which proves our claim. In particular, this shows that for the sequence defined in (3), $\lim_{n\to\infty} w_n = 1$, and then for the sequence defined in (2), $\lim_{n\to\infty} v_n = -1$.

Finally, let us observe that the study of the families of recursive sequences

$$\tilde{u}_{n+1}^{\beta,\gamma} = \min\left\{\tilde{u}_n^{\beta,\gamma}, \cos\left(\gamma n + \beta\right)\right\}, \qquad n \geqslant 1,$$

$$z_{n+1}^{\beta,\gamma} = \max\left\{z_n^{\beta,\gamma}, \sin\left(\gamma n + \beta\right)\right\}, \qquad n \geqslant 1,$$

and

$$\tilde{z}_{n+1}^{\beta,\gamma} = \min\left\{\tilde{z}_n^{\beta,\gamma}, \sin\left(\gamma n + \beta\right)\right\}, \qquad n \geqslant 1,$$

reduces to the obvious equalities:

$$\tilde{u}_n^{\beta,\gamma} = -u_n^{\beta+\pi,\gamma},\ z_n^{\beta,\gamma} = u_n^{\beta-\frac{1}{2}\pi,\gamma} \text{ and } \tilde{z}_n^{\beta,\gamma} = -u_n^{\beta+\frac{1}{2}\pi,\gamma}.$$

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108.42 On matrices whose elements are integers with given determinant

Introduction

For matrices with large positive integer elements with a small determinant is an interesting question in a linear algebra course. In this paper, we investigate matrices of order n with large positive integer elements and having a small determinant. In [1], the author explains the method for finding an infinite family of square matrices of order 2 with large positive integer entries and small positive integer determinant. Motivated by this fact, we generalise it for the case of square matrices of any arbitrary order $n \geqslant 2$. More precisely, we prove the following result.

Theorem 1: Given positive integers d and M, there exist infinitely many matrices $A = [a_{ij}]_{1 \leqslant i,j \leqslant n}$ with integer elements satisfying $a_{ij} \geqslant M$ and $\det A = d$.

For the proof, first we consider matrices of order 3. For this purpose, we employ a linear Diophantine equation. A linear Diophantine equation (in three variables) is an equation of the general form $ax + by + cz = d$ where a, b and c are given integers and x, y and z are unknown integers. It is a classical fact that (see [2] for more detail) this Diophantine equation has integer solution in x, y and z if, and only if, d is a multiple of the greatest common divisor of a, b and c. In order to deal with matrices of higher order, instead of considering Diophantine equation on more variables, we try to build it by using the notion of the Kronecker product, running over an inductive procedure. We recall that for the matrices $A = \left[a_{ij}\right]$ and B of order $m \times n$ and $p \times q$, respectively, the Kronecker product of the matrices A and B, denoted by $A \otimes B$, is the $mp \times nq$ matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where the block $a_{ij}B$ is formed by multiplying each a_{ij} element by the entire matrix B.

Theorem 1: (for the case $n = 3$)

In this section, we prove Theorem 1 for the case $n = 3$. We give a numerical example to ensure that the method of proof is constructive.

Lemma:

Given positive integers d and M, there exist infinitely many matrices $A = [a_{ij}]_{1 \leqslant i,j \leqslant 3}$ with integer elements satisfying $a_{ij} \geqslant M$ and $\det A = d$.

Proof: For the given integer $M > 0$ we consider the following 3×3 matrix with integer entries

$$A = \begin{bmatrix} M & M + 1 & x \\ M + 1 & M + 2 & y \\ 2M + 1 & 2M + 2 & z \end{bmatrix}.$$

Note that $\det A = -Mx + (M + 1)y - z$. Thus, $\det A = d$ reads as the following linear Diophantine equation in three variables

$$-Mx + (M + 1)y - z = d. \tag{1}$$

Since $\gcd(M, M + 1, 1) = 1$, (1) has infinitely many solutions. We can solve (1) by reducing it into a two variable equation. Let $(M + 1)y - z = W$. By (1) we have the following linear Diophantine equations in two variables

$$-Mx + W = d, \tag{2}$$

$$(M + 1)y - z = W. \tag{3}$$

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We observe that $x_0 = 0$ and $W_0 = d$ is a particular integer solution of (2). Thus, all integer solutions of (2), are

$$x = t, \qquad W = d + Mt, \qquad (t \in \mathbb{Z}). \tag{4}$$

From (3) and (4) we have $(M + 1)y - z = d + Mt$. Since $\gcd(M + 1, 1) = 1$, this equation has infinitely many solutions for all integers M. It is easy to check that $y_0 = t$ and $z_0 = t - d$ is a particular integer solution. So, all of its integer solutions are

$$y = t - k, \qquad z = t - d - (M + 1)k, \qquad \left(t, k \in \mathbb{Z}\right).$$

Attaching $x = t$ to this, we obtain the general solution of (1) as well. To obtain conditional matrices of the Lemma, it is sufficient to take $t > M$ and $k < \left\lfloor \frac{t - d - M}{M + 1} \right\rfloor$, simultaneously. This completes the proof.

Remark: We can investigate the Lemma above with several Diophantine equations. Let p, q and r be different prime numbers greater than M, and

$$A = \begin{bmatrix} p & p & x \\ q & q + 1 & y \\ r & r & z \end{bmatrix}.$$

Then, the equality $\det A = d$ reads as the Diophantine equation $pz - rx = d$. The condition $\gcd(p, r) = 1$ ensures the existence of infinitely many solutions for this equation. Therefore, the Lemma can be considered with at least 5 of 9 prime entries.

Example

Consider three prime numbers 15485863, 32452843 and 49979687 greater than 15000000. Let

$$a_{11} = a_{12} = 15485863,$$

$$a_{21} = a_{22} - 1 = 32452843,$$

$$a_{31} = a_{32} = 49979687.$$

By the above Remark, we obtain

$$15485863z - 49979687x = 2 \qquad \text{with} \qquad M = 15000000 \qquad \text{and} \qquad d = 2.$$

It is easy to check that

$$x_0 = 13228138 \qquad \text{and} \qquad z_0 = 42693016$$

is a particular integer solution and so all integer solutions to the equation are of the form

$$x = 13228138 + 15485863t, \qquad z = 42693016 + 49979687t, \qquad t \in \mathbb{Z}.$$

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We take $y = 160481183$, which is a prime number, and $t = 1$. Thus

$$A = \begin{bmatrix} 15485863 & 15485863 & 28714001 \\ 32452843 & 32452844 & 160481183 \\ 49979687 & 49979687 & 92672703 \end{bmatrix},$$

with $\det A = 2$.

Theorem 1 (for the general case)

We give some notations to simplify our description of the proof of Theorem 1 for the general case $n \geqslant 3$. We let J_n and I_n denote the $n \times n$ all-one matrix and the identity matrix, respectively and $\mathbf{0}_{m \times n}$ be the $m \times n$ zero-matrix. For square matrices $A_1, \ldots, A_n$ we define $\operatorname{diag}(A_1, \ldots, A_n)$ as the block matrix where the blocks along the diagonal are $A_1, \ldots, A_n$ and all other blocks are $\mathbf{0}$. Now we give the proof of Theorem 1.

Proof of Theorem 1: First, we assume that n is even, and we let $n = 2k$ for some integer k. By the result of [1], for $l \in \mathbb{N}$, there are infinitely many sets $\mathcal{A}_l = \{A_{l1}, \ldots, A_{lk}\}$ of 2×2 matrices with integer elements greater than M that satisfy the condition $\prod_{i=1}^{k} \det A_{li} = d$. We consider the following block matrix

$$B_l = J_{2k} \otimes A_{l1} + \operatorname{diag}(\mathbf{0}_{2 \times 2}, A_{l2}, \ldots, A_{lk}).$$

By the definition of $B_l = [b_{ij}]_{1 \leqslant i,j \leqslant 2k}$, we observe that $b_{ij} \geqslant M$. It is easy to see that B_l is equivalent to $\operatorname{diag}(A_{l1}, \ldots, A_{lk})$ by using row operations. Since $\det(\operatorname{diag}(A_{l1}, \ldots, A_{lk})) = \prod_{i=1}^{k} \det A_{li}$, we deduce that $\det(B_l) = d$.

For odd values of n we follow an inductive procedure. The case $n = 3$ has been considered in the Lemma. For $n = 2k + 1$ with $k \geqslant 2$ we let A_{2k+1} be the $(2k + 1) \times (2k + 1)$ block matrix defined recurrently by

$$A_{2k+1} = \begin{bmatrix} A_{2k-1} & B \\ C & A_2 + D \end{bmatrix}$$

where A_2 is the 2×2 matrix by methods of [1], B, C and D obtained from A_{2k-1} as follows:

- B is the first two columns of matrix A_{2k-1},
- C is the first two rows of matrix A_{2k-1},
- D is the upper left block of size 2×2 of the matrix A_{2k-1}.

Accordingly, A_{2k+1} and $\operatorname{diag}(A_{2k-1}, A_2)$ are equivalent, and consequently

$$\det A_{2k+1} = \det A_{2k-1} \det A_2 = d.$$

Note that we can take infinitely many 2×2 and $(2k - 1) \times (2k - 1)$ matrices same as A_2 and A_{2k-1} with integer elements larger than M and $\det A_{2k-1} \det A_2 = d$. Therefore, there exist infinitely many matrices A_{2k+1} satisfying the conditions of the theorem.

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108.43 An alternating recursion: proof of a conjecture by Erik Vigren

The following construction was considered by Erik Vigren in [1]. With positive numbers a_0, a_1 given, a_n is defined for $n \geqslant 2$ by an alternating recursion:

$$a_{2n} = \sqrt{a_{2n-2}a_{2n-1}},$$

the geometric mean of the previous two terms, while

$$a_{2n+1} = \frac{2a_{2n-1}a_{2n}}{a_{2n-1} + a_{2n}},$$

the harmonic mean of the previous two terms (which we denote by $H\left(a_{2n-1}, a_{2n}\right)$).

It was conjectured in [1], with support from numerical calculations, that a_n converges to $\gamma(a_0, a_1)$, where for $x < y$,

$$\gamma(x, y) = \frac{y}{\sqrt{\frac{y}{x} - 1}} \tan^{-1}\sqrt{\frac{y}{x} - 1} \tag{1}$$

while for $x > y$,

$$\gamma(x, y) = \frac{y}{\sqrt{1 - \frac{y}{x}}} \tanh^{-1}\sqrt{1 - \frac{y}{x}}. \tag{2}$$

Here we give a proof for the case where $a_0 < a_1$, so that (1) applies. The case $a_0 > a_1$ can then be proved similarly, or derived from (1) using $\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$ and $\ln\left(1 + ic\right) - \ln(1 - ic) = 2i\tan^{-1} c$.

Lemma 1: (a_n) tends to a limit.

Proof: First, since either type of mean of x and y lies between x and y, an easy induction shows that $a_0 < a_n < a_1$ for all $n \geqslant 2$. Now $a_2 = \sqrt{a_0 a_1}$, so