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ON BEST SIMULTANEOUS APPROXIMATION IN NORMED LINEAR SPACES

BY

D. S. GOEL, A. S. B. HOLLAND, C. NASIM AND B. N. SAHNEY

1. Let S be a non-empty family of real valued continuous functions on $[a, b]$. Diaz and McLaughlin [1], [2], and Dunham [3] have considered the problem of simultaneously approximating two continuous functions f_1 and f_2 by elements of S. If $\|\cdot\|$ denotes the supremum norm, then the problem is to find an element $s^* \in S$, if it exists, for which

$$\max(\|f_1-s^*\|, \|f_2-s^*\|) = \inf_{s\in S} \max(\|f_1-s\|, \|f_2-s\|).$$

We shall study the above problem in general normed linear spaces.

Definition 1.1. Let X be a normed linear space and K a subset of X. Given any two elements $x_1, x_2 \in X$ define:

$$d(x_1, x_2; k) = \inf_{k\in K} \max(\|x_1-k\|, \|x_2-k\|).$$

An element $k^* \in K$ is said to be a best simultaneous approximation to x_1 and x_2 if:

$$d(x_1, x_2; k) = \max(\|x_1-k^*\|, \|x_2-k^*\|)$$

2. First we show that the best simultaneous approximation exists if the set K is a finite dimensional subspace of the normed linear space X.

Lemma 2.1. *Let $x_1, x_2 \in X$ and let $k \in X$. Then $\phi(k) \equiv \max(\|x_1-k\|, \|x_2-k\|)$ is a continuous functional on X.*

Proof. Since the norms $\|x_1-k\|$, $\|x_2-k\|$ are continuous functionals of k on X, $\phi(k)$ is clearly a continuous functional.

Lemma 2.2. *If K is a finite dimensional subspace of a normed linear space X, then there exists a best simultaneous approximation $k^* \in K$ to $x_1, x_2 \in X$.*

Proof. Let $\rho = \max(\|x_1\|, \|x_2\|)$. Consider the spheres $S(x_1, \rho)$, $S(x_2, \rho)$ in K and write:

$$S = S(x_1, \rho) \cup S(x_2, \rho).$$

Then

$$\inf_{k\in S} \max(\|x_1-k\|, \|x_2-k\|) = \inf_{k\in K} \max(\|x_1-k\|, \|x_2-k\|) \leq \rho.$$

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Since S is compact, the continuous functional $\phi(k)$ defined on S attains its minimum over S. If $\min \phi(k)=\phi(k^*)$ then the element k^* is a best simultaneous approximation to x_1 and x_2, and the lemma is proved.

LEMMA 2.3. *Let K be a convex subset of X, and $x_1, x_2 \in X$. If $k_1, k_2 \in K$ are best simultaneous approximations to x_1, x_2 by the elements of K, then: $\lambda k_1+(1-\lambda)k_2=\bar{k} \in K$, $0\leq\lambda\leq 1$, is also a best simultaneous approximation to x_1, x_2.*

Proof. Since

$$\begin{aligned}
&\max(\|x_1-\bar{k}\|, \|x_2-\bar{k}\|)\\
&= \max(\|\lambda(x_1-k_1)+(1-\lambda)(x_1-k_2)\|, \|\lambda(x_2-k_1)+(1-\lambda)(x_2-k_2)\|\\
&\leq \max(\lambda \|x_1-k_1\|+(1-\lambda) \|x_1-k_2\|, \lambda \|x_2-k_1\|+(1-\lambda) \|x_2-k_2\|)\\
&\leq \lambda \max(\|x_1-k_1\|, \|x_2-k_1\|)+(1-\lambda)\max(\|x_1-k_2\|, \|x_2-k_2\|)\\
&\leq \lambda d(x_1, x_2; k)+(1-\lambda)d(x_1, x_2; k)\\
&= d(x_1, x_2; k)
\end{aligned}$$

and the reverse inequality always holds, we conclude that:

$$\max(\|x_1-\bar{k}\|, \|x_2-\bar{k}\|) = d(x_1, x_2, k).$$

3. If K is a subspace of a strictly convex normed linear space X, then it is known that there is at most one best approximation to any element $x \in X-K$. In this section we shall prove a similar result for the best simultaneous approximation.

PROPOSITION 3.1. *Let K be a subspace of a strictly convex normed linear space X. Then there is at most one best simultaneous approximation from the elements of K, to any two elements $x_1, x_2 \in X$.*

Proof. Suppose k_1 and k_2 are best simultaneous approximations to x_1, x_2. Let $d=\max(\|x_1-k_i\|, \|x_2-k_i\|)$, $(i=1, 2)$. Then there are two cases to consider.

(a) Let $\|x_1-k_1\|=d$ and $\|x_2-k_1\|=l<d$ (or vice-versa), and write $d-l=\varepsilon$. We can find a convex neighbourhood $U\subset K$ of k_1 such that:

$$d-\varepsilon/4 \leq \|x_1-k\| \leq d+\varepsilon/4$$

and

$$l-\varepsilon/4 \leq \|x_2-k\| \leq l+\varepsilon/4, \qquad \forall\, k \in U.$$

Thus $\max(\|x_1-k\|, \|x_2-k\|)=\|x_1-k\|$ whenever $k \in U$. Further, $\|x_1-k\|\geq d$. The element $\bar{k}=\lambda k_2+(1-\lambda)k_1 \in U$ provided λ is sufficiently small and non-zero. Since $\bar{k}$ is also a best simultaneous approximation by lemma 2.3, we have $\|x_1-\bar{k}\|= d$. However $\|x_1-k_1\|=d$ and $\|x_1-(k_1+\bar{k})/2\|=d$. From these last three relations and the strict convexity of the norm we deduce that $k_1=\bar{k}$, thus $k_1=k_2$.

(b) Assume $\|x_1-k_1\|=\|x_2-k_1\|=d$, and also $\|x_1-k_2\|=\|x_2-k_2\|=d$ (if not then the previous argument holds). Write: $\bar{k}=(k_1+k_2)/2$, then there are three possibilities, either

(i) $\|x_1-\bar{k}\|=\|x_2-\bar{k}\|=d$,
(ii) $\|x_1-\bar{k}\|=d$ and $\|x_2-\bar{k}\|<d$, or
(iii) $\|x_1-\bar{k}\|<d$, $\|x_2-\bar{k}\|=d$.

In all the three cases we have either:

$$\|x_1-k_1\| = \|x_1-k_2\| = \|x_1-(k_1+k_2)/2\|, \quad \text{or}$$
$$\|x_2-k_1\| = \|x_2-k_2\| = \|x_2-(k_1+k_2)/2\|$$

or both. Using the strict convexity of the norm we deduce that $k_1=k_2$.

4. Let K be a closed and convex subset of a Banach Space X. If X is uniformly convex, then every element in X has a unique best approximation from the elements of K. In this section we show that a similar result holds for best simultaneous approximation.

PROPOSITION 4.1. *Let K and X be as above, then any two elements $x_1, x_2 \in X$ have a unique best simultaneous approximation from the elements of K.*

Proof. Let

$$d = \inf_{k\in K} \max(\|x_1-k\|, \|x_2-k\|)$$

and $\{k_n\}$ be a sequence of elements in K such that:

$$\lim_{n\to\infty} \max(\|x_1-k_n\|, \|x_2-k_n\|) \to d.$$

We can assume without loss of generality that $d=1$.

Let $d_m=\max(\|x_1-k_m\|, \|x_2-k_m\|)$, then $d_m\geq 1$ and

$$\frac{\|x_i-k_m\|}{d_m} \leq 1. \tag{4.1}$$

Consider

$$\frac{1}{2}\left(\frac{k_m}{d_m}+\frac{k_n}{d_n}\right) = \frac{d_nk_m+d_mk_n}{d_m+d_n}\cdot\frac{d_m+d_n}{2d_md_n}$$

and write

$$y_{mn} = \frac{d_nk_m+d_mk_n}{d_m+d_n}.$$

Since K is convex $y_{mn}\in K$. Hence $\max(\|x_1-y_{mn}\|, \|x_2-y_{mn}\|)\geq 1$ and consequently

$$\max\left(\left\|\frac{d_m+d_n}{2d_md_n}x_1-\frac{1}{2}\left(\frac{k_m}{d_m}+\frac{k_n}{d_n}\right)\right\|, \left\|\frac{d_m+d_n}{2d_md_n}x_2-\frac{1}{2}\left(\frac{k_m}{d_m}+\frac{k_n}{d_n}\right)\right\|\right)$$
$$= \max(\|x_1-y_{mn}\|, \|x_2-y_{mn}\|)\frac{d_m+d_n}{2d_md_n} \geq \frac{d_m+d_n}{2d_md_n}.$$

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Therefore at least one of the following is true:

$$\left\|\frac{x_1-k_m}{d_m}+\frac{x_1-k_n}{d_n}\right\| \ge \frac{d_m+d_n}{d_m d_n} \tag{4.2}$$

$$\left\|\frac{x_2-k_m}{d_m}+\frac{x_2-k_n}{d_n}\right\| \ge \frac{d_m+d_n}{d_m d_n}. \tag{4.3}$$

Suppose (4.2) is true, then, from (4.1) and the uniform convexity of the norm it follows that for any given $\varepsilon>0$, there exists a N such that

$$\left\|\frac{x_1-k_m}{d_m}-\frac{x_1-k_n}{d_n}\right\| < \varepsilon \quad \text{for } m, n > N. \tag{4.4}$$

Using (4.4) and the fact that $d_m \to 1$ it can be shown that the sequence $\{k_n\}$ is a Cauchy sequence, hence it converges to some k in X. Since K is closed, $k \in K$. The element k is the unique best simultaneous approximation.

5. In an inner produce space the problem of best simultaneous approximation is relatively much easier. Let H be a real inner produce space and G a subspace of H. Consider two elements $x_1, x_2 \in H$, which have best approximations, say g_1, g_2 from the elements of G. If $\|x_1-g_2\| \le \|x_2-g_2\|$, then g_2 is also a best simultaneous approximation. Similarly if $\|x_2-g_1\| \le \|x_1-g_1\|$, then g_1 is also a best simultaneous approximation. If the above two conditions are not satisfied then

$$\bar{g} = \lambda g_1+(1-\lambda)g_2, \qquad (0<\lambda<1),$$

is the best simultaneous approximation, where λ is given by

$$\|x_1-\bar{g}\| = \|x_2-\bar{g}\|. \tag{5.1}$$

For this we need to show that

$$\max(\|x_1-\bar{g}+g\|, \|x_2-\bar{g}+g\|) \ge \|x_1-\bar{g}\| = (\|x_2-\bar{g}\|) \qquad \forall g \in G.$$

On the contrary suppose that there exists a $g \in G$ such that

$$\|x_1-\bar{g}+g\| < \|x_1-\bar{g}\|, \quad \text{and} \tag{5.2}$$

$$\|x_2-\bar{g}+g\| < \|x_1-\bar{g}\|. \tag{5.3}$$

From (5.2) and (5.3) we obtain

$$(x_1-g_2, g) < -\frac{(g, g)}{2(1-\lambda)} \quad \text{and}$$

$$(x_2-g_1, g) < -\frac{(g, g)}{2\lambda}.$$

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Adding these two we get

$$(x_1-g_2+x_2-g_1, g) < -\frac{(g, g)}{2}\left[\frac{1}{\lambda}+\frac{1}{1-\lambda}\right]$$

or

$$(x_1-g_1, g)+(x_2-g_2, g) < -\frac{(g, g)}{2}\left[\frac{1}{\lambda}+\frac{1}{1-\lambda}\right]$$

i.e. $0<-(g,g)/2[(1/(1-\lambda))+1/\lambda]$ since by hypothesis x_1-g_1, $x_2-g_2\perp G$ which is a contradiction.

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Department of Mathematics,
The University of Calgary,
Calgary, Alberta, T2N 1N4,
Canada