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ON COSINE FAMILIES CLOSE TO SCALAR COSINE FAMILIES

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Abstract

We prove that if two normed-algebra-valued cosine families indexed by a single Abelian group, of which one is bounded and comprised solely of scalar elements of the underlying algebra, differ in norm by less than 1 uniformly in the parametrising index, then these families coincide.

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1. Introduction

A classic result, in its early form due to Cox [2], states that if $\mathbf{A}$ is a normed algebra with a unity denoted by 1 and a is an element of $\mathbf{A}$ such that $\sup_{n\in\mathbb{N}} \|a^n - 1\| < 1$, then $a = 1$. Cox's version concerned the case of square matrices of a given size. This was later extended to bounded operators on Hilbert space by Nakamura and Yoshida [6], and to an arbitrary normed algebra by Hirschfeld [4] and Wallen [9]. The latter author in fact proved a stronger result, namely that $\|a^n - 1\| = o(n)$ and $\liminf_{n\to\infty} n^{-1}(\|a-1\| + \|a^2 - 1\| + \cdots + \|a^n - 1\|) < 1$ imply $a = 1$, and he achieved this by using an elementary argument.

An immediate consequence of the Cox–Nakamura–Yoshida–Hirschfeld–Wallen theorem is that if $\{S(t)\}_{t\geq 0}$ is a semigroup on a Banach space X such that

$$\sup_{t\geq 0} \|S(t) - I_X\| < 1,$$

then $S(t) = I_X$ for each $t \geq 0$; here I_X denotes the identity operator on X. Recently, Bobrowski and Chojnacki [1] established an analogue of this result for one-parameter cosine families: if $a \in \mathbb{R}$ and $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family on a Banach space X such that

$$\sup_{t\in\mathbb{R}} \|C(t) - (\cos at)I_X\| < \tfrac{1}{2}, \tag{1.1}$$

then $C(t) = (\cos at)I_X$ for each $t \in \mathbb{R}$. This conclusion was further refined by Schwenninger and Zwart [7] who showed that condition (1.1) can be replaced by the condition

$$\sup_{t\in\mathbb{R}} \|C(t) - (\cos at)I_X\| < 1.$$

For the case $a = 0$, the same authors later showed that the even weaker condition

$$\sup_{t\in\mathbb{R}} \|C(t) - I_X\| < 2$$

suffices [8]. The result of Bobrowski and Chojnacki and those of Schwenninger and Zwart rely on rather involved arguments, drawing on ideas from operator theory and semigroup theory.

In this note we extend the first result of Schwenninger and Zwart (that is, the result of [7]) to cover the case of cosine families that are not necessarily indexed by real numbers, not necessarily operator-valued, and not necessarily continuous in any particular sense. A crucial step towards proving the relevant result will be the establishment of an analogue of the Cox–Nakamura–Yoshida–Hirschfeld–Wallen theorem for cosine sequences. The proof of the latter result will use only elementary means.

2. Preliminaries and results

Let $\mathbf{A}$ be a normed algebra, real or complex, with a unity 1. An element of $\mathbf{A}$ is called *scalar* if it is a scalar multiple of the unity 1. A family in $\mathbf{A}$ is termed *scalar* if every member of this family is scalar. Given a scalar λ, the symbol λ will be employed to denote both the scalar itself and the element of $\mathbf{A}$ obtained by multiplying the unity element of $\mathbf{A}$ by λ. In particular, if $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ is a family of scalars, then $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ will also denote the corresponding family of scalar elements of $\mathbf{A}$.

We recall that an $\mathbf{A}$-valued family $\{a_\lambda\}_{\lambda\in\Lambda}$ is said to be *bounded* if $\sup_{\lambda\in\Lambda} \|a_\lambda\| < \infty$.

Let G be an Abelian group, written additively, with a neutral element 0. A family $\{C(g)\}_{g\in G}$ in $\mathbf{A}$ is called a *cosine family* if

(i) $2C(g)C(h) = C(g + h) + C(g - h)$ for all $g, h \in G$ (d'Alembert's functional equation, also called the cosine functional equation),

(ii) $C(0) = 1$.

With this minimum of preparation, we are ready to state the main result of the paper.

THEOREM 2.1. *Let $\mathbf{A}$ be a normed algebra with a unity 1, let G be an Abelian group, let $\{c(g)\}_{g\in G}$ be a bounded scalar-valued cosine family, and let $\{C(g)\}_{g\in G}$ be an $\mathbf{A}$-valued cosine family such that*

$$\sup_{g\in G} \|C(g) - c(g)\| < 1.$$

Then $C(g) = c(g)$ for each $g \in G$.

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We shall deduce this theorem from a seemingly weaker result presented below as Theorem 2.2.

We continue with preliminary definitions and results. A cosine sequence is a cosine family for which the indexing group is the additive group of integers $\mathbb{Z}$. Every cosine sequence $\{c_n\}_{n\in\mathbb{Z}}$ is even: the equality $c_{-n} = c_n$ holds for all $n \in \mathbb{Z}$. Furthermore, every cosine sequence $\{c_n\}_{n\in\mathbb{Z}}$ is uniquely determined by its element indexed by 1, namely,

$$c_n = T_{|n|}(c_1) \quad (n \in \mathbb{Z}),$$

where, for $n \in \mathbb{N} \cup \{0\}$, $T_n(x)$ is the nth Chebyshev polynomial of the first kind,

$$T_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{n-2k}(x^2 - 1)^k.$$

The element c_1 of a cosine sequence $\{c_n\}_{n\in\mathbb{Z}}$ is commonly termed the *generator* of the sequence. Every element of $\mathbf{A}$ generates a unique cosine sequence. The cosine sequence generated by $a \in \mathbf{A}$ is given by $c_n(a) = T_{|n|}(a)$ for every $n \in \mathbb{Z}$.

For each $\gamma \in \mathbb{C} \setminus \{0\}$, let $\{c_n^{(\gamma)}\}_{n\in\mathbb{Z}}$ be the $\mathbb{C}$-valued cosine sequence given by

$$c_n^{(\gamma)} = \frac{\gamma^n + \gamma^{-n}}{2} \quad (n \in \mathbb{Z}).$$

Denote by $\mathbb{T}$ the unit circle in the complex plane. If $\gamma \in \mathbb{T}$, then $c_n^{(\gamma)} = \operatorname{Re} \gamma^n$ for all $n \in \mathbb{Z}$, so that all elements of $\{c_n^{(\gamma)}\}_{n\in\mathbb{Z}}$ are real numbers with modulus no greater than 1.

We are now ready to present the result that is the main ingredient needed to prove Theorem 2.1.

THEOREM 2.2. *Let $\mathbf{A}$ be a normed algebra with a unity 1, let $\gamma \in \mathbb{T}$, and let $\{c_n\}_{n\in\mathbb{Z}}$ be an $\mathbf{A}$-valued cosine sequence such that*

$$\sup_{n\in\mathbb{Z}} \|c_n - c_n^{(\gamma)}\| < 1.$$

Then $c_n = c_n^{(\gamma)}$ for each $n \in \mathbb{Z}$.

Theorem 2.2 can be viewed as a counterpart of the Cox–Nakamura–Yoshida–Hirschfeld–Wallen theorem for cosine sequences. Its proof will be much in the spirit of the work of Wallen, although the details will be more complicated.

3. Proof of Theorem 2.2

This section is devoted to proving Theorem 2.2. We begin by establishing a key algebraic identity.

Let $\mathbf{A}$ be a normed algebra with a unity 1, let $\gamma \in \mathbb{T}$, and let $\{c_n\}_{n\in\mathbb{Z}}$ be an $\mathbf{A}$-valued cosine sequence. Then

$$2(c_1^{(\gamma)} - c_1) \sum_{k=0}^{n-1} \gamma^k c_k = \gamma^n c_{n-1} - \gamma^{n-1} c_n - c_1 + \gamma^{-1}. \tag{3.1}$$

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Indeed, by the cosine functional equation,

$$2c_1 \sum_{k=0}^{n-1} \gamma^k c_k = \sum_{k=0}^{n-1} \gamma^k c_{k-1} + \sum_{k=0}^{n-1} \gamma^k c_{k+1} = \sum_{k=-1}^{n-2} \gamma^{k+1} c_k + \sum_{k=1}^{n} \gamma^{k-1} c_k.$$

On the other hand,

$$2c_1^{(\gamma)} \sum_{k=0}^{n-1} \gamma^k c_k = (\gamma + \gamma^{-1}) \sum_{k=0}^{n-1} \gamma^k c_k = \sum_{k=0}^{n-1} \gamma^{k+1} c_k + \sum_{k=0}^{n-1} \gamma^{k-1} c_k.$$

Hence

$$\begin{aligned}2(c_1^{(\gamma)} - c_1) \sum_{k=0}^{n-1} \gamma^k c_k &= \Big(\sum_{k=0}^{n-1} - \sum_{k=-1}^{n-2}\Big)\gamma^{k+1} c_k + \Big(\sum_{k=0}^{n-1} - \sum_{k=1}^{n}\Big)\gamma^{k-1} c_k \\ &= \gamma^n c_{n-1} - c_{-1} - \gamma^{n-1} c_n + \gamma^{-1},\end{aligned}$$

which, given that $c_{-1} = c_1$, immediately yields (3.1).

We next observe that the sequence $\{c_n\}_{n\in\mathbb{Z}}$ from the statement of Theorem 2.2 is necessarily bounded, because the sequence $\{c_n^{(\gamma)}\}_{n\in\mathbb{Z}}$ from the same statement is bounded and $\sup_{n\in\mathbb{Z}} \|c_n - c_n^{(\gamma)}\| < 1$.

Lemma 3.1. *Under the assumptions of Theorem 2.2, if, for each $n \in \mathbb{N}$, we put*

$$P_n := \frac{1}{n} \sum_{k=0}^{n-1} c_k,$$

then $P_n = 1$ for each $n \in \mathbb{N}$ if $\gamma = 1$, and $\lim_{n\to\infty} P_n = 0$ if $\gamma \neq 1$.

Proof. Let $0 < \delta < 1$ be such that $\|c_n - c_n^{(\gamma)}\| \le \delta$ for each $n \in \mathbb{Z}$. We break the proof up into three cases.

Case $\gamma = 1$. Assuming $\gamma = 1$ in (3.1), we obtain, for each $n \in \mathbb{N}$,

$$2(1 - c_1) \sum_{k=0}^{n-1} c_k = c_{n-1} - c_n - c_1 + 1.$$

We can rewrite this as

$$(1 - c_1)P_n = e_n \quad \text{with } e_n = \frac{1}{2n}(c_{n-1} - c_n - c_1 + 1). \tag{3.2}$$

Note that $\lim_{n\to\infty} e_n = 0$, as $\{c_n\}_{n\in\mathbb{Z}}$ is bounded. Given that $c_k^{(1)} = 1$ for every $k \in \mathbb{Z}$, we have, for each $n \in \mathbb{N}$,

$$1 - P_n = \frac{1}{n} \sum_{k=0}^{n-1} (c_k^{(1)} - c_k)$$

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and hence

$$\|1 - P_n\| \leq \frac{1}{n}\sum_{k=0}^{n-1}\|c_k^{(1)} - c_k\| \leq \delta.$$

Writing

$$1 - c_1 = (1 - c_1)(1 - P_n) + e_n,$$

we find that

$$\|1 - c_1\| \leq \delta\|1 - c_1\| + \|e_n\|,$$

whence, letting $n \to \infty$,

$$\|1 - c_1\| \leq \delta\|1 - c_1\|.$$

As $\delta < 1$, we see that $\|1 - c_1\| = 0$ and further that $c_1 = 1$. Consequently, $c_n = 1$ for each $n \in \mathbb{Z}$, and thus $P_n = 1$ for each $n \in \mathbb{N}$.

Case $\gamma = -1$. Assuming $\gamma = -1$ in (3.1), we deduce that, for each $n \in \mathbb{N}$,

$$2(1 + c_1)\sum_{k=0}^{n-1}(-1)^k c_k = (-1)^{n-1}(c_n + c_{n-1}) + c_1 + 1.$$

Letting

$$Q_n := \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k c_k,$$

we see that

$$(1 + c_1)Q_n = f_n \quad \text{with } f_n = \frac{1}{2n}((-1)^{n-1}(c_n + c_{n-1}) + c_1 + 1).$$

Clearly, $\lim_{n\to\infty} f_n = 0$. Taking into account that $c_k^{(-1)} = (-1)^k$ for every $k \in \mathbb{Z}$, we have, for each $n \in \mathbb{N}$,

$$1 - Q_n = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(c_k^{(-1)} - c_k)$$

and hence

$$\|1 - Q_n\| \leq \frac{1}{n}\sum_{k=0}^{n-1}\|c_k^{(-1)} - c_k\| \leq \delta.$$

Writing

$$1 + c_1 = (1 + c_1)(1 - Q_n) + f_n,$$

we see that

$$\|1 + c_1\| \leq \delta\|1 + c_1\| + \|f_n\|,$$

whence, letting $n \to \infty$,

$$\|1 + c_1\| \leq \delta\|1 + c_1\|.$$

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As $\delta < 1$, we conclude that $\|1 + c_1\| = 0$ and further that $c_1 = -1$. Consequently, $c_n = (-1)^n$ for each $n \in \mathbb{Z}$, and thus

$$P_n = \frac{1 + (-1)^{n-1}}{2n}$$

for each $n \in \mathbb{N}$, which immediately yields $\lim_{n\to\infty} P_n = 0$.

Case $\gamma \notin \{-1, 1\}$. In this case $\operatorname{Im}\gamma \neq 0$, and, as $|c_1^{(\gamma)}| = |\operatorname{Re}\gamma| = \sqrt{1 - (\operatorname{Im}\gamma)^2}$, we have $|c_1^{(\gamma)}| < 1$. Let $\epsilon > 0$ be such that $\delta + \epsilon < 1$. Choose $l \in \mathbb{N}$ sufficiently large so that $|c_1^{(\gamma)}|^l < \epsilon$. It is readily proved by induction that

$$c_1^l = \frac{1}{2^{l-1}} \sum_{k=0}^{(l-1)/2} \binom{l}{k} c_{l-2k}$$

if l is odd, and

$$c_1^l = \frac{1}{2^l}\binom{l}{\frac{l}{2}} + \frac{1}{2^{l-1}} \sum_{k=0}^{(l/2)-1} \binom{l}{k} c_{l-2k}$$

if l is even, with similar formulae holding for $(c_1^{(\gamma)})^l$ (see [3, formulae 1.320, 5 and 1.320, 7]). Hence

$$\|c_1^l - (c_1^{(\gamma)})^l\| \leq \frac{1}{2^{l-1}} \sum_{k=0}^{(l-1)/2} \binom{l}{k} \|c_{l-2k} - c_{l-2k}^{(\gamma)}\| \leq \delta$$

if l is odd, and

$$\|c_1^l - (c_1^{(\gamma)})^l\| \leq \frac{1}{2^{l-1}} \sum_{k=0}^{(l/2)-1} \binom{l}{k} \|c_{l-2k} - c_{l-2k}^{(\gamma)}\| \leq \left(1 - \frac{1}{2^l}\binom{l}{\frac{l}{2}}\right)\delta < \delta$$

if l is even. In either case $\|c_1^l - (c_1^{(\gamma)})^l\| \leq \delta$. It follows that $\|c_1^l\| \leq \|c_1^l - (c_1^{(\gamma)})^l\| + |c_1^{(\gamma)}|^l \leq \delta + \epsilon$. At this stage, we shall exploit (3.2) once again. Multiplying both sides of $(1 - c_1)P_n = e_n$ by $1 + c_1 + \cdots + c_1^{l-1}$, we get

$$(1 - c_1^l)P_n = (1 + c_1 + \cdots + c_1^{l-1})e_n,$$

or equivalently,

$$P_n = c_1^l P_n + (1 + c_1 + \cdots + c_1^{l-1})e_n.$$

Hence

$$\|P_n\| \leq (\delta + \epsilon)\|P_n\| + l \max\{1, \|c_1\|^{l-1}\}\|e_n\|,$$

and, as $\lim_{n\to\infty} e_n = 0$,

$$\limsup_{n\to\infty} \|P_n\| \leq (\delta + \epsilon) \limsup_{n\to\infty} \|P_n\|.$$

Remembering that $\delta + \epsilon < 1$, we conclude that $\limsup_{n\to\infty} \|P_n\| = 0$, whence $\lim_{n\to\infty} P_n = 0$. □

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We now proceed to the proof of Theorem 2.2 proper.

PROOF OF THEOREM 2.2. By the cosine functional equation,

$$c_k^2 = \tfrac{1}{2}(1 + c_{2k}) \quad \text{and} \quad (c_k^{(\gamma)})^2 = \tfrac{1}{2}(1 + c_{2k}^{(\gamma)})$$

for every $k \in \mathbb{Z}$. Hence, for each $n \in \mathbb{N}$,

$$\begin{aligned}(c_1 - c_1^{(\gamma)}) \sum_{k=0}^{n-1} (c_k - c_k^{(\gamma)})^2 &= (c_1 - c_1^{(\gamma)}) \sum_{k=0}^{n-1} (c_k^2 + (c_k^{(\gamma)})^2 - 2c_k^{(\gamma)} c_k) \\ &= (c_1 - c_1^{(\gamma)}) \Big[n + \frac{1}{2} \sum_{k=0}^{n-1} c_{2k} + \frac{1}{2} \sum_{k=0}^{n-1} c_{2k}^{(\gamma)} - 2 \sum_{k=0}^{n-1} c_k^{(\gamma)} c_k \Big]. \quad (3.3)\end{aligned}$$

As a first step in exploiting the above relation, we replace γ by γ^{-1} in (3.1), whereupon, taking into account that $c_1^{(\gamma)} = c_1^{(\gamma^{-1})}$, we find that

$$2(c_1^{(\gamma)} - c_1) \sum_{k=0}^{n-1} \gamma^{-k} c_k = \gamma^{-n} c_{n-1} - \gamma^{1-n} c_n - c_1 + \gamma$$

for each $n \in \mathbb{N}$. Adding this identity to (3.1) and dividing by 2 yields

$$2(c_1^{(\gamma)} - c_1) \sum_{k=0}^{n-1} c_k^{(\gamma)} c_k = c_n^{(\gamma)} c_{n-1} - c_{n-1}^{(\gamma)} c_n - c_1 + c_1^{(\gamma)}.$$

Hence, as both $\{c_n\}_{n \in \mathbb{Z}}$ and $\{c_n^{(\gamma)}\}_{n \in \mathbb{Z}}$ are bounded,

$$\lim_{n \to \infty} \frac{c_1 - c_1^{(\gamma)}}{n} \sum_{k=0}^{n-1} c_k^{(\gamma)} c_k = 0. \quad (3.4)$$

Next, we apply Lemma 3.1 to the cosine sequences $\{c_{2n}\}_{n \in \mathbb{Z}}$ and $\{c_{2n}^{(\gamma)}\}_{n \in \mathbb{Z}} = \{c_n^{(\gamma^2)}\}_{n \in \mathbb{Z}}$, obtaining

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} c_{2k} = \begin{cases} 1 & \text{if } \gamma^2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By applying Lemma 3.1 to two copies of $\{c_{2n}^{(\gamma)}\}_{n \in \mathbb{Z}}$, or, alternatively, by taking into account that

$$\sum_{k=0}^{n-1} c_{2k}^{(\gamma)} = \frac{1}{2} \Big[\sum_{k=0}^{n-1} \gamma^{2k} + \sum_{k=0}^{n-1} \gamma^{-2k} \Big] = \begin{cases} 1 & \text{if } \gamma^2 = 1, \\ \dfrac{1 - \gamma^{2n}}{2(1 - \gamma^2)} + \dfrac{1 - \gamma^{-2n}}{2(1 - \gamma^{-2})} & \text{otherwise,} \end{cases}$$

we also get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} c_{2k}^{(\gamma)} = \begin{cases} 1 & \text{if } \gamma^2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Hence

$$\lim_{n\to\infty} \frac{c_1 - c_1^{(\gamma)}}{n}\Big[n + \frac{1}{2}\sum_{k=0}^{n-1} c_{2k} + \frac{1}{2}\sum_{k=0}^{n-1} c_{2k}^{(\gamma)}\Big] = \begin{cases} 2(c_1 - c_1^{(\gamma)}) & \text{if } \gamma^2 = 1,\\ c_1 - c_1^{(\gamma)} & \text{otherwise.}\end{cases}$$

If we now combine the above relation with (3.3) and (3.4), we find that

$$\lim_{n\to\infty} \frac{c_1 - c_1^{(\gamma)}}{n}\sum_{k=0}^{n-1}(c_k - c_k^{(\gamma)})^2 = \begin{cases} 2(c_1 - c_1^{(\gamma)}) & \text{if } \gamma^2 = 1,\\ c_1 - c_1^{(\gamma)} & \text{otherwise.}\end{cases}$$

Hence

$$\lim_{n\to\infty} \Big\| \frac{c_1 - c_1^{(\gamma)}}{n}\sum_{k=0}^{n-1}(c_k - c_k^{(\gamma)})^2 \Big\| \geq \|c_1 - c_1^{(\gamma)}\|.$$

On the other hand, if $0 < \delta < 1$ is such that $\|c_k - c_k^{(\gamma)}\| \leq \delta$ for every $k \in \mathbb{Z}$, then

$$\Big\| \frac{c_1 - c_1^{(\gamma)}}{n}\sum_{k=0}^{n-1}(c_k - c_k^{(\gamma)})^2 \Big\| \leq \frac{\|c_1 - c_1^{(\gamma)}\|}{n}\sum_{k=0}^{n-1}\|c_k - c_k^{(\gamma)}\|^2 \leq \delta^2\|c_1 - c_1^{(\gamma)}\|$$

for each $n \in \mathbb{N}$. Therefore,

$$\|c_1 - c_1^{(\gamma)}\| \leq \delta^2\|c_1 - c_1^{(\gamma)}\|,$$

which implies $\|c_1 - c_1^{(\gamma)}\| = 0$ and further $c_1 = c_1^{(\gamma)}$. Hence, finally, $c_n = c_n^{(\gamma)}$ for all $n \in \mathbb{Z}$. □

4. Proof of Theorem 2.1

Here we finally deduce Theorem 2.1 from Theorem 2.2.

Proof of Theorem 2.1. Fix $g \in G$ arbitrarily and define two sequences $\{c_n\}_{n\in\mathbb{Z}}$ and $\{\tilde{c}_n\}_{n\in\mathbb{Z}}$ by

$$c_n = C(ng) \quad \text{and} \quad \tilde{c}_n = c(ng)$$

for every $n \in \mathbb{Z}$. By a result of Kannappan [5], there exists $\gamma \in \mathbb{C} \setminus \{0\}$ such that $\tilde{c}_n = c_n^{(\gamma)}$ for all $n \in \mathbb{N}$. Now, γ has unit modulus, for otherwise, should $|\gamma| \neq 1$ hold, $\gamma^n + \gamma^{-n}$ would diverge in modulus to infinity as $n \to \infty$, contradicting the boundedness of $\{c(g)\}_{g\in G}$. Clearly,

$$\sup_{n\in\mathbb{Z}} \|c_n - c_n^{(\gamma)}\| < 1,$$

so we can apply Theorem 2.2 to conclude that $c_n = c_n^{(\gamma)}$ for all $n \in \mathbb{Z}$. In particular, $C(g) = c_1 = c_1^{(\gamma)} = c(g)$. As g was chosen arbitrarily, the theorem is established. □

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