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On the computation of resonant tidal dissipation in the liquid layers of planets and stars

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Abstract. Planets and stars have liquid layers that can support internal gravity waves and inertial waves respectively restored by the buoyancy and Coriolis forces. Both types of waves are excited by tides, leading to resonantly amplified dissipation. We review the theoretical formalism to compute these resonances and present some challenges and methods to overcome them.

Keywords. liquid layers - buoyancy - Coriolis - tides - resonances

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1. Introduction

The long-term orbital evolution of stars, planets and moons around a central object is directly affected by energy dissipation in both the orbiting object and the central body. This mechanism is mediated by tidal gravitational forces as they deform periodically the rigid layers of the given body, generating heat in the process. Fluid layers, if present, respond differently to periodic tidal forces since they can support inertial waves, i.e. waves whose main restoring force is the Coriolis force, as well as internal gravity waves, i.e. whose main restoring force is buoyancy. Whenever the tidal forcing frequency is close to one of the normal mode frequencies associated with inertial or internal gravity modes of the fluid layer, resonant amplification might occur. This amplified fluid motion is usually accompanied by comparatively large energy dissipation rates, ultimately impacting the long-term evolution of the orbit. Thus it is crucial to determine the resonant mode frequencies in these fluid layers as well as their damping rates. Together with observations, this knowledge also allows us to better characterize the internal structure of the fluid layers in the same way as done in asteroseismology, helioseismology or planetary ring seismology. Computing the normal modes of a rotating fluid body can be challenging. The use of a spherical geometry, together with the anelastic or Boussinesq approximation to eliminate acoustic waves (which due to their much higher frequency are unlikely to be involved in tidally forced systems), leads a numerically tractable problem. In the following we present in some detail the formalism needed to obtain a set of differential equations suitable for numerical studies.

2. Mathematical formalism

2.1. *Momentum equation and hydrostatic equilibrium*

The momentum equation governing the motion in the rotating fluid layers reads:

$$\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right) = -\nabla P - \rho\nabla\Phi + \mathbf{f}, \qquad (2.1)$$

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where $\mathbf{r}$ and $\mathbf{v}$ are the position vector and flow velocity measured in the frame rotating at constant angular velocity $\mathbf{\Omega}$ (henceforth *planetary frame*), ρ is the fluid's mass density, P is the isotropic pressure, and $\mathbf{f}$ represents the sum of additional forces (viscosity, Lorentz, etc.). The gravitational potential Φ must satisfy Poisson equation:

$$\nabla^2\Phi = -4\pi G\rho. \tag{2.2}$$

This may be written as the sum of an equilibrium and a time-varying contributions (see eq. (2.26) below). The equilibrium potential satisfies the hydrostatic condition obtained by setting $\mathbf{v}=\mathbf{0}$ (and assuming $\mathbf{f}=\mathbf{0}$) into eq. (2.1):

$$\nabla P_0 = \rho_0\mathbf{g}_0, \tag{2.3}$$

$$\text{with,} \quad \mathbf{g}_0 = \nabla\left(\Phi_0 + \frac{1}{2}|\mathbf{\Omega}\times\mathbf{r}|\right), \tag{2.4}$$

where $\mathbf{g}_0$ is the equilibrium gravity acceleration in which we include the centrifugal potential. Taking the curl of eq. (2.3) we see that the equipotentials of ρ_0 and P_0 coincide (baroclinic fluid) and are everywhere perpendicular to $\mathbf{g}_0$.

2.2. *First order perturbations*

In the limit of small velocities, the advection term can be omitted from eq. (2.1). In general, the fluid motion will cause variations in the pressure and density fields:

$$P = P_0 + \delta P, \qquad \rho = \rho_0 + \delta\rho, \tag{2.5}$$

where δP and $\delta\rho$ denote the *Eulerian* increments of pressure and density, respectively. Those are related to their *Lagrangian* counterparts, ΔP and $\delta\rho$, via (e.g. Dahlen and Tromp 1998):

$$\Delta P = \delta P + \mathbf{x}\cdot\nabla P_0, \qquad \Delta\rho = \delta\rho + \mathbf{x}\cdot\nabla\rho_0, \tag{2.6}$$

where $\mathbf{x}$ is the displacement vector related to the velocity via $\mathbf{v} = \partial_t\mathbf{x}$ (different to the position vector $\mathbf{r}$). To first order in the perturbations and in the displacement, the momentum eq. (2.1) reduces to:

$$\partial_t\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} = -\frac{1}{\rho_0}\nabla\delta P + \frac{\delta\rho}{\rho_0}\mathbf{g}_0 - \nabla\delta\Phi + \frac{1}{\rho_0}\mathbf{f}. \tag{2.7}$$

The increment of gravitational potential is the sum of internal and external perturbations and satisfies:

$$\nabla^2\delta\Phi = 4\pi G\delta\rho. \tag{2.8}$$

The Eulerian increment of density, $\delta\rho$, must also satisfy the continuity equation expressing mass conservation (e.g. Dahlen and Tromp 1998):

$$\delta\rho = -\nabla\cdot(\rho_0\mathbf{x}). \tag{2.9}$$

In principle eq. (2.9) can be used into eqs. (2.7) and (2.8) to solve for $\mathbf{x}$ and $\delta\Phi$. This method however proves challenging and additional simplifications are desirable.

2.3. *Anelastic approximation*

The anelastic approximation assumes that the Eulerian increments of density and pressure are negligible in eq. (2.6):

$$\Delta P \approx \mathbf{x}\cdot\nabla P_0, \qquad \Delta\rho \approx \mathbf{x}\cdot\nabla\rho_0. \tag{2.10}$$

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Moreover, from eqs. (2.6) and (2.9), we find $\Delta\rho/\rho_0 = -(\nabla \cdot \mathbf{x})$, and the continuity equation reduces to:

$$\nabla \cdot (\rho_0 \mathbf{x}) = 0. \tag{2.11}$$

This equation effectively filters out the rapid oscillations in density and pressure typically associated with acoustic waves of frequencies much higher than typical tidal frequencies (Dintrans and Rieutord 2001). After some rearrangements, we can rewrite the momentum equation as:

$$\partial_t^2 \mathbf{x} + 2\mathbf{\Omega} \times \partial_t \mathbf{x} = -\nabla \left(\frac{\delta P}{\rho_0} + \delta\Phi \right) - N^2 \frac{\mathbf{g}_0}{|\mathbf{g}_0|^2} (\mathbf{g}_0 \cdot \mathbf{x}) + \frac{1}{\rho_0} \mathbf{f}, \tag{2.12}$$

where the second term on the right-hand side is the buoyancy force per unit mass, where N is the *Brunt-Väisälä frequency* defined as:

$$N^2 = -\mathbf{g}_0 \cdot \left(\frac{\mathbf{g}_0}{c^2} - \frac{\nabla\rho_0}{\rho_0} \right), \tag{2.13}$$

and where we have introduced the *isentropic* speed of sound (squared): $c^2 \equiv \Delta P/\Delta\rho$. Dintrans and Rieutord (2001) observed that the pressure and gravity increments can be eliminated by taking the curl of eq. (2.12). The resulting equation coupled with the constraint eq. (2.11) form a complete system in the displacement $\mathbf{x}$. Once solved, the gravity increment can be recovered from Poisson eq. (2.8) which reduces to:

$$\nabla^2 \delta\Phi = 4\pi G \rho_0 \frac{N^2}{|\mathbf{g}_0|^2} (\mathbf{g}_0 \cdot \mathbf{x}). \tag{2.14}$$

2.4. *Incompressible fluid limit*

When the fluid is taken as incompressible, we have $\Delta\rho = 0$, corresponding to the limit of $c \to \infty$. From eqs. (2.10) and (2.11) we then find:

$$\nabla \cdot \mathbf{x} = 0, \tag{2.15}$$

while the Brunt-Väisälä frequency reduces to:

$$N^2 = \mathbf{g}_0 \cdot \frac{\nabla\rho_0}{\rho_0}. \tag{2.16}$$

Notice that eq. (2.15) combined with eq. (2.9), and compared to eq. (2.10) gives:

$$\delta\rho = \mathbf{x} \cdot \nabla\rho_0 \approx 0. \tag{2.17}$$

Imposing eq. (2.17) exactly would make the buoyancy force vanish entirely. This force is proportional to $\delta\rho/\rho_0$ which, though small, is multiplied by $\mathbf{g}_0$ which might be large. This is the physical reason behind the common prescription known as the *Boussinesq approximation* according to which one should neglect density gradient and retain the Eulerian increment of density in the momentum equation only where it is multiplied by gravity acceleration. By the same token, Poisson eq. (2.14) reduces to a Laplace equation in the perturbation:

$$\nabla^2 \delta\Phi \approx 0. \tag{2.18}$$

In this limit, the gravity potential increment inside the fluid therefore reduces to a harmonic function of the coordinates, related to the external potential by boundary conditions (see eqs. (2.21 and (2.22) below).

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2.5. *Viscous dissipation*

Dissipation in the fluid layers can have many origins (viscous, ohmic, etc.). For simplicity, we focus on the effect of viscosity, introduced into eq. (2.12) as an additional force: $\mathbf{f} = \nabla \cdot \boldsymbol{\tau}$, where the *deviatoric stress tensor*, $\boldsymbol{\tau}$ writes:

$$\boldsymbol{\tau} = \zeta(\nabla \cdot \mathbf{v})\mathbb{1} + \mu \left[\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}} - \frac{2}{3}(\nabla \cdot \mathbf{v})\mathbb{1}\right], \tag{2.19}$$

with ζ and μ respectively denoting the *bulk* and *dynamic* viscosities ($\mathbb{1}$ is the identity). ζ conveniently disappears from the curl of the momentum equation, leaving only μ as a parameter. The power dissipated by viscous forces can then be shown equal to:

$$\mathcal{D}_{\text{visc.}} = 2\nu \int_{\mathcal{V}} \widehat{\nabla \mathbf{v}} : \widehat{\nabla \mathbf{v}}, \tag{2.20}$$

where $\widehat{\nabla \mathbf{v}} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}})$, and $:$ denotes the double tensor contraction. $\nu = \mu/\rho_0$ is the kinematic viscosity. The value of this parameter is typically not well known. Its molecular value can safely be assumed as very small–of the order of that of water $10^{-6}\text{m}^2/\text{s}$–in planetary and stellar interior where it is generally treated as a constant.

2.6. *Boundary conditions*

The gravity potential must satisfy the following constraints at the equilibrium surface (Dahlen and Tromp 1998):

$$[\delta\Phi]_{-}^{+} = 0 \tag{2.21}$$

$$[\hat{\mathbf{n}} \cdot (\nabla\delta\Phi + 4\pi G\rho_0 \mathbf{x})]_{-}^{+} = 0, \tag{2.22}$$

where $\hat{\mathbf{n}}$ is the unit normal vector and the notation $[\cdot]_{-}^{+}$ denotes the difference in the values of the enclosed quantity across the boundary. The remaining conditions on the displacement field depend on the nature of the boundary. A free boundary will move so as to minimise stresses at its surface. To first order in the perturbations, this condition applies to the equilibrium surface and writes:

$$[\hat{\mathbf{n}} \cdot \Delta\boldsymbol{\sigma}]_{-}^{+} = \mathbf{0} \tag{2.23}$$

$$\text{with, } \Delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma} + \mathbf{x} \cdot \nabla\boldsymbol{\sigma}_0,$$

where $\Delta\boldsymbol{\sigma}$ and $\delta\boldsymbol{\sigma}$ are the Lagrangian and Eulerian increments of stress, respectively. The latter is equal to the sum of the isotropic increment of pressure and deviatoric tensor in the fluid, $\delta\boldsymbol{\sigma} = -\delta P\mathbb{1} + \boldsymbol{\tau}$, while the equilibrium stress tensor is purely isotropic: $\boldsymbol{\sigma}_0 = -P_0\mathbb{1}$. The two components of eq. (2.23) tangential to the boundary vanish identically in the absence of viscosity. At the interface with a solid boundary, eq. (2.23) is replaced by the simpler 'no-slip' condition:

$$[\mathbf{x}]_{-}^{+} = \mathbf{0}. \tag{2.24}$$

In the absence of viscosity, this constraint is too strong and must be replaced by the weaker 'no-penetration' condition:

$$[\hat{\mathbf{n}} \cdot \mathbf{x}]_{-}^{+} = 0. \tag{2.25}$$

2.7. *Tidal perturbations*

Tidal forcing enters the picture via the gravity potential:

$$\Phi = \Phi_0 + \delta\Phi(t), \tag{2.26}$$

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where $\delta\Phi(t) = \delta\Phi(t)^{\text{ext}} + \delta\Phi(t)^{\text{int}}$ is the sum of the external time-varying part of the tidal potential plus the internal contribution from deformation and must satisfy eq. (2.14). The external forcing induced by a single object can be decomposed in terms of spherical harmonics and Fourier modes (e.g. Ogilvie 2014):

$$\delta\Phi(t)^{\text{ext}} = \sum_n \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \delta\Phi^{\text{ext}}_{\ell,m,n} \mathrm{Y}_\ell^m(\theta,\varphi) e^{in\Omega_o t}, \tag{2.27}$$

where θ and φ are the colatitude and azimutal coordinates, and Ω_o is the mean orbital frequency. Tides with angular frequency $n\Omega_o$, and azimutal number m are perceived as oscillations at frequency $\omega = n\Omega_o - m\Omega$ in the planetary frame. ω corresponds to the response frequency of the fluid to the orbital forcing.

2.8. *Simplification: Spherical symmetry*

In principle the fluid configuration should satisfy the hydrostatic condition eqs. (2.3) and (2.4). If $\mathbf{\Omega} \neq \mathbf{0}$, the equilibrium shape is an ellipsoid of revolution around that axis (taken along the z-axis). If the equilibrium potential Φ_0 is not axisymmetric–as it is the case for objects tidally locked in low orbit around their parent body–the equilibrium ellipsoid is triaxial. Such ellipsoidal shapes do not readily lend themselves to spherical harmonics discretization. It may be tempting to resort to ellipsoidal harmonics. However, in addition to the practical difficulties associated with their use, they can only accommodate exact ellipsoids, i.e. those whose sections along their principal axes are true ellipses, and this precise type of figure can only exist for an object of homogeneous density (Chandrasekhar 1987). This situation does not lend itself well to the exploration of internal structures of differentiated objects, which typically starts with the adoption of a radial density profile. Theoretically, it is possible to approximate the equilibrium figure of such a body by means of a series expansion in flattening (Clairaut's method). Axisymmetric bodies can also be represented by a series of concentric Maclaurin ellipsoids (Hubbard 2013). These methods all rely on the use of complicated coordinate systems, making it all the more difficult to achieve the high numerical resolution required for viscous fluids. Moreoever, many partially fluid bodies such as terrestrial planets or icy moons have shapes far removed from hydrostatic equilibrium.

For all the above reasons, it is often valuable to stick to the simple spherical shape as a first order approximation. This assumption alone simplifies eq. (2.12) immensely by making $\mathbf{g}_0 = -g_0\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit radial vector, thus making the buoyancy force radial and equal to $-N^2(\mathbf{x}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}$, while the Brunt-Viäsälä frequency reduces to its more common expression (De Boeck et al. 1992):

$$N^2 = -\frac{g_0^2}{c^2} - \frac{g_0}{\rho_0}\frac{d\rho_0}{dr}. \tag{2.28}$$

As we can see, two inputs are needed to calculate the fluid motion: a radial density profile, ρ_0, and another profile giving the evolution of the speed of sound, c with depth. As always, one recovers the incompressible limit by letting $c \to \infty$.

3. Free solutions

A detailed resolution of eq. (2.12) is beyond the scope of this short communication, we present only a general outline. For definiteness, we focus on the case where the fluid fills the whole sphere. Since we expect the tidal dissipation to be amplified by gravito-inertial modes, an important first step is to compute these modes. This is done by setting $\mathbf{x} = \tilde{\mathbf{x}}e^{\lambda t}$, and solving the resulting differential eigenvalue problem in $\hat{\mathbf{x}}$ and λ with the external forcing $\delta\Phi^{\text{ext}}$ set to zero.

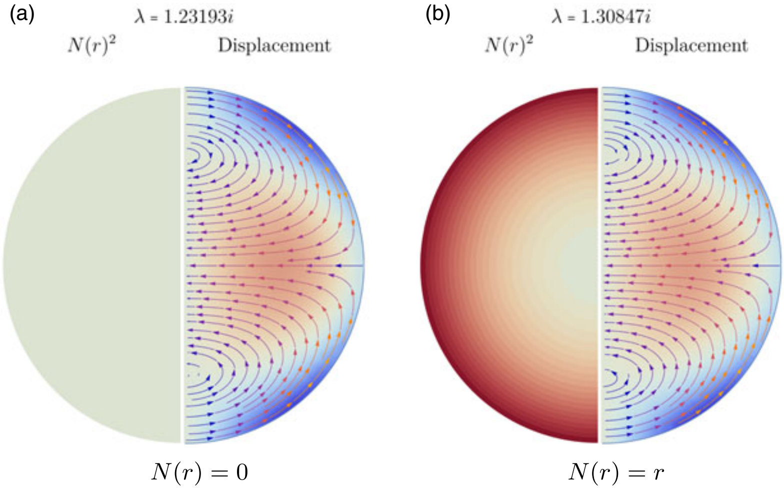

Figure 1. Example of free solutions to eq. (2.12) for two different buoyancy profiles ($m = 2$). (a) Inertial mode. (b) gravito-inertial mode. The arrows represent the displacement in the meridional plane. They scale from blue to red based on the value of the displacement's norm. The background color represents the azimutal displacement with blue and red corresponding to a flow coming out and in the page, respectively. The left parts of each figure show the equipotentials of the function $N^2(r)$.

When buoyancy is set to zero, the spectrum is populated by inertial modes with frequencies $\omega = \mathrm{Im}(\lambda)$ satisfying $|\omega| < 2\Omega$. Solutions to the momentum equation can then be arrived at analytically (Greenspan 1968) based on the boundary condition eq. (2.25). Alternatively, one may discretise the domain and solve the resulting algebraic eigenvalue problem numerically (Rekier et al. 2019). The second option can be readily adapted to more complex boundary conditions and to the case where $N(r) \neq 0$. In such a case, inertial modes get mixed with gravity modes. These mixed modes can be classified depending on the relative values of ω, Ω, and the maximum value of $N(r)$ (Triana et al. 2021). Figure 1a shows an example of inertial mode with azimutal wave number $m = 2$. Figure 1b shows its gravito-inertial mode counterpart when the Brunt-Väisälä frequency grows linearly from the centre.

4. Outlook

The shapes and frequencies of the solutions to eq. (2.12) strongly depend on the internal fluid structure. While on the one hand, this makes their computations more challenging, on the other hand it opens up the possibility of using these to probe planets and stars interior. Although viscous forces are expected to affect the frequencies of the modes very weakly due to their typical weakness compared to the dominant Coriolis force, taking viscosity into account is not only crucial in order to estimate tidal dissipation, it also contributes to regularize modes that are otherwise singular, by turning surfaces of singularities across the inviscid fluid into thin internal shear layers contributing to resonantly enhance tidal dissipation. The aim of the present work was to present the mathematical formalism governing the problem of tidal dissipation in some details. A more detailed account of its solution will be presented in a future work.

References

Chandrasekhar, S. (1987). *Ellipsoidal Figures of Equilibrium*. Dover Books on Mathematics. Dover.

Dahlen, F. A. and Tromp, J. (1998). *Theoretical Global Seismology*. Princeton University Press.

De Boeck, I., Van Hoolst, T., and Smeyers, P. (1992). The subseismic approximation for low-frequency modes of the Earth applied to low-degree, low-frequency g-modes of non-rotating stars. *Astronomy and Astrophysics*, 259:167–174.

Dintrans, B. and Rieutord, M. (2001). A comparison of the anelastic and subseismic approximations for low-frequency gravity modes in stars. *Monthly Notices of the Royal Astronomical Society*, 324(3):635–642.

Greenspan, H. P. (1968). *The Theory of Rotating Fluids*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge U.P, London.

Hubbard, W. B. (2013). Concentric Maclaurin Spheroid Models Of Rotating Liquid Planets. *ApJ*, 768(1):43.

Ogilvie, G. I. (2014). Tidal dissipation in stars and giant planets. *Annual Review of Astronomy and Astrophysics*, 52(1):171–210.

Rekier, J., Trinh, A., Triana, S. A., and Dehant, V. (2019). Inertial modes in near-spherical geometries. *Geophysical Journal International*, 216(2):777–793.

Triana, S. A., Dumberry, M., Cébron, D., Vidal, J., Trinh, A., Gerick, F., and Rekier, J. (2021). Core Eigenmodes and their Impact on the Earth's Rotation. *Surv Geophys*.