AN ELEMENTARY PROOF OF SOME CHARACTER SUM IDENTITIES OF APOSTOL

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Let χ denote a primitive character modulo k. Using two different representations for Dirichlet L-functions, Apostol [1] recently derived a representation for

$$M_m(\chi) = \sum_{r=1}^{k-1} \chi(r) r^m$$

involving the sums

$$T_m(\bar{\chi}) = \sum_{r=1}^{k-1} \bar{\chi}(r) \cot^m(\pi r/k),$$

where m is a positive integer. Furthermore, if $\chi(r) = (r \mid p)$, the residue class character modulo the odd prime p, he derived a representation for $M_m(\chi)$ involving the sums

$$S_m = \sum_{r=1}^{p-1} \cot^m(\pi r^2/p).$$

A completely elementary proof of these identities is given here.

We shall use the simple facts,

$$\sum_{r=1}^{k} e^{2\pi i r h/k} = \begin{cases} k, & \text{if } \ k \mid h, \\ 0, & \text{if } \ k \nmid h, \end{cases} \tag{1}$$

and

$$\sum_{r=1}^{k} \chi(r) = 0. \tag{2}$$

Let $G(m, \chi)$ denote the Gaussian sum

$$G(m, \chi) = \sum_{r=1}^{k-1} \chi(r) e^{2\pi i r m/k},$$

and put $G(\chi) = G(1, \chi)$. We shall need the factorization theorem for Gaussian sums associated with a primitive character [2, p. 67],

$$G(m, \bar{\chi}) = \chi(m) G(\bar{\chi}). \tag{3}$$

Throughout the sequel, χ denotes a primitive character.

THEOREM 1. *If n is a positive integer, define*

$$f(\chi, n) = \sum_{h=1}^{k-1} \bar{\chi}(h) \left\{ \frac{e^{2\pi i h/k}}{1 - e^{2\pi i h/k}} \right\}^n.$$

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Then

$$(-k)^n f(\chi, n) = G(\bar{\chi}) \sum_{j_1, j_2, \ldots, j_n = 1}^{k} j_1 j_2 \ldots j_n \chi(j_1 + j_2 + \ldots + j_n).$$

Proof. If $k \nmid h$, for any positive integer r,

$$\sum_{j=1}^{r} e^{2\pi i j h/k} = \frac{e^{2\pi i h/k} - e^{2\pi i (r+1)h/k}}{1 - e^{2\pi i h/k}}.$$

Hence

$$f(\chi, n) = \sum_{h=1}^{k-1} \bar{\chi}(h) \prod_{m=1}^{n} \left\{ \sum_{j_m=1}^{r_m} e^{2\pi i j_m h/k} + \frac{e^{2\pi i (r_m+1)h/k}}{1 - e^{2\pi i h/k}} \right\}, \tag{4}$$

where $1 \leqq r_m \leqq k$, $1 \leqq m \leqq n$. Now sum both sides of (4) over r_m, $1 \leqq r_m \leqq k$, $1 \leqq m \leqq n$. Upon using (1), we find that

$$k^n f(\chi, n) = \sum_{h=1}^{k-1} \bar{\chi}(h) \sum_{r_1=1}^{k} \ldots \sum_{r_n=1}^{k} \sum_{j_1=1}^{r_1} \ldots \sum_{j_n=1}^{r_n} e^{2\pi i (j_1 + j_2 + \ldots + j_n)h/k}.$$

Invert the order of summation on r_m and j_m $(1 \leqq m \leqq n)$ and use (1). We obtain

$$\begin{aligned} k^n f(\chi, n) &= \sum_{h=1}^{k-1} \bar{\chi}(h) \sum_{j_1=1}^{k} (k - j_1 + 1) e^{2\pi i j_1 h/k} \ldots \sum_{j_n=1}^{k} (k - j_n + 1) e^{2\pi i j_n h/k} \\ &= (-1)^n \sum_{j_1, j_2, \ldots, j_n = 1}^{k} j_1 j_2 \ldots j_n \sum_{h=1}^{k-1} \bar{\chi}(h) e^{2\pi i (j_1 + j_2 + \ldots + j_n)h/k} \\ &= (-1)^n \sum_{j_1, j_2, \ldots, j_n = 1}^{k} j_1 j_2 \ldots j_n G(j_1 + j_2 + \ldots + j_n, \bar{\chi}) \\ &= (-1)^n G(\bar{\chi}) \sum_{j_1, j_2, \ldots, j_n = 1}^{k} j_1 j_2 \ldots j_n \chi(j_1 + j_2 + \ldots + j_n), \end{aligned}$$

by (3), and the proof is complete.

For $1 \leqq m \leqq 4$, Apostol [1] expressed $M_m(\chi)$ as a linear combination of $T_1(\bar{\chi}), \ldots, T_m(\bar{\chi})$. From his calculations it became clear that the same result is valid for an arbitrary positive integer m. These representations for $M_m(\chi)$ can be derived from Theorem 1. We shall work out the details for the first two examples.

Example 1. If x is not an integer, then

$$\tfrac{1}{2} i \cot \pi x = \tfrac{1}{2} + e^{2\pi i x}/(1 - e^{2\pi i x}). \tag{5}$$

Hence, by the use of Theorem 1 and (2), we have

$$\tfrac{1}{2} i k T_1(\bar{\chi}) = k f(\chi, 1) = -G(\bar{\chi}) \sum_{j=1}^{k} \chi(j) j,$$

i.e.,

$$G(\bar{\chi}) M_1(\chi) = -\tfrac{1}{2} i k T_1(\bar{\chi}).$$

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Example 2. Upon using (5) and (2), we find that

$$-\tfrac{1}{4}k^2T_2(\bar{\chi}) = k^2f(\chi, 1)+k^2f(\chi, 2). \tag{6}$$

To evaluate $f(\chi, 2)$ we use Theorem 1. Letting $j_2 = r-j_1$ and $j_1 = j$, we obtain, with the use of (2),

$$k^2f(\chi, 2) = G(\bar{\chi}) \sum_{j=1}^{k} j \sum_{r=j+1}^{j+k} r\chi(r).$$

If we invert the order of summation, we find that

$$\begin{aligned} k^2f(\chi, 2) &= G(\bar{\chi})\left\{ \sum_{r=2}^{k} r\chi(r) \sum_{j=1}^{r-1} j + \sum_{r=k+1}^{2k} r\chi(r) \sum_{j=r-k}^{k} j \right\} \\ &= G(\bar{\chi})\left\{ \sum_{r=2}^{k} \tfrac{1}{2}r^2(r-1)\chi(r) + \sum_{r=1}^{k} (r+k)\chi(r)[\tfrac{1}{2}k(k+1)-\tfrac{1}{2}r(r-1)] \right\} \\ &= G(\bar{\chi})\{\tfrac{1}{2}k^2M_1(\chi)+kM_1(\chi)-\tfrac{1}{2}kM_2(\chi)\}, \end{aligned} \tag{7}$$

upon simplification and the use of (2). We now substitute (7) into (6) and use the results of Example 1. After a little simplification we arrive at

$$G(\bar{\chi})M_2(\chi) = -\tfrac{1}{2}ik^2T_1(\bar{\chi})+\tfrac{1}{2}kT_2(\bar{\chi}).$$

Next, we show that the second class of identities given by Apostol [1] can be derived in an elementary manner.

THEOREM 2. *If p is an odd prime and n is a positive integer, define*

$$g(p, n) = \sum_{r=1}^{p-1} \left\{ \frac{e^{2\pi i r^2/p}}{1-e^{2\pi i r^2/p}} \right\}^n.$$

If $\chi(r) = (r\,|\,p)$, then

$$(-p)^n g(p, n) = G(\chi) \sum_{j_1,j_2,\ldots j_n=1}^{p} j_1j_2\cdots j_n\chi(j_1+j_2+\ldots+j_n)+\sum_{h=1}^{p-1}\left\{\sum_{j=1}^{p} j\,e^{2\pi i jh/p}\right\}^n. \tag{8}$$

Since

$$\sum_{j=1}^{p} j\,e^{2\pi i jh/p} = \tfrac{1}{2}p\{1-i\cot(\pi h/p)\},$$

the second expression on the right side of (8) *may be written as*

$$(\tfrac{1}{2}p)^n \sum_{h=1}^{p-1} \{1-i\cot(\pi h/p)\}^n.$$

Proof. Proceeding as in the proof of Theorem 1, we arrive at

$$(-p)^n g(p, n) = \sum_{j_1,j_2,\ldots,j_n=1}^{p} j_1j_2\cdots j_n \sum_{r=1}^{p-1} e^{2\pi i(j_1+j_2+\ldots+j_n)r^2/p}.$$

Since each congruence $r^2 \equiv h \pmod p$ has either 0 or 2 solutions modulo p, we have

$$\begin{aligned}(-p)^n g(p, n) &= 2 \sum_{j_1, j_2, \ldots, j_n = 1}^{p} j_1 j_2 \cdots j_n \sum_{\substack{h=1 \\ (h \mid p) = 1}}^{p-1} e^{2\pi i (j_1 + j_2 + \ldots + j_n) h/p} \\ &= \sum_{j_1, j_2, \ldots, j_n = 1}^{p} j_1 j_2 \cdots j_n \sum_{h=1}^{p-1} \{(h \mid p) + 1\} e^{2\pi i (j_1 + j_2 + \ldots + j_n) h/p} \\ &= G(\chi) \sum_{j_1, j_2, \ldots, j_n = 1}^{p} j_1 j_2 \cdots j_n \chi(j_1 + j_2 + \ldots + j_n) + \sum_{h=1}^{p-1} \left\{ \sum_{j=1}^{p} j\, e^{2\pi i j h/p} \right\}^n,\end{aligned}$$

upon the use of (3).

Theorem 2 may be employed to show that $M_m(\chi)$ can be written as the sum of a polynomial in p and a linear combination of $S_1, \ldots, S_m$. We shall work out the details for the first two cases.

Example 3. Upon the use of (5) and Theorem 2,

$$\begin{aligned}\tfrac{1}{2} i p S_1 &= \tfrac{1}{2} p(p-1) + p g(p, 1) \\ &= \tfrac{1}{2} p(p-1) - G(\chi) M_1(\chi) - \sum_{j=1}^{p} j \sum_{h=1}^{p-1} e^{2\pi i j h/p} \\ &= \tfrac{1}{2} p(p-1) - G(\chi) M_1(\chi) + \sum_{j=1}^{p-1} j - p(p-1),\end{aligned}$$

or, upon simplification,

$$G(\chi) M_1(\chi) = -\tfrac{1}{2} i p S_1.$$

Example 4. Employing (5) and the value of $g(p, 1)$ from Example 3, we have

$$-\tfrac{1}{4} S_2 = \tfrac{1}{2} i S_1 - \tfrac{1}{4}(p-1) + g(p, 2).$$

Using Theorem 2 and Example 2, we find after simplification that

$$G(\chi) M_2(\chi) = \tfrac{1}{2} p S_2 - i p^2 S_1 - \tfrac{1}{2} p(p-1) + \frac{2}{p} \sum_{j_1, j_2 = 1}^{p} j_1 j_2 \sum_{h=1}^{p-1} e^{2\pi i (j_1 + j_2) h/p}. \tag{9}$$

This last expression may be evaluated by separating out the terms when $j_1 + j_2 = p$ or $2p$. Upon doing this, we find that the triple sum in (9) becomes

$$(p-1) \sum_{\substack{j_1, j_2 = 1 \\ j_1 + j_2 = p}}^{p} j_1 j_2 + p^2(p-1) - \sum_{\substack{j_1, j_2 = 1 \\ j_1 + j_2 \neq p, 2p}}^{p} j_1 j_2 = -p^4/12 + p^3/3 - 5p^2/12.$$

Upon substituting the above into (9) and simplifying, we obtain

$$G(\chi) M_2(\chi) = \tfrac{1}{2} p S_2 - \tfrac{1}{2} i p^2 S_1 - \tfrac{1}{6} p(p-1)(p-2).$$

REFERENCES

1. T. M. Apostol, Dirichlet L-functions and character power sums, *J. Number Theory* **2** (1970), 223–234.

2. H. Davenport, *Multiplicative number theory* (Chicago, 1967).

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