24
Oscillators

24.1 Introduction

Our aim in this chapter is to show how quantized detector networks (QDN) describe the quantized one-dimensional bosonic (harmonic) oscillator. The adjective *bosonic* here refers to the fact that in the standard quantum theory of this system under observation (SUO), the phase space operators of position and momentum satisfy commutation properties, in contrast to the *fermionic* oscillator, whose corresponding variables satisfy anticommutation properties and which is studied at the end of this chapter.

The one-dimensionality referred to here is somewhat misleading: it refers to the classical mechanics (CM) theory of a single particle moving in one spatial dimension under the influence of an attractive quadratic force potential. In the quantized version of the same SUO, there are infinitely many degrees of freedom, requiring the use of an infinite-rank quantum register. This is one of the few occasions in QDN in which we refer to the concept of infinity.

Because the rank of the oscillator's quantum register is infinite, we find that the QDN representation comes with a tremendous amount of mathematical overkill and redundancy. Specifically, the Hilbert space dimension of such a quantum register is far bigger than that actually needed to describe a quantized bosonic oscillator, the vast bulk of states in the register being what we shall call *transbosonic*.

As a Hilbert space, the infinite-dimension quantum register is nonseparable, which means that it has no *denumerable* (that is, countable) basis, unlike the Hilbert space $\mathcal{H}^{HO}$ of energy eigenstates used to describe the standard QM bosonic oscillator. While the improper position coordinate basis $\{|x\rangle : x \text{ real}\}$ is nondenumerable, the normalized energy eigenstate basis for $\mathcal{H}^{HO}$ is countable. We will show how the QDN labstates that are the analogues of standard QM oscillator states are restricted to a subspace of the register of measure zero and remain there from stage to stage.

The degree of "overkill" in the QDN description of the oscillator is no different in essence to that encountered in the quantum field theory (QFT) description

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of the oscillator. QFT is a *many particle theory* that allows for the possibility of more than one oscillator to be excited at the same time. This is in contrast to Heisenberg's matrix mechanics (Heisenberg, 1925) and Schrödinger's wave mechanics (Schrödinger, 1926), which are one-particle descriptions of the oscillator. Being based on signal detector principles, QDN can readily describe arbitrary numbers of oscillator-type signals, over and above quantum superpositions of one-particle signal states. This underlines the point that QDN looks much more like a halfway house to QFT rather than another representation of standard QM. The transbosonic states referred to above are really a reflection of that fact.

24.2 The Classical Oscillator Register

The first step in the QDN description of the harmonic oscillator is to define an infinite-rank classical bit register $\mathcal{R}^{[\infty]}$. This is an infinite, countable collection of classical bits, each bit being labeled by a distinct nonnegative integer n running from zero to infinity. We shall call $\mathcal{R}^{[\infty]}$ the *classical oscillator register* and write $\mathcal{R}^{[\infty]} \equiv B^0 B^1 B^2 \ldots$, where B^i is the ith classical bit.

Here we adopt the convention that $\mathcal{R}^{[\infty]}$ is the *Cartesian product* of the binary sets B^0, B^1, and so on.

Our QDN approach to the harmonic oscillator raises a significant issue. Up to this point, we have argued that no experiment deals with actual infinities, so QDN has exploited that fact. Indeed, finiteness seems to be a strength of QDN rather than a weakness. We should explain why we now introduce an infinite-rank quantum register to describe one of the simplest of SUOs, the harmonic oscillator.

We have found that whenever such issues arise in QDN, the answers are to be found by looking at what happens in the laboratory and what the observer actually does. QDN discusses detectors, not the supposed objects being detected. There are two contrasting facts about apparatus, however. The first fact is that relative to any real observer, all apparatus comes in discrete packages, at any empirical level.[1] At the highest empirical level, which we may call the emergent level, a detector has a single objectivized identity, which is as a detector.[2] In QDN we model this by a single binary detector. At the lowest empirical level, which we may call the reductionist level, a real detector is described as a vast but countable number of discrete components called molecules and atoms. At that level, it need not be recognizable as a detector at all.

The second fact is that despite the countability associated with the emergent and reductionist levels, there are three scenarios where infinity makes an appearance.

[1] By this we mean what is observable, not what is theorized.

[2] If we view reductionism as the counterbalance to emergence, then the highest empirical level corresponds to the lowest reductionist level, where minimal mathematical details are given.

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Modeling Space Is Not Easy

The great mathematical physicist Schwinger gave the following succinct statement on how he viewed QM, apparatus, space, and time:

> The mathematical machinery of quantum mechanics is a symbolic expression of the laws of atomic measurement, abstracted from the specific properties of individual techniques of measurement. In particular, the space-time manifold that is the background of any quantum-mechanical description is an idealization of the function of a measurement apparatus to define a macroscopic frame of reference. (Schwinger, 1958)

This is in accord with the principles of QDN. If we wish to model space-time itself, rather than just apparatus that is *in* space-time (a rather different proposition altogether), then we are faced with the concept of *indefiniteness* rather than infinity. By this we mean that there no natural, obvious, or observable limits to space and time as far as any observer is concerned. Even if we discretized space and time coordinates, thereby eliminating continuity, how many points would we include in our modeling?

Apparatus Depends on Continuous Parameters

Continuity cannot be eliminated even when we have very simple apparatus. For example, in the Stern–Gerlach experiment, the orientation of the main magnetic field is parametrized by three angles in a continuum of angles, and there is no natural way of discretizing any of those.

Quantum Process in the Information Void

QDN uses, but does not derive from reductionist principles, any transition amplitudes from stage to stage: it is designed to work with the architecture of processes and how those amplitudes are related to observable signals. The calculation of such amplitudes will usually (but not invariably) involve working models of empty space (the vacuum) that depend on continuity.

Every classical bit B^i has two elements, $\mathbf{0}^i$ and $\mathbf{1}^i$, representing "*no signal*" and "*signal*" respectively. A rank-r classical register has 2^r elements, so $\mathcal{R}^{[\infty]}$ contains an infinite number of classical states. Representing these requires dealing with infinite sets.

The mathematical difficulty here is that these sets are nondenumerable, that is, are not countable. The paradox is that while the bits making up the register are each of finite cardinality, and the bits themselves can be counted (our labeling proves this), the number of states is not countable. Mathematically, this is the same phenomenon that allows mathematicians to express every rational real number in terms of a recurring decimal expansion, but none of the irrationals.

The Signal Basis Representation

The most natural representation of a classical state in $\mathcal{R}^{[\infty]}$ is the signal basis representation (SBR), which we have met before. An arbitrary state $\boldsymbol{\Psi}$ in $\mathcal{R}^{[\infty]}$ can be defined in the form

$$\boldsymbol{\Psi} \equiv \boldsymbol{i}^0 \boldsymbol{i}^1 \boldsymbol{i}^2 \ldots = \prod_{a=0}^{\infty} \boldsymbol{i}^a, \qquad \boldsymbol{i}^a = \mathbf{0}^a \text{ or } \mathbf{1}^a \text{ for } a = 0, 1, 2, \ldots \tag{24.1}$$

Every such state therefore corresponds to a unique *binary sequence* $S_{\boldsymbol{\Psi}} \equiv \{i^0, i^1, i^2, \ldots\}$, consisting of an infinite string of ones and zeros.

Example 24.1 The register state

$$\boldsymbol{\Psi} \equiv \mathbf{0}^0\mathbf{0}^1\mathbf{1}^2\mathbf{1}^3\mathbf{0}^4\mathbf{1}^5\mathbf{0}^6\mathbf{1}^7\mathbf{1}^8\mathbf{0}^9\mathbf{1}^{10}\mathbf{1}^{11}\mathbf{0}^{12}\mathbf{1}^{13}\mathbf{1}^{14}\mathbf{0}^{15}\mathbf{1}^{16}\ldots \tag{24.2}$$

corresponds to the binary sequence

$$S_{\boldsymbol{\Psi}} \equiv \{0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, \ldots\}. \tag{24.3}$$

The nondenumerability of $\mathcal{R}^{[\infty]}$ creates a potential problem when we come to quantization, because the Hilbert space $\mathcal{Q}^{[\infty]}$ corresponding to $\mathcal{R}^{[\infty]}$ is an infinite tensor product. Such Hilbert spaces are always nonseparable (Streater and Wightman, 1964), which means that they have no countable basis. This contrasts with the fact that the Hilbert space of the standard QM quantized bosonic oscillator has a complete, countable basis set, consisting of energy eigenstates.

Fortunately, in the QDN analysis for the harmonic oscillator, we can restrict our attention to a set of special operators, referred to here as *bosonic operators* over the infinite-rank quantum register $\mathcal{Q}^{[\infty]}$, which have the merit that, in physical applications dealing with a single oscillator, the problems of nonseparability can be avoided. A similar phenomenon occurs in relativistic quantum field theory (Streater and Wightman, 1964; Klauder and Sudarshan, 1968).

To understand how this comes about, we first classify each state in $\mathcal{R}^{[\infty]}$ as one of three possible types. Two of these types form countable subsets of the register, while the third type forms a nondenumerable subset. These types correspond, roughly speaking, to the integers, the rationals, and the irrationals, respectively, in the real number system. This can be seen by the following heuristic arguments.

Finite Countable States

The first type, the set of all *finite countable* states in $\mathcal{R}^{[\infty]}$, consists of states associated with binary sequences that consist of zeros after some given finite element J, which depends on the sequence. For example, the state $\mathbf{1}^0\mathbf{1}^1\mathbf{0}^2\mathbf{1}^3\mathbf{0}^4\mathbf{0}^5\mathbf{0}^6\mathbf{0}^7\mathbf{0}^8\mathbf{0}^9\mathbf{0}^{10}\ldots$ is finite countable ($J = 4$), whereas the state corresponding to the infinitely recurring sequence

$$\mathbf{1}^0\mathbf{0}^1\mathbf{1}^2\mathbf{0}^3\mathbf{1}^4\mathbf{0}^5\mathbf{1}^6\mathbf{0}^7\mathbf{1}^8\mathbf{0}^9\mathbf{1}^{10}\ldots \tag{24.4}$$

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is not finite countable. For a finite countable state $\boldsymbol{i}^0\boldsymbol{i}^1\boldsymbol{i}^2\ldots\boldsymbol{i}^J\mathbf{0}^{J+1}\mathbf{0}^{J+2}\ldots$ a modification of the computational basis map (5.14) maps this state to the integer $i^0 + 2i^1 + 2^2i^2 + \cdots + 2^J i^j$, which is finite.

Recurring Sequence States

The second type, the *recurring sequence* states in $\mathcal{R}^{[\infty]}$, consists of those sequences that would be finite countable sequences but for the fact that the infinite string of zeros after J is replaced by some nontrivial recurring binary sequence. Recurring sequence states cannot be classified by finite integers using the computational map (5.14). However, we can use another map, defined by

$$\boldsymbol{i}^0\boldsymbol{i}^1\boldsymbol{i}^2\ldots \to i^0 + \frac{i^1}{2^1} + \frac{i^2}{2^2} + \cdots \tag{24.5}$$

to map such states into the interval $[0,2]$. We shall call this the *continuum map*. It is easy to see that, in fact, *all* states in $\mathcal{R}^{[\infty]}$ can be mapped into the interval $[0,2]$ via the continuum map. The signal ground state $\mathbf{0}^0\mathbf{0}^1\mathbf{0}^2\ldots$ maps into 0, while the *fully occupied state* $\mathbf{1}^0\mathbf{1}^1\mathbf{1}^2\ldots$ maps into 2. All other states necessarily map into the open interval $(0,2)$.

It is not hard to see that finite countable states and recurring sequence states map into the rationals via the continuum map, but not in a one-to-one way. For example, the finite countable state $\mathbf{1}^0\mathbf{0}^1\mathbf{0}^2\mathbf{0}^3\ldots$ maps into the number 1 by the continuum map, which is also the value mapped from the recurring sequence state $\mathbf{0}^0\mathbf{1}^1\mathbf{1}^2\mathbf{1}^3\ldots$

Remark 24.2 In standard decimal-based arithmetic, it is generally asserted that the infinitely recurring decimal $0.\dot{9} \equiv 0.9999\ldots$ is "equal" to the number 1. While the register states $\mathbf{1}^0\mathbf{0}^1\mathbf{0}^2\mathbf{0}^3\ldots$ and $\mathbf{0}^0\mathbf{1}^1\mathbf{1}^2\mathbf{1}^3\ldots$ map to the same value 1 via the continuum map, physically, these are two very different states. This underlines the difference between pure mathematics ($0.\dot{9} = 1$) and physics ($\mathbf{1}^0\mathbf{0}^1\mathbf{0}^2\mathbf{0}^3\ldots \neq \mathbf{0}^0\mathbf{1}^1\mathbf{1}^2\mathbf{1}^3\ldots$).

Not all pure mathematicians would agree that $0.\dot{9} = 1$ is an absolute equality; some would argue that the difference $1 - 0.\dot{9}$ is an infinitesimal. But not every mathematician accepts the concept of infinitesimals as sound.

Irrational Sequence States

The problem with nondenumerability arises because of the existence of the third type of infinite binary sequence. This consists of all those binary sequences that are not recurring, such as

$$\{1,1,1,0,1,1,1,1,0,1,1,1,1,1,0,0,0,0,0,0,0,0,0,1,1,\ldots\}, \tag{24.6}$$

an example based on the successive digits in the decimal representation of π. There are infinitely many such sequences and they cannot be counted, as they correspond to the irrationals, which is easy to prove.

Our conclusion is, therefore, that states in $\mathcal{R}^{[\infty]}$ cannot be classified by the integers alone. Instead, we may use the sequence corresponding to each state as an index. Specifically, if S is the binary sequence

$$S \equiv \{s^a : s^a = 0 \text{ or } 1 \text{ for } a = 0, 1, 2, \ldots\}, \tag{24.7}$$

then the corresponding state $\boldsymbol{S}$ in $\mathcal{R}^{[\infty]}$ is given uniquely by the expression

$$\boldsymbol{S} \equiv \boldsymbol{s}^0 \boldsymbol{s}^1 \boldsymbol{s}^2 \ldots = \prod_{a=0}^{\infty} \boldsymbol{s}^a. \tag{24.8}$$

Remark 24.3 The *zero sequence* $Z \equiv \{0, 0, 0, \ldots\}$ corresponds to the **signal ground state**, denoted $\mathbf{0}^0\mathbf{0}^1\mathbf{0}^2\ldots$ in the occupation representation and $\mathbf{0}$ in the computation representation. This state is **not** the QDN analogue of the conventional oscillator ground state $|0\rangle$.

Given two binary sequences S, T corresponding to register states $\boldsymbol{S}$ and $\boldsymbol{T}$, respectively, their "inner product" $\overline{\boldsymbol{S}}\boldsymbol{T}$ is defined in the obvious way, viz.

$$\overline{\boldsymbol{S}}\boldsymbol{T} = \left\{ \prod_{a=0}^{\infty} \overline{\boldsymbol{s}}^a \right\} \left\{ \prod_{b=0}^{\infty} \boldsymbol{t}^b \right\} \equiv \prod_{a=0}^{\infty} \overline{\boldsymbol{s}}^a \boldsymbol{t}^a \equiv \delta^{ST}, \tag{24.9}$$

where the generalized Kronecker symbol δ^{ST} takes the value unity if and only if the binary sequences S and T are identical; otherwise, it is zero.

Definition 24.4 Two binary sequences $S \equiv \{s^a : a = 0, 1, 2, \ldots\}, T \equiv \{t^a : a = 0, 1, 2, \ldots\}$ are identical if and only if $s^a = t^a$ for all nonnegative integers, that is, for $a = 0, 1, 2, \ldots$

We take it as an axiom that, given two binary sequences S and T, the product $\prod_{a=0}^{\infty} \overline{\boldsymbol{s}}^a \boldsymbol{t}^a$ is 1 if the sequences are identical and 0 otherwise.

24.3 Quantization

Quantization amounts to associating the set of states in $\mathcal{R}^{[\infty]}$ as a (preferred) basis $B^{[\infty]}$ for a nonseparable Hilbert space $\mathcal{Q}^{[\infty]}$. States in $\mathcal{Q}^{[\infty]}$ will be called *quantum register states* and are of the form

$$\boldsymbol{\Psi} = \sum_S \Psi(S)\, \boldsymbol{S}, \tag{24.10}$$

where the summation is over all possible infinite binary sequences S and the coefficients $\Psi(S)$ are complex. The Hilbert space inner product is defined in the obvious way, that is,

$$\begin{aligned} \overline{\boldsymbol{\Psi}}\boldsymbol{\Phi} &= \left(\sum_S \overline{\Psi}(S) \overline{\boldsymbol{S}} \right) \left(\sum_T \Phi(T) \boldsymbol{T} \right) \\ &= \sum_S \sum_T \overline{\Psi}(S)\Phi(T) \underbrace{\overline{\boldsymbol{S}}\boldsymbol{T}}_{\delta^{ST}} = \sum_S \Psi^*(S)\, \Phi(S). \end{aligned} \tag{24.11}$$

As discussed above, finite countable sequences can be mapped into the integers via the computational map (5.14). For a sequence S that maps into integer n, we can use the notation $\boldsymbol{n}$ rather than $\boldsymbol{S}$ to denote the corresponding quantum register state.

24.4 Bosonic Operators

The quantum register states in QDN required to represent the standard discrete set of quantized bosonic oscillator energy eigenstates $\{|n\rangle : n = 0, 1, 2, 3, \ldots\}$ form a subset of the finite countable states. We shall call any normalized element of this subset a *bosonic state*. To identify this subset, we need to filter out of the set of all finite countable states those states that are redundant. To do this, we first define the *bosonic projection operators* as follows.

Each component B^a of the classical oscillator register $\mathcal{R}^{[\infty]}$ is a classical bit with two elements. Quantization starts with the interpretation of the two bit states $\mathbf{0}^a$, $\mathbf{1}^a$ in B^a as the two preferred basis quantum outcomes states of a quantum bit, Q^a, representing the two states, *ground* and *signal*, of a detector. Each quantum bit Q^a is a two-dimensional Hilbert space with its own set of projection and signal operators $\left\{\boldsymbol{P}^a, \widehat{\boldsymbol{P}}^a, \boldsymbol{A}^a, \widehat{\boldsymbol{A}}^a\right\}$ satisfying Table 24.1, a generalization of Table 4.1.

The *quantized bosonic register* $\mathcal{Q}^{[\infty]}$ is the infinite-dimensional Hilbert space given by the commuting tensor product of all the individual qubits, i.e.,

$$\mathcal{Q}^{[\infty]} \equiv Q^0 \otimes Q^1 \otimes Q^2 \otimes \cdots = Q^0 Q^1 Q^2 \ldots \tag{24.12}$$

Here and below we shall drop the tensor product symbol $\otimes$, it being understood we are dealing with commuting tensor products.

Definition 24.5 The set of *bosonic filter operators* $\{\widehat{\mathbb{P}}^a_B : a = 0, 1, 2, \ldots\}$ is defined by the commuting tensor product

$$\widehat{\mathbb{P}}^a_B \equiv \left\{\prod_{b \neq a}^{\infty} \boldsymbol{P}^b\right\} \widehat{\boldsymbol{P}}^a = \boldsymbol{P}^0 \boldsymbol{P}^1 \ldots \boldsymbol{P}^{a-1} \widehat{\boldsymbol{P}}^a \boldsymbol{P}^{a+1} \ldots, \quad a = 0, 1, 2, \ldots \tag{24.13}$$

Each bosonic projection operator $\widehat{\mathbb{P}}^a_B$ defines a one-dimensional Hilbert subspace of $\mathcal{Q}^{[\infty]}$ with basis $\{\mathbf{2}^a\}$ in the computational basis representation. Eigenstates of these operators with eigenvalue $+1$ will be called *bosonic eigenstates*.

Table 24.1 *Products of signal bit operators*

	$\boldsymbol{P}^a$	$\widehat{\boldsymbol{P}}^a$	$\boldsymbol{A}^a$	$\widehat{\boldsymbol{A}}^a$
$\boldsymbol{P}^a$	$\boldsymbol{P}^a$	$\mathbf{0}$	$\boldsymbol{A}^a$	$\mathbf{0}$
$\widehat{\boldsymbol{P}}^a$	$\mathbf{0}$	$\widehat{\boldsymbol{P}}^a$	$\mathbf{0}$	$\widehat{\boldsymbol{A}}^a$
$\boldsymbol{A}^a$	$\mathbf{0}$	$\boldsymbol{A}^a$	$\mathbf{0}$	$\boldsymbol{P}^a$
$\widehat{\boldsymbol{A}}^a$	$\widehat{\boldsymbol{A}}^a$	$\mathbf{0}$	$\widehat{\boldsymbol{P}}^a$	$\mathbf{0}$

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Theorem 24.6 *The bosonic filter operators satisfy the multiplicative rule*

$$\widehat{\mathbb{P}}_B^a \widehat{\mathbb{P}}_B^b = \delta^{ab} \widehat{\mathbb{P}}_B^a, \qquad a, b \in \mathbb{N} \equiv \{0, 1, 2, \ldots\}. \tag{24.14}$$

The element $\mathbf{2}^n$ in $\mathcal{Q}^{[\infty]}$ corresponds to the energy eigenstate $|n\rangle$ in the standard quantum theory of the oscillator.

The bosonic filter operator $\widehat{\mathbb{P}}_B^a$ should not be confused with the signal projection operators $\widehat{\mathbb{P}}^a \equiv \left\{\prod_{b \neq a}^{\infty} \boldsymbol{I}^b\right\} \widehat{\boldsymbol{P}}^a$, where $\boldsymbol{I}^b$ is the identity operator for qubit Q^b. The difference is based on logic: an eigenstate of $\widehat{\mathbb{P}}^a$ will return a positive signal if detector a is examined, regardless of whatever signal state any other detector is in, whereas an eigenstate of $\widehat{\mathbb{P}}_B^a$ will return a positive signal in detector a only if all the other detectors are each in their signal ground state.

Given the bosonic projection operators $\widehat{\mathbb{P}}^a$, the next step is to define the *bosonic identity operator*

$$\mathbb{I}_B \equiv \sum_{a=0}^{\infty} \widehat{\mathbb{P}}_B^a. \tag{24.15}$$

This operator satisfies the idempotency condition required of any projection operator, that is,

$$\mathbb{I}_B \mathbb{I}_B = \mathbb{I}_B \tag{24.16}$$

and plays the role of a "bosonic filter," passing only those states and operators associated with the quantum register that have the desired properties associated with the harmonic oscillator.

Definition 24.7 A quantum register operator $\mathbb{O}$ is *bosonic* if and only if it commutes with the bosonic identity, i.e.,

$$\mathbb{O} \text{ bosonic } \Leftrightarrow [\mathbb{O}, \mathbb{I}_B] = 0. \tag{24.17}$$

Definition 24.8 A *bosonic state* is defined to be any vector in $\mathcal{Q}^{[\infty]}$ that is an eigenstate of $\mathbb{I}_B$ with eigenvalue $+1$. All other states in $\mathcal{Q}^{[\infty]}$ will be referred to as *transbosonic*.

The set of all bosonic states is denoted $\mathcal{Q}_B$ and is the QDN analogue of the Hilbert space of quantum oscillator states in standard QM.

Examples of transbosonic states are the signal ground state $\mathbf{0}$ and all those finite countable elements $\boldsymbol{p}$ of the computational basis B where p is not some power of two. In fact, almost all elements in the quantum register are transbosonic. It can be readily verified that linear superpositions of bosonic states are always bosonic states, while the linear superposition of any transbosonic state with any other state in the register is always transbosonic.

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The importance of the bosonic operators is that when they are applied to bosonic states, the result is always a bosonic state, which can be easily proved.

24.5 Quantum Register Oscillator Operators

We now in a position to discuss how we map the standard quantum oscillator into the quantum register.

In the standard QM of the bosonic oscillator, the most important operators are the ladder operators a and $a^\dagger$. These satisfy the commutation relation

$$\left[a, a^\dagger\right] = 2I, \tag{24.18}$$

where I is the identity operator in the standard oscillator Hilbert space $\mathcal{H}$ and we take Planck's constant $\hbar$ and the oscillator constant ω to be unity. These ladder operators have the representations

$$\begin{aligned} a &= \sum_{n=0}^{\infty} |n\rangle \sqrt{(n+1)\,2} \langle n+1|, \\ a^\dagger &= \sum_{n=0}^{\infty} |n+1\rangle \sqrt{(n+1)\,2} \langle n|, \end{aligned} \tag{24.19}$$

where the states $|n\rangle$, $n = 1, 2, 3, \ldots$ are the usual orthonormal excited states of the oscillator ground state $|0\rangle$. The key to the QDN description is the observation that these states are identified one-to-one with the bosonic states $\mathbf{2}^n$ discussed above, namely,

$$|n\rangle \leftrightarrow \mathbf{2}^n, \quad n = 0, 1, 2, \ldots \tag{24.20}$$

We note that

$$|0\rangle \leftrightarrow \mathbf{1}^0\mathbf{0}^1\mathbf{0}^2\mathbf{0}^3 \ldots = \mathbf{1}, \quad |1\rangle \leftrightarrow \mathbf{0}^0\mathbf{1}^1\mathbf{0}^2\mathbf{0}^3 \ldots = \mathbf{2}, \tag{24.21}$$

and so on. A particularly significant observation is that the quantum register signal ground state $\mathbf{0}$ is *not* the oscillator ground state $|0\rangle$. This is one of the reasons we felt it necessary to introduce nonstandard notation for labstates: keeping a clear distinction between those and conventional QM states is more than a matter of notion but reflects deeper interpretational issues.

To find a quantum register representation of the ladder operators, we first introduce some auxiliary notation. We define the register operators

$$\mathbb{P} \equiv \left\{ \prod_{a=0}^{\infty} \boldsymbol{P}^a \right\}, \quad \mathbb{A}^a \equiv \left\{ \prod_{b \neq a}^{\infty} \boldsymbol{I}^b \right\} \boldsymbol{A}^a, \quad \widehat{\mathbb{A}}^a \equiv \left\{ \prod_{b \neq a}^{\infty} \boldsymbol{I}^b \right\} \widehat{\boldsymbol{A}}^a. \tag{24.22}$$

None of these operators is bosonic.

Exercise 24.9 Prove that the operators $\mathbb{P}$, $\mathbb{A}^a$, and $\widehat{\mathbb{A}}^a$ do not commute with the bosonic identity $\mathbb{I}_B$ defined by Eq. (24.15).

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The operators $\widehat{\mathbb{A}}^a$ can be used to generate bosonic states from the signal ground state $\mathbf{0}$, which is transbosonic.[3] Specifically, we have

$$\mathbf{2}^a = \widehat{\mathbb{A}}^a \mathbf{0}, \quad a = 0, 1, 2, \ldots \tag{24.23}$$

With these definitions we construct the operators

$$\widehat{\mathbb{B}}^a \equiv \widehat{\mathbb{A}}^a \mathbb{P} \mathbb{A}^{a+1}, \quad \mathbb{B}^a \equiv \widehat{\mathbb{A}}^{a+1} \mathbb{P} \mathbb{A}^a. \tag{24.24}$$

Remarkably, these operators are bosonic, as can be readily proved from the fact that

$$\mathbb{A}^a \mathbb{I}_B = \mathbb{A}^a. \tag{24.25}$$

We find for $a = 0, 1, 2, \ldots$

$$\begin{aligned}
\mathbb{B}^a &\equiv \left\{ \bigotimes_{b \neq a, a+1}^{\infty} \boldsymbol{P}^b \right\} \boldsymbol{A}^a \widehat{\boldsymbol{A}}^{a+1} = \boldsymbol{P}^0 \boldsymbol{P}^1 \ldots \boldsymbol{P}^{a-1} \boldsymbol{A}^a \widehat{\boldsymbol{A}}^{a+1} \boldsymbol{P}^{a+2} \boldsymbol{P}^{a+3} \ldots \\
\widehat{\mathbb{B}}^a &\equiv \left\{ \bigotimes_{b \neq a, a+1}^{\infty} \boldsymbol{P}^b \right\} \widehat{\boldsymbol{A}}^a \boldsymbol{A}^{a+1} = \boldsymbol{P}^0 \boldsymbol{P}^1 \ldots \boldsymbol{P}^{a-1} \widehat{\boldsymbol{A}}^a \boldsymbol{A}^{a+1} \boldsymbol{P}^{a+1} \boldsymbol{P}^{a+3} \ldots
\end{aligned} \tag{24.26}$$

These operators satisfy the relations

$$\widehat{\mathbb{B}}^a \mathbb{B}^b = \delta^{ab} \mathbb{P}^{a+1}, \quad \mathbb{B}^a \widehat{\mathbb{B}}^b = \delta^{ab} \mathbb{P}^a. \tag{24.27}$$

Then the standard ladder operators $a, a^\dagger$ take the quantum register representation

$$\begin{aligned}
a \leftrightarrow a_B &\equiv \sum_{n=0}^{\infty} \sqrt{2\,(n+1)} \mathbb{B}^n, \\
a^\dagger \leftrightarrow a_B^\dagger &\equiv \sum_{n=0}^{\infty} \sqrt{2\,(n+1)} \widehat{\mathbb{B}}^n.
\end{aligned} \tag{24.28}$$

These operators are bosonic and have the commutation relation

$$\left[a_B, a_B^\dagger \right] = 2 \mathbb{I}_B. \tag{24.29}$$

Because these register ladder operators commute with the bosonic identity $\mathbb{I}_B$, any states that they create from bosonic states are also bosonic. We find for example

$$|n\rangle \equiv \frac{\left(a^\dagger\right)^n}{\sqrt{2^n n!}} |0\rangle \leftrightarrow \frac{\left(a_B^\dagger\right)^n}{\sqrt{2^n n!}} \mathbf{1} = \mathbf{2}^n, \quad n = 0, 1, 2, \ldots \tag{24.30}$$

Note that a_B annihilates the bosonic ground state $\mathbf{1}$, giving the zero vector 0 in the quantum register $\mathcal{Q}^{[\infty]}$, not the signal ground state $\mathbf{0}$.

[3] The signal ground state $\mathbf{0}$ is not the same as the oscillator ground state $|0\rangle$, which is identified with the quantum register state $\mathbf{2}^0 = \mathbf{1}$ in the computational basis representation and $\mathbf{1}^0 \mathbf{0}^1 \mathbf{0}^2 \mathbf{0}^3 \ldots$ in the occupation basis representation.

The standard QM bosonic oscillator Hamiltonian operator $\widehat{H} \equiv \frac{1}{2}a^\dagger a + \frac{1}{2}$ has quantum register representation

$$\widehat{H} \leftrightarrow \mathbb{H}_B \equiv \tfrac{1}{2}a_B^\dagger a_B + \tfrac{1}{2}\mathbb{I}_B = \sum_{n=0}^{\infty}\left(n + \tfrac{1}{2}\right)\mathbb{P}_B^n. \tag{24.31}$$

This operator commutes with the bosonic identity $\mathbb{I}_B$, and therefore, any states in the quantum register that start off bosonic at initial time remain bosonic, if they evolve under this Hamiltonian.

24.6 Comparison with Quantum Field Theory

Several factors indicate that the QDN formalism is a halfway house between fixed-particle number Schrödinger–Dirac quantum mechanics and the multiparticle formalism of quantum field theory (QFT). First, the QDN strategy focuses not on particles but on signals. There is no intrinsic requirement to conserve signality except in those experiments where there is a physical reason, such as the conservation of some quantum number, such as electric charge. Second, the existence of transbosonic states in the QDN register $\mathcal{Q}^{[\infty]}$ is a clear marker that the QDN formalism can accommodate the equivalent of QFT multiparticle states.

An important similarity between QDN and QFT is the existence of the Fock vacuum state $|0\rangle$ in QFT and the signal ground state $\mathbf{0}$ in QDN. We recall that in QFT, Fock space $\mathcal{F}$ is defined by expansions of the form

$$\mathcal{F} \equiv \{|0\rangle\} \oplus \mathcal{H} \oplus \mathfrak{S}(\mathcal{H} \otimes \mathcal{H}) \oplus \cdots, \tag{24.32}$$

that is, as the infinite direct sum of appropriately symmetrized tensor products of copies of a single-particle Hilbert space $\mathcal{H}$. Looking at the above presented QDN version of the bosonic oscillator, we see that the QDN signal ground state $\mathbf{0}$ corresponds to the QFT vacuum $|0\rangle$; the QDN bosonic state space $\mathcal{Q}_B$ corresponds to $\mathcal{H}$; and the transbosonic states in QDN correspond to the multiparticle states in QFT.

The QDN formalism may be interpreted as an attempt to encode Fock's vision of QFT into a mathematical formalism that is based on detectors rather than SUOs. In the next chapter, we go further in this respect by extending the QDN formalism to allow for the possibility of constructing the signal ground state $\mathbf{0}$ itself through the creation of apparatus in the laboratory from a state of nonexistence, denoted $\emptyset$, that corresponds to the information void that we have focused on in other chapters. The information void represents the state of a laboratory in which there are no detectors whatsoever. However, this certainly does not mean that there is no observer or laboratory.

24.7 Fermionic Oscillators

In this section we extend the above discussion of the bosonic oscillator to the fermionic oscillator, that is, a system under observation (SUO) that requires anticommuting degrees of freedom in its classical (Martin, 1959b; Casalbuoni, 1976a) and quantum formulations (Candlin, 1956; Martin, 1959a; Casalbuoni, 1976b).

Not long after the discovery of quantum mechanics by Heisenberg and Schrödinger, Jordan and Wigner showed how to describe fermions in quantum register terms (Jordan and Wigner, 1928; Bjorken and Drell, 1965). Their construction of local fermionic quantum field operators requires tensor product contributions from all of the qubits in a quantum register. In a QDN approach to fermionic quantum fields (Eakins and Jaroszkiewicz, 2005), their techniques were used to describe fermionic quantum fields using an infinite-rank quantum register associated with a net of detectors distributed throughout all of physical space. Because the Jordan–Wigner construction requires nontrivial contributions from all qubits in the register, fermionic fields are manifestly and inherently nonlocal in QDN.

In contrast with the bosonic oscillator studied above, we may restrict attention to a finite-rank quantum register $\mathcal{Q}^{[N]} \equiv Q^1 Q^2 \dots Q^N$.

The most convenient basis here is the signal basis representation. We follow the approach outlined in Bjorken and Drell (1965) for the construction of fermionic operators.

We use all the bit operators discussed previously and introduce a new one, denoted $\boldsymbol{\sigma}^a$, for each qubit $a = 1, 2, \dots, N$ in the register, defined by

$$\boldsymbol{\sigma}^a \equiv \boldsymbol{P}^a - \widehat{\boldsymbol{P}}^a, \quad a = 1, 2, \dots, N. \tag{24.33}$$

Next, we define a set of nonlocal operators, α^a, $\widehat{\alpha}^a$, $a = 1, 2, \dots, N$:

$$\begin{aligned}
\alpha^1 &\equiv \boldsymbol{A}^1 \boldsymbol{I}^2 \boldsymbol{I}^3 \dots \boldsymbol{I}^N, \\
\alpha^a &\equiv \boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2 \dots \boldsymbol{\sigma}^{a-1} \boldsymbol{A}^a \boldsymbol{I}^{a+1} \dots \boldsymbol{I}^N, \\
\widehat{\alpha}^1 &\equiv \widehat{\boldsymbol{A}}^1 \boldsymbol{I}^2 \boldsymbol{I}^3 \dots \boldsymbol{I}^N, \\
\widehat{\alpha}^a &\equiv \boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2 \dots \boldsymbol{\sigma}^{a-1} \widehat{\boldsymbol{A}}^a \boldsymbol{I}^{a+1} \dots \boldsymbol{I}^N, \quad a = 1, 2, \dots, N.
\end{aligned} \tag{24.34}$$

These are the required “fermionic field operators.” They satisfy the anticommutation relations

$$\begin{aligned}
\alpha^a \alpha^b + \alpha^b \alpha^a &= 0, \qquad \widehat{\alpha}^a \widehat{\alpha}^b + \widehat{\alpha}^b \widehat{\alpha}^a = 0, \\
\alpha^a \widehat{\alpha}^b + \widehat{\alpha}^b \alpha^a &= \delta^{ab} \mathbb{I}, \qquad 1 \leqslant a, b, \leqslant N.
\end{aligned} \tag{24.35}$$

where $\mathbb{I} \equiv \boldsymbol{I}^1 \boldsymbol{I}^2 \dots \boldsymbol{I}^N$ is the register identity operator.

Exercise 24.10 Use the signal bit algebra listed in Table 4.1 to prove (24.35).

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The α^a, $\widehat{\alpha}^b$ operators have the following properties, which are easy to prove:

$$\begin{aligned} \alpha^a \mathbf{0} &= 0, \\ \widehat{\alpha}^a \mathbf{0} &\neq 0, \\ \widehat{\alpha}^a \widehat{\alpha}^a \mathbf{0} &= 0, \quad a = 1, 2, \ldots, N. \end{aligned} \tag{24.36}$$

This means that we can reconstruct all the relevant structure of fermionic field theory, the operators $\widehat{\alpha}^a$ and α^a playing the role of fermionic creation and annihilation operators.

A significant feature of the above anticommutation relations is that they are achieved via the use of the nonlocal operators and do not invoke Grassmannian (anticommuting) numbers to do so.