m - O_f^{1+\frac{1}{m}})'.$$

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Since $\|O_f\|_{\mathcal{A}_\alpha} \le C_\alpha \|f\|_{\mathcal{A}_\alpha} \le C_\alpha$, we obtain

$$\begin{aligned}
&\|fO_f^{\frac{1}{m}}k_m - fO_f^{\frac{1}{m}}\|_{\mathcal{A}_\alpha} \\
&\quad = \|fO_f^{\frac{1}{m}}k_m - fO_f^{\frac{1}{m}}\|_\infty + \sup_{z\in\mathbb{D}}\{(1-|z|)^{1-\alpha}|(fO_f^{\frac{1}{m}}k_m - fO_f^{\frac{1}{m}})'(z)|\} \\
&\qquad\qquad + D^{1/2}(fO_f^{\frac{1}{m}}k_m - fO_f^{\frac{1}{m}}) \\
&\quad \le \|fO_f^{\frac{1}{m}}k_m - fO_f^{\frac{1}{m}}\|_\infty + C_\alpha\|f\|_\alpha\|O_f^{\frac{1}{m}}k_m - O_f^{\frac{1}{m}}\|_\infty \\
&\qquad\qquad + \sup_{z\in\mathbb{D}}\{(1-|z|)^{1-\alpha}|(O_f^{1+\frac{1}{m}}k_m - O_f^{1+\frac{1}{m}})'(z)|\} \\
&\qquad\qquad + C\|O_f^{\frac{1}{m}}k_m - O_f^{\frac{1}{m}}\|_\infty D^{1/2}(f) + CD^{1/2}(O_f^{1+\frac{1}{m}}k_m - O_f^{1+\frac{1}{m}}) \\
&\quad \le C_\alpha\|O_f^{\frac{1}{m}}k_m - O_f^{\frac{1}{m}}\|_\infty + C\|O_f^{1+\frac{1}{m}}k_m - O_f^{1+\frac{1}{m}}\|_{\mathcal{A}_\alpha} \le \frac{C_\alpha}{m}.
\end{aligned}$$

Then, fix $\eta_1 \ge \eta_0$ such that

$$\|fO_f^{\frac{1}{m}}k_m - fO_f^{\frac{1}{m}}\|_{\mathcal{A}_\alpha} < \epsilon/3 \qquad (m \ge \eta_1).$$

We have $k_m = \sum_{i\le j_m} c_i f^N_{\Gamma_i}$, where $\sum_{i\le j_m} c_i = 1$. Set $E'_m = \bigcup_{i<j_m} \partial\gamma_i$. Using Lemma 2.3, we obtain an outer function $F_m \in \mathcal{J}(E'_m)$ such that $|F_m(\zeta)| \le C_m d^M(\zeta, E'_m)$ for $\zeta \in \mathbb{T}$ and

$$\|fO_f^{\frac{1}{m}}k_mF_m - fO_f^{\frac{1}{m}}k_m\|_{\mathcal{A}_\alpha} < \frac{1}{m} \qquad (m \ge 1).$$

Then fix $\eta_2 \ge \eta_1$ such that

$$\|fO_f^{\frac{1}{m}}k_mF_m - fO_f^{\frac{1}{m}}k_m\|_{\mathcal{A}_\alpha} < \epsilon/3 \qquad (m \ge \eta_2).$$

Consequently we obtain

$$\|fO_f^{\frac{1}{m}}k_mF_m - f\|_{\mathcal{A}_\alpha} < \epsilon \qquad (m \ge \eta_2).$$

It is not hard to see that

$$|O_f^{\frac{1}{m}}k_mF_m(\xi)| \le |k_mF_m(\xi)| \le C_m d^M(\xi, E_f) \qquad (\xi \in \mathbb{T}).$$

Therefore $g_m = O_f^{\frac{1}{m}}k_mF_m$ is the desired sequence, which completes the proof of Theorem 1.2. ■

3 Beurling–Carleman–Domar Resolvent Method

Since $\mathcal{A}_\alpha \subset \mathrm{lip}_\alpha$, then for all $f \in \mathcal{A}_\alpha$, E_f satisfies the Carleson condition

$$\int_{\mathbb{T}} \log \frac{1}{d(e^{it}, E_f)} dt < +\infty.$$

For $f \in \mathcal{A}_\alpha$, we denote by B_f the Blashke product with zeros $Z_f \setminus E_f$, where $Z_f := \{z \in \overline{\mathbb{D}} : f(z) = 0\}$. We begin with following lemma.

Lemma 3.1 *Let $\mathcal{I}$ be a closed ideal of $\mathcal{A}_\alpha$. Define $B_{\mathcal{I}}$ to be the Blashke product with zeros $Z_{\mathcal{I}} \setminus E_{\mathcal{I}}$. There is a function $f \in \mathcal{I}$ such that $B_f = B_{\mathcal{I}}$.*

Proof Let $g \in \mathcal{I}$ and let B_n be the Blashke product with zeros $Z_g \cap \mathbb{D}_n$, where $\mathbb{D}_n := \{z \in \mathbb{D} : |z| < \frac{n-1}{n},\ n \in \mathbb{N}\}$. Set $g_n = g/K_n$, where $K_n = B_n/I_n$ and I_n is the Blashke product with zeros $Z_{\mathcal{I}} \cap \mathbb{D}_n$. We have $g_n \in \mathcal{I}$ for every n. Indeed, fix $n \in \mathbb{N}$. It is permissible to assume that Z_{K_n} consists of a single point, say $Z_{K_n} = \{w\}$. Let $\pi \colon \mathcal{A}_\alpha \to \mathcal{A}_\alpha/\mathcal{I}$ be the canonical quotient map. First suppose $w \notin Z_{\mathcal{I}}$, then $\pi(K_n)$ is invertible in $\mathcal{A}_\alpha/\mathcal{I}$. It follows that $\pi(g_n) = \pi(g)\pi^{-1}(K_n) = 0$, hence $g_n \in \mathcal{I}$. If $w \in Z_{\mathcal{I}}$, we consider the following ideal $\mathcal{J}_w := \{f \in \mathcal{A}_\alpha : fI_n \in \mathcal{I}\}$. It is clear that $\mathcal{J}_w$ is closed. Since $w \notin Z_{\mathcal{J}_w}$, it follows that K_n is invertible in the quotient algebra $\mathcal{A}_\alpha/\mathcal{J}_w$, and so $g/(I_nK_n) \in \mathcal{J}_w$. Hence $g_n \in \mathcal{I}$.

It is clear that g_n converges uniformly on compact subsets of $\mathbb{D}$ to $f = (g/B_g)B_{\mathcal{I}}$, and we have $B_f = B_{\mathcal{I}}$. In the sequel we prove that $f \in \mathcal{I}$. If we obtain

$$|(g_n)'(z)| \le o\Big(\frac{1}{(1-r)^{1-\alpha}}\Big) \qquad (z \in \mathbb{D}),$$

uniformly with respect to n, then $\lim_{n\to+\infty} \|g_n - f\|_\alpha = 0$ by [7, Lemma 1]. Indeed, by the Cauchy integral formula

$$(g_n)'(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(\zeta)\overline{K_n(\zeta)}}{(\zeta - z)^2} d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(g(\zeta) - g(z/|z|))\overline{K_n(\zeta)}}{(\zeta - z)^2} d\zeta \qquad (z \in \mathbb{D}).$$

Then, for $z = re^{i\theta} \in \mathbb{D}$

$$|(g_n)'(z)| \le \frac{\|K_n\|_\infty}{2\pi}\int_{\mathbb{T}} \frac{|g(\zeta) - g(z/|z|)|}{|\zeta - z|^2}|d\zeta| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|g(e^{i(t+\theta)}) - g(e^{i\theta})|}{1 - 2r\cos t + r^2} dt.$$

For all $\varepsilon > 0$, there is $\eta > 0$ such that if $|t| \le \eta$, we have

$$|g(e^{i(t+\theta)}) - g(e^{i\theta})| \le \varepsilon |t|^\alpha \qquad (\theta \in [-\pi, +\pi]).$$

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Then

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{|g(e^{i(t+\theta)}) - g(e^{i\theta})|}{1-2r\cos t + r^2} dt \\
&\leq \varepsilon \int_{|t|\leq \eta} \frac{|t|^\alpha}{(1-r)^2 + 4rt^2/\pi^2} dt + \|g\|_\alpha \int_{|t|\geq \eta} \frac{|t|^\alpha}{(1-r)^2 + 4rt^2/\pi^2} dt \\
&\leq \frac{\varepsilon}{r^{\frac{1+\alpha}{2}}(1-r)^{1-\alpha}} \int_0^{+\infty} \frac{u^\alpha}{1+(2u/\pi)^2} du + \frac{\|g\|_\alpha}{r^{\frac{1+\alpha}{2}}(1-r)^{1-\alpha}} \int_{|u|\geq \frac{\eta\sqrt{r}}{1-r}} \frac{u^\alpha}{1+(2u/\pi)^2} du \\
&\leq \varepsilon O\Big(\frac{1}{(1-r)^{1-\alpha}}\Big) + \|g\|_\alpha o\Big(\frac{1}{(1-r)^{1-\alpha}}\Big).
\end{aligned}$$

We obtain

$$\int_{-\pi}^{\pi} \frac{|g(e^{i(t+\theta)}) - g(e^{i\theta})|}{1-2r\cos t + r^2} dt \leq \|g\|_\alpha o\Big(\frac{1}{(1-r)^{1-\alpha}}\Big).$$

Consequently

$$|(g_n)'(z)| \leq \|g\|_\alpha o\Big(\frac{1}{(1-r)^{1-\alpha}}\Big) \qquad (z \in \mathbb{D}).$$

By the F-property of $\mathcal{A}_\alpha$, we have $\|g_n\| \leq C_\alpha \|g\|_{\mathcal{A}_\alpha}$. Using the Hilbertian structure of $\mathcal{D}$, we deduce that there is a sequence $h_n \in co(\{g_k\}_{k=n}^{\infty})$ converging to f in $\mathcal{D}$. It is clear that $h_n \in \mathfrak{I}$ and $\lim_{n\to+\infty} \|h_n - f\|_\alpha = 0$. Then $\lim_{n\to+\infty} \|h_n - f\|_{\mathcal{A}_\alpha} = 0$. Thus $f \in \mathfrak{I}$. This completes the proof of the lemma. ∎

As a consequence of Theorem 1.2, we can prove Theorem 1.1 and deduce that each closed ideal of $\mathcal{A}_\alpha$ is standard. For the sake of completeness, we sketch the proof here.

Proof of Theorem 1.1 Define γ on $\mathbb{D}$ by $\gamma(z) = z$, and let $\pi\colon \mathcal{A}_\alpha \to \mathcal{A}_\alpha/\mathfrak{I}$ be the canonical quotient map. Also, let $f \in \mathcal{J}(E_{\mathfrak{I}})$ be such that $f/U_{\mathfrak{I}} \in \mathcal{H}^\infty(\mathbb{D})$ and $(f_n)_n$ be the sequence in Theorem 1.2 associated to f with $M \geq 3$. More exactly, we have $f_n = fg_n$, where $|g_n(\xi)| \leq d^3(\xi, E_f) \leq d^3(\xi, E_{\mathfrak{I}})$. Define

$$\mathrm{L}_\lambda(f)(z) := \begin{cases} \dfrac{f(z)-f(\lambda)}{z-\lambda} & \text{if } z \neq \lambda, \\ f'(\lambda) & \text{if } z = \lambda. \end{cases}$$

Then

$$\pi(f)(\pi(\gamma)-\lambda)^{-1} = f(\lambda)(\pi(\gamma)-\lambda)^{-1} + \pi\big(\mathrm{L}_\lambda(f)\big). \tag{3.1}$$

It is clear that $(\pi(\gamma)-\lambda)^{-1}$ is an analytic function on $\mathbb{C}\setminus Z_{\mathfrak{I}}$. Note that the multiplicity of the pole $z_0 \in Z_{\mathfrak{I}} \cap \mathbb{D}$ of $(\pi(\gamma)-\lambda)^{-1}$ is equal to the multiplicity of the zero z_0 of $U_{\mathfrak{I}}$.

Since $U_{\mathcal{J}}$ divides f, then according to (3.1) we can deduce that $\pi(f)(\pi(\gamma)-\lambda)^{-1}$ is an analytic function on $\mathbb{C}\setminus E_{\mathcal{J}}$. Let $|\lambda|>1$, we have

$$\|\pi(f)(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_\alpha} \le \|f\|_{\mathcal{A}_\alpha}\sum_{n=0}^{\infty}\|\gamma^n\|_{\mathcal{A}_\alpha}|\lambda|^{-n-1} \le \|f\|_{\mathcal{A}_\alpha}\frac{C}{(|\lambda|-1)^{3/2}}.$$

By Lemma 3.1, there is $g\in\mathcal{I}$ such that $B_g = B_{\mathcal{J}}$. Let $k = f(g/B_g)$. Then $k = (f/B_{\mathcal{J}})g \in \mathcal{I}$, and for $|\lambda|<1$, we have

$$k(\lambda)(\pi(\gamma)-\lambda)^{-1} = -\pi\big(\mathrm{L}_\lambda(k)\big).$$

Therefore

$$\begin{aligned}\|\pi(f)(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_\alpha} &\le |f(\lambda)|\|(\pi(\gamma)-\lambda)^{-1}\|_{\mathcal{A}_\alpha} + \|\mathrm{L}_\lambda(f)\|_{\mathcal{A}_\alpha}\\ &\le \frac{\|\mathrm{L}_\lambda(k)\|_{\mathcal{A}_\alpha}}{|g/B_g|(\lambda)} + \|\mathrm{L}_\lambda(f)\|_{\mathcal{A}_\alpha}\\ &\le \frac{C(f,k)}{(1-|\lambda|)|g/B_g|(\lambda)}\\ &\le C(f,k)e^{\frac{C}{1-|\lambda|}} \quad (|\lambda|<1).\end{aligned}$$

We use [11, Lemmas 5.8 and 5.9] to deduce

$$\|\pi(f)(\pi(\gamma)-\xi)^{-1}\| \le \frac{C(f,k)}{d(\xi,E_{\mathcal{J}})^3} \qquad (1\le|\xi|\le 2,\ \xi\notin E_{\mathcal{J}}).$$

Then, we obtain

$$\xi \mapsto |(g_n)(\xi)|\|\pi(f)(\pi(\gamma)-\xi)^{-1}\| \in L^\infty(\mathbb{T}).$$

With a simple calculation as in [3, Lemma 2.4], we can deduce that

$$\pi(f_n) = \frac{1}{2\pi i}\int_{\mathbb{T}}(g_n)(\xi)\pi(f)(\pi(\gamma)-\xi)^{-1}d\xi.$$

Denote $\mathcal{I}^\infty_{U_{\mathcal{J}}}(E_{\mathcal{J}}) := \{h\in A(\mathbb{D}) : h_{|E_{\mathcal{J}}} = 0 \text{ and } h/U_{\mathcal{J}}\in A(\mathbb{D})\}$. From [5, p. 81], we know that $\mathcal{I}^\infty_{U_{\mathcal{J}}}(E_{\mathcal{J}})$ has an approximate identity $(e_m)_{m\ge1}\in\mathcal{I}^\infty_{U_{\mathcal{J}}}(E_{\mathcal{J}})$ such that $\|e_m\|_\infty\le 1$. $\mathcal{I}$ is dense in $\mathcal{I}^\infty_{U_{\mathcal{J}}}(E_{\mathcal{J}})$ with respect to the sup norm $\|\cdot\|_\infty$, so there exists $(u_m)_{m\ge1}\in\mathcal{I}$ with $\|u_m\|_\infty\le 1$ and $\lim_{m\to\infty}u_m(\xi)=1$ for $\xi\in\mathbb{T}\setminus E_{\mathcal{J}}$. Therefore $\pi(f_n)=\pi(f_n-f_nu_m)\to 0$ as $m\to\infty$. Then $f_n\in\mathcal{I}$ and $f\in\mathcal{I}$. ■

4 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on a series of lemmas. In what follows, C_ρ will denote a positive number that depends only on ρ, not necessarily the same at each occurrence. For an open subset Δ of $\mathbb{D}$, we put

$$\|f'\|^2_{L^2(\Delta)} := \int_\Delta |f'(z)|^2 dA(z).$$

We begin with the following key lemma.

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Lemma 4.1 *Let $f \in \mathcal{A}_\alpha$ be such that $\|f\|_{\mathcal{A}_\alpha} \leq 1$, and let $\rho > 1$ be given. Then*

$$\int_\gamma \frac{|f(e^{it})|^{2\rho}}{d(e^{it})} dt \leq C_\rho \|f'\|^2_{L^2(\Delta_\gamma)},$$

where $a, b \in E_f$, $\gamma = (a, b) \subset \mathbb{T} \setminus E_f$, $d(z) := \min\{|z-a|, |z-b|\}$ and $\Delta_\gamma := \{z \in \mathbb{D} : z/|z| \in \gamma\}$.

Proof Let $e^{it} \in \gamma$ and define $z_t := (1 - d(e^{it}))e^{it}$. Since $|\gamma| < 1/2$, we obtain $|z_t| > 1/2$. We have

$$|f(e^{it})|^{2\rho} \leq 2^{2\rho-1}\Big(|f(e^{it}) - f(z_t)|^{2\rho} + |f(z_t)|^{2\rho}\Big). \tag{4.1}$$

By Hölder's inequality combined with the fact that $\|f\|_\infty \leq \|f\|_{\mathcal{A}_\alpha} \leq 1$, we get

$$\begin{aligned} |f(e^{it}) - f(z_t)|^{2\rho} &= |f(e^{it}) - f(z_t)|^{2\rho-2}|f(e^{it}) - f(z_t)|^2 \\ &\leq 2^{2\rho-2}(1-|z_t|)\int_{|z_t|}^1 |f'(re^{it})|^2 dr \\ &\leq 2^{2\rho-1} d(e^{it}) \int_0^1 |f'(re^{it})|^2 r dr. \end{aligned}$$

Hence

$$\begin{aligned} \int_\gamma \frac{|f(e^{it}) - f(z_t)|^{2\rho}}{d(e^{it})} dt &\leq 2^{2\rho-1} \int_\gamma \int_0^1 |f'(re^{it})|^2 r dr dt \\ &\leq 2^{2\rho-1}\pi \|f'\|^2_{L^2(\Delta_\gamma)}. \end{aligned} \tag{4.2}$$

Since $d(e^{it}) \leq 1/2$, we obtain $\frac{d(e^{it})}{\sqrt{2}} \leq d(z_t) \leq \sqrt{2} d(e^{it})$. Put $d(z_t) = |z_t - \xi|$ and note that either $\xi = a$ or $\xi = b$. Let

$$z_t(u) = (1-u)z_t + u\xi \qquad (0 \leq u \leq 1).$$

With a simple calculation, we can prove that for all $e^{it} \in \gamma$ and for all u, $0 \leq u \leq 1$, we have

$$|z_t(u) - w| > \frac{1}{2}(1-u)d(e^{it}) \qquad (w \in \partial\Delta_\gamma),$$

where $\partial\Delta_\gamma$ is the boundary of Δ_γ. Then $\mathbb{D}_{t,u} := \{z \in \mathbb{D} : |z - z_t(u)| \leq \frac{1}{2}(1-u)d(e^{it})\} \subset \Delta_\gamma$, for all $e^{it} \in \gamma$ and for all u, $0 \leq u \leq 1$. Since $|f'(z)|$ is subharmonic on $\mathbb{D}$, it follows that

$$|f'(z_t(u))| \leq \frac{4}{\pi(1-u)^2 d^2(e^{it})} \int_{\mathbb{D}_{t,u}} |f'(z)| dA(z) \leq \frac{2}{\pi^{1/2}(1-u)d(e^{it})} \|f'\|_{L^2(\Delta_\gamma)}.$$

Set $\varepsilon_\rho = 2\alpha(\rho - 1)$. We have

$$\begin{aligned}|f^\rho(z_t)|^2 &= |f^\rho(z_t) - f^\rho(\xi)|^2\\ &= \rho^2|z_t - \xi|^2| \int_0^1 f^{\rho-1}(z_t(u))f'(z_t(u))du|^2\\ &\leq C_\rho d^2(e^{it}) \Big(\int_0^1 |z_t(u) - \xi|^{\frac{\varepsilon_\rho}{2}} |f'(z_t(u))|du \Big)^2\\ &\leq C_\rho d^{\varepsilon_\rho}(e^{it}) \Big(\int_0^1 \frac{1}{(1-u)^{1-\frac{\varepsilon_\rho}{2}}} du \Big)^2 \|f'\|^2_{L^2(\Delta_\gamma)}\\ &\leq C_\rho d^{\varepsilon_\rho}(e^{it})\|f'\|^2_{L^2(\Delta_\gamma)}.\end{aligned}$$

Hence

$$\int_\gamma \frac{|f(z_t)|^{2\rho}}{d(e^{it})} dt \leq C_\rho \|f'\|^2_{L^2(\Delta_\gamma)}. \tag{4.3}$$

Therefore the result follows from (4.1), (4.2), and (4.3). ■

In the sequel we denote by f an outer function in $\mathcal{A}_\alpha$ such that $\|f\|_{\mathcal{A}_\alpha} \leq 1$, and we fix a constant ρ, $1 < \rho \leq 2$. By [7, Theorem B], we have $f^\rho f_\Gamma^N \in \mathrm{lip}_\alpha$ and $\|f^\rho f_\Gamma^N\|_{\mathrm{lip}_\alpha} \leq C_{N,\rho}$. To prove Theorem 2.1 we need to estimate the integral $\int_{\mathbb{D}} |(f^\rho f_\Gamma^N)'|^2 dA(z)$. Define

$$g_\Gamma(z) := \frac{1}{\pi} \int_\Gamma \frac{e^{i\theta}}{(e^{i\theta} - z)^2} \log|f(e^{i\theta})| d\theta.$$

Clearly we have $f' = f(g_\Gamma + g_{\Gamma\setminus\Gamma})$ and $(f_\Gamma^N)' = N f_\Gamma^N g_\Gamma$, so

$$f^\rho (f_\Gamma^N)' = N f^\rho f_\Gamma^N g_\Gamma \tag{4.4}$$

$$= f^{\rho-1} N f' f_\Gamma^N - N f^\rho f_\Gamma^N g_{\Gamma\setminus\Gamma}. \tag{4.5}$$

Since $\|f\|_\infty \leq 1$, it is obvious that $\|f_\Gamma^N\|_\infty \leq 1$ and $\|f^{\rho-1}\|_\infty \leq 1$. Hence, by (4.4) we get

$$\int_{\mathbb{D}} |(f^\rho f_\Gamma^N)'|^2 dA(z) \leq \rho^2 + N^2 \int_{\mathbb{D}} |f^\rho (f_\Gamma)'|^2 dA(z).$$

We fix $\gamma = (a, b) \subset \mathbb{T} \setminus E_f$ such that $f(a) = f(b) = 0$. Our purpose in what follows is to estimate the integral $\int_{\Delta_\gamma} |f^\rho (f_\Gamma)'|^2 dA(z)$, which we can rewrite as

$$\int_{\Delta_\gamma} |f^\rho (f_\Gamma)'|^2 dA(z) = \int_{\Delta_\gamma^1} + \int_{\Delta_\gamma^2},$$

where

$$\Delta_\gamma^1 := \big\{ z \in \Delta_\gamma :\ d(z) < 2(1 - |z|) \big\}$$
$$\Delta_\gamma^2 := \big\{ z \in \Delta_\gamma :\ d(z) \geq 2(1 - |z|) \big\}.$$

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4.1 The Integral on the Region Δ_γ^1

We begin with the following lemma.

Lemma 4.2

$$\int_{\Delta_\gamma} \frac{|f(z) - f(z/|z|)|^{2\rho}}{(1-|z|)^2} dA(z) \le \frac{1}{2\alpha(\rho-1)} \|f'\|^2_{L^2(\Delta_\gamma)}.$$

Proof Let $z = re^{it} \in \Delta_\gamma$ and put $\varepsilon_\rho = 2\alpha(\rho-1)$. We have

$$\begin{aligned} r|f(re^{it}) - f(e^{it})|^{2\rho} &= r|f(re^{it}) - f(e^{it})|^{2\rho-2}|f(re^{it}) - f(e^{it})|^2 \\ &\le r(1-r)^{1+\varepsilon_\rho} \int_r^1 |f'(se^{it})|^2 ds \\ &\le (1-r)^{1+\varepsilon_\rho} \int_0^1 |f'(se^{it})|^2 sds. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\Delta_\gamma} \frac{|f(z) - f(z/|z|)|^{2\rho}}{(1-|z|)^2} dA(z) &= \int_0^1 \Big(\int_\gamma |f(re^{it}) - f(e^{it})|^{2\rho} \frac{rdt}{\pi} \Big) \frac{dr}{(1-r)^2} \\ &\le \|f'\|^2_{L^2(\Delta_\gamma)} \int_0^1 \frac{1}{(1-r)^{1-\varepsilon_\rho}} dr. \end{aligned}$$

This completes the proof. ∎

Now, we can state the following result.

Lemma 4.3

$$\int_{\Delta_\gamma^1} |f(z)|^{2\rho} |f_\Gamma'(z)|^2 dA(z) \le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)}.$$

Proof By Cauchy's estimate, it follows that $|f_\Gamma'(re^{it})| \le \frac{1}{1-r}$. Using Lemma 4.2, we get

$$\begin{aligned} \text{(4.6)} \qquad \int_{\Delta_\gamma^1} |f(z)|^{2\rho} |f_\Gamma'(z)|^2 dA(z) &\le \int_{\Delta_\gamma^1} \frac{|f(z)|^{2\rho}}{(1-|z|)^2} dA(z) \\ &\le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)} + 2^{2\rho-1} \int_{\Delta_\gamma^1} \frac{|f(z/|z|)|^{2\rho}}{(1-|z|)^2} dA(z). \end{aligned}$$

Using Lemma 4.1, we obtain

$$\begin{aligned} \text{(4.7)} \qquad \int_{\Delta_\gamma^1} \frac{|f(z/|z|)|^{2\rho}}{(1-|z|)^2} dA(z) &= \frac{1}{\pi} \int_{\Delta_\gamma^1} \frac{|f(e^{it})|^{2\rho}}{(1-r)^2} rdrdt \\ &\le \frac{C}{\pi} \int_\gamma \frac{|f(e^{it})|^{2\rho}}{d(e^{it})} dt \le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)}. \end{aligned}$$

The result of our lemma follows by combining the estimates (4.6) and (4.7). ∎

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4.2 The Integral on the Region Δ_γ^2

In this subsection, we estimate the integral $\int_{\Delta_\gamma^2} |f(z)|^{2\rho}|f_\Gamma'(z)|^2 dA(z)$. Before this, we make some remarks. For $z \in \mathbb{D}$ define

$$a_\gamma(z) := \begin{cases} \frac{1}{2\pi}\int_\Gamma \frac{-\log|f(e^{i\theta})|}{|e^{i\theta}-z|^2} d\theta & \text{if } \gamma \not\subseteq \Gamma, \\ \frac{1}{2\pi}\int_{\mathbb{T}\setminus\Gamma} \frac{-\log|f(e^{i\theta})|}{|e^{i\theta}-z|^2} d\theta & \text{if } \gamma \subseteq \Gamma. \end{cases}$$

Using equation (4.4), it is easy to see that

$$|f(z)^\rho f_\Gamma'(z)|^2 \le 4 \left| f(z)^\rho \frac{1}{2\pi}\int_\Gamma \frac{-\log|f(e^{i\theta})|}{|e^{i\theta}-z|^2} d\theta \right|^2 .$$

Using equation (4.5), it is clear that

$$|f(z)^\rho f_\Gamma'(z)|^2 \le 2|f'(z)|^2 + 8 \left| f(z)^\rho \frac{1}{2\pi}\int_{\mathbb{T}\setminus\Gamma} \frac{-\log|f(e^{i\theta})|}{|e^{i\theta}-z|^2} d\theta \right|^2 .$$

Then

$$\int_{\Delta_\gamma^2} |f(z)|^{2\rho}|f_\Gamma'(z)|^2 dA(z) \le 2\|f'\|^2_{L^2(\Delta_\gamma)} + 8\int_{\Delta_\gamma^2} |f(z)|^{2\rho} a_\gamma^2(z) dA(z). \tag{4.8}$$

Since $\log|f| \in L^1(\mathbb{T})$, we have

$$a_\gamma(z) \le \frac{C}{d^2(z)} \qquad (z \in \Delta_\gamma). \tag{4.9}$$

Given such inequality, it is not easy to estimate immediately the integral of the function $|f(z)|^{2\rho}a_\gamma^2(z)$ on the whole Δ_γ^2. In what follows, we give a partition of Δ_γ^2 into three parts so that one can estimate the integral $\int |f(z)|^{2\rho}a_\gamma^2(z)dA(z)$ on each part. Let $z \in \Delta_\gamma^2$; three situations are possible:

$$a_\gamma(z) \le 8\frac{|\log(d(z))|}{d(z)}, \tag{4.10}$$

$$8\frac{|\log(d(z))|}{d(z)} < a_\gamma(z) < 8\frac{|\log(d(z))|}{1-r}, \tag{4.11}$$

$$8\frac{|\log(d(z))|}{1-r} \le a_\gamma(z). \tag{4.12}$$

We can now divide Δ_γ^2 into the following three parts

$$\Delta_\gamma^{21} := \left\{ z \in \Delta_\gamma^2 : z \text{ satisfying } (4.10) \right\},$$
$$\Delta_\gamma^{22} := \left\{ z \in \Delta_\gamma^2 : z \text{ satisfying } (4.11) \right\},$$
$$\Delta_\gamma^{23} := \left\{ z \in \Delta_\gamma^2 : z \text{ satisfying } (4.12) \right\}.$$

4.2.1 The Integral on the Regions Δ_γ^{21} and Δ_γ^{23}

In this case we begin with the following.

Lemma 4.4

$$\int_{\Delta_\gamma^{21}} |f(z)|^{2\rho} a_\gamma^2(z)\,dA(z) \le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)}.$$

Proof Using Lemma 4.2, we get

$$\begin{aligned}
\int_{\Delta_\gamma^{21}} |f(z)|^{2\rho} a_\gamma^2(z)\,dA(z)
&\le 2^\rho \int_{\Delta_\gamma^{21}} |f(z)|^{\rho-1} |f(z) - f(z/|z|)|^{\rho+1} a_\gamma^2(z)\,dA(z) \\
&\qquad + 2^\rho \int_{\Delta_\gamma^{21}} |f(z)|^{\rho-1} |f(z/|z|)|^{\rho+1} a_\gamma^2(z)\,dA(z) \\
&\le C_\rho \int_{\Delta_\gamma} \frac{|f(z) - f(z/|z|)|^{\rho+1}}{(1-|z|)^2}\,dA(z) + C_\rho \int_{\Delta_\gamma^{21}} \frac{|f(e^{it})|^{\rho+1}}{d^2(e^{it})}\,rdrdt \\
&\le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)} + C_\rho \int_{\Delta_\gamma^{21}} \frac{|f(e^{it})|^{\rho+1}}{d^2(e^{it})}\,drdt = I_{2,1}.
\end{aligned}$$

Let $e^{it} \in \gamma$ and denote by ζ_t the point of $\partial\Delta_\gamma^2 \cap \mathbb{D}$ such that $\zeta_t/|\zeta_t| = e^{it}$. We have

$$|e^{it} - \zeta_t| = 1 - |\zeta_t| = \frac{d(\zeta_t)}{2} \le d(e^{it}).$$

Then

$$\begin{aligned}
\int_{\Delta_\gamma^{21}} \frac{|f(e^{it})|^{\rho+1}}{d^2(e^{it})}\,drdt &\le \int_{\Delta_\gamma^2} \frac{|f(e^{it})|^{\rho+1}}{d^2(e^{it})}\,drdt \\
&= \int_\gamma \frac{|f(e^{it})|^{\rho+1}}{d^2(e^{it})} \int_{|\zeta_t|}^1 drdt \le \int_\gamma \frac{|f(e^{it})|^{\rho+1}}{d(e^{it})}\,dt.
\end{aligned}$$

Using Lemma 4.1, we get $I_{2,1} \le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)}$. This proves the result. ∎

Lemma 4.5

$$\int_{\Delta_\gamma^{23}} |f(z)|^{2\rho} a_\gamma^2(z)\,dA(z) \le CA(\Delta_\gamma),$$

where $A(\Delta_\gamma)$ is the area measure of Δ_γ.

Proof Set

$$\Lambda_\gamma := \begin{cases} \Gamma & \text{for } \gamma \not\subseteq \Gamma, \\ \mathbb{T} \setminus \Gamma & \text{for } \gamma \subseteq \Gamma. \end{cases}$$

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Let $z \in \Delta_\gamma^{23}$. We have

$$
\begin{aligned}
|f(z)| &= \exp\left\{ \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|e^{i\theta}-z|^2}\log|f(e^{i\theta})|d\theta \right\} \\
&\leq \exp\left\{ \frac{1}{2\pi}\int_{\Lambda_\gamma} \frac{1-r}{|e^{i\theta}-z|^2}\log|f(e^{i\theta})|d\theta \right\} \\
&= \exp\left\{ -(1-r)a_\gamma(z) \right\} \leq d^8(z).
\end{aligned}
$$

Using (4.9), we obtain the result. ■

4.2.2 The Integral on the Region Δ_γ^{22}

Here, we will give an estimate of the following integral

$$\int_{\Delta_\gamma^{22}} |f(z)|^{2\rho} a_\gamma^2(z)\,dA(z).$$

Before doing this, we begin with some lemmas. The next one is essential for what follows. Note that a similar result is used by various authors: Korenblum [6], Matheson [7], Shamoyan [8], and Shirokov [9, 10].

Lemma 4.6 *Let $z \in \Delta_\gamma^{22}$ and let $\mu_z = 1 - \frac{8|\log(d(z))|}{a_\gamma(z)}$. Then*

$$|f(\mu_z z)| \leq d^2(z). \tag{4.13}$$

Proof Let $z \in \Delta_\gamma$ and let $\mu < 1$. We have

$$
\begin{aligned}
|f(\mu z)| &= \exp\left\{ \frac{1}{2\pi}\int_0^{2\pi} \frac{1-(\mu r)^2}{|e^{i\theta}-\mu z|^2}\log|f(e^{i\theta})|d\theta \right\} \\
&\leq \exp\left\{ \frac{1}{2\pi}\int_{\Lambda_\gamma} \frac{1-(\mu r)^2}{|e^{i\theta}-\mu z|^2}\log|f(e^{i\theta})|d\theta \right\} \\
&\leq \exp\left\{ -(1-\mu r)\inf_{\theta\in\Lambda_\gamma}\left|\frac{e^{i\theta}-z}{e^{i\theta}-\mu z}\right|^2 a_\gamma(z) \right\}.
\end{aligned}
$$

For $z \in \Delta_\gamma^{22}$, it is clear that $1-\mu_z \leq d(z) \leq |e^{i\theta}-z|$ for all $e^{i\theta} \in \Lambda_\gamma$. Then

$$\inf_{\theta\in\Lambda_\gamma}\left|\frac{e^{i\theta}-z}{e^{i\theta}-\mu_z z}\right| \geq \frac{1}{2} \qquad (z \in \Delta_\gamma^{22}).$$

Thus

$$|f(\mu_z z)| \leq \exp\left\{ -\frac{1-\mu_z}{4}a_\gamma(z) \right\} \qquad (z \in \Delta_\gamma^{22}).$$

Then, we have

$$|f(\mu_z z)| \leq \exp\left\{ -\frac{1}{4}(1-\mu_z)a_\gamma(z) \right\} = d^2(z) \qquad (z \in \Delta_\gamma^{22}),$$

which yields (4.13). ■

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For $r < 1$, define

$$\gamma_r := \{z \in \mathbb{D} : |z| = r \text{ and } z/|z| \in \gamma\}.$$

Without loss of generality, we can suppose that $d(z) \leq \frac{1}{2}$, $z \in \Delta_\gamma^2$. We need the following.

Lemma 4.7 *Let $r < 1$. Then*

$$\int_{\gamma_r \cap \Delta_\gamma^{22}} |f(re^{it}) - f(\mu_{re^{it}} re^{it})|^{2\rho} a_\gamma^2(re^{it}) r dt \leq \frac{C_\rho}{(1-r)^{1-\varepsilon_\rho}} \|f'\|^2_{L^2(\Delta_\gamma)},$$

where $\varepsilon_\rho = \alpha(\rho - 1)$.

Proof Let $re^{it} \in \Delta_\gamma^{22}$. Then

$$\begin{aligned} &|f(re^{it}) - f(\mu_{re^{it}} re^{it})|^{\rho-1}[(1-\mu_{re^{it}})a_\gamma(re^{it})]^2 \\ &\qquad \leq 64(1-\mu_{re^{it}})^{\varepsilon_\rho} \log^2(d(re^{it})) \leq C_\rho. \end{aligned}$$

It is clear that $1 - r \leq 1 - \mu_{re^{it}} \leq d(re^{it}) \leq \frac{1}{2}$ and so $\frac{1}{2} \leq \mu_{re^{it}} \leq r$. We have

$$\begin{aligned} &\int_{\gamma_r \cap \Delta_\gamma^{22}} |f(re^{it}) - f(\mu_{re^{it}} re^{it})|^{2\rho} a_\gamma^2(re^{it}) r dt \\ &\leq C_\rho \int_{\gamma_r \cap \Delta_\gamma^{22}} \frac{|f(re^{it}) - f(\mu_{re^{it}} re^{it})|^{\rho+1}}{(1-\mu_{re^{it}})^2} r dt \\ &\leq \frac{C_\rho}{(1-r)^{1-\varepsilon_\rho}} \int_{\gamma_r \cap \Delta_\gamma^{22}} \frac{|f(re^{it}) - f(\mu_{re^{it}} re^{it})|^2}{1-\mu_{re^{it}}} r dt \\ &\leq \frac{C_\rho}{(1-r)^{1-\varepsilon_\rho}} \int_{\gamma_r \cap \Delta_\gamma^{22}} \Big(\int_{\mu_{re^{it}} r}^{r} |f'(se^{it})|^2 ds \Big) r dt \\ &\leq \frac{C_\rho}{(1-r)^{1-\varepsilon_\rho}} \int_{S_r} |f'(se^{it})|^2 s ds dt \\ &\leq \frac{C_\rho}{(1-r)^{1-\varepsilon_\rho}} \int_{S_r} |f'(w)|^2 dA(w), \end{aligned}$$

where

$$S_r := \left\{ w \in \mathbb{D} : 0 \leq |w| \leq r \text{ and } \frac{w}{|w|} \in \gamma \right\}.$$

The proof is therefore completed. ■

The last result that we need before giving the proof of Theorem 2.1 is the following one.

Lemma 4.8

$$\int_{\Delta_\gamma^{22}} |f(z)|^{2\rho} a_\gamma^2(z) dA(z) \le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)} + CA(\Delta_\gamma).$$

Proof Using (4.9) and Lemmas 4.6 and 4.7, we find that

$$\begin{aligned}
&\int_{\Delta_\gamma^{22}} |f(z)|^{2\rho} a_\gamma^2(z) dA(z)\\
&\quad = \frac{1}{\pi} \int_0^1 \Big(\int_{\gamma_r \cap \Delta_\gamma^{22}} |f(re^{it})|^{2\rho} a_\gamma^2(re^{it}) r dt \Big)\, dr\\
&\quad \le CA(\Delta_\gamma) + 2^{2\rho-1} \int_0^1 \Big(\int_{\gamma_r \cap \Delta_\gamma^{22}} |f(re^{it}) - f(\mu_{re^{it}} re^{it})|^{2\rho} a_\gamma^2(re^{it}) r dt \Big)\, dr\\
&\quad \le CA(\Delta_\gamma) + C_\rho \|f'\|^2_{L^2(\Delta_\gamma)}.
\end{aligned}$$

This completes the proof of the lemma. ■

4.2.3 Conclusion

Now, according to (4.8) and Lemmas 4.4, 4.5, and 4.8, we obtain

$$\begin{aligned}
\int_{\Delta_\gamma^2} |f(z)|^{2\rho} |f_\Gamma'(z)|^2 dA(z) &\le 2\|f'\|^2_{L^2(\Delta_\gamma)} + 8 \int_{\Delta_\gamma^2} |f(z)|^{2\rho} a_\gamma^2(z) dA(z)\\
&\le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)} + CA(\Delta_\gamma).
\end{aligned}$$

Combining this with Lemma 4.3, we deduce that

$$\int_{\Delta_\gamma} |f(z)|^{2\rho} |f_\Gamma'(z)|^2 dA(z) \le C_\rho \|f'\|^2_{L^2(\Delta_\gamma)} + CA(\Delta_\gamma).$$

Hence

$$\begin{aligned}
\int_{\mathbb{D}} |f(z)|^{2\rho} |f_\Gamma'(z)|^2 dA(z) &= \sum_{n=1}^{\infty} \int_{\Delta_{\gamma_n}} |f(z)|^{2\rho} |f_\Gamma'(z)|^2 dA(z)\\
&\le C_\rho \sum_{n=1}^{\infty} \|f'\|^2_{L^2(\Delta_{\gamma_n})} + C \sum_{n=1}^{\infty} A(\Delta_{\gamma_n}) \le C_\rho.
\end{aligned}$$

This completes the proof of Theorem 2.1. ■

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