SOME NOZZLE FLOWS FOUND BY THE HODOGRAPH METHOD

T. M. CHERRY

(rec. 13 Oct. 1958)

1. Introduction

For investigating the steady irrotational isentropic flow of a perfect gas in two dimensions, the hodograph method is to determine in the first instance the position coordinates x, y and the stream function ψ as functions of velocity components, conveniently taken as q (the speed) and θ (direction angle). Inversion then gives ψ, q, θ as functions of x, y. The method has the great advantage that its field equations are linear, so that it is practicable to obtain exact solutions, and from any two solutions an infinity of others are obtainable by superposition. For problems of flow past fixed boundaries the linearity of the field equations is usually offset by non-linearity in the boundary conditions, but this objection does not arise in problems of trans-sonic nozzle design, where the rigid boundary is the end-point of the investigation.

Accordingly, this paper aims at showing something of the potentialities of the hodograph method for constructing nozzle flows.

It is well known that, for a trans-sonic nozzle, supersonic velocities which are near (in magnitude and direction) to the sonic velocity at the centre of the throat occur at three distinct points (x, y); so on inversion x, y must be three-valued functions of q, θ. For applying the hodograph method to nozzle-flows, therefore, the essential problem is to find solutions of the hodograph equations that are three-valued in the neighbourhood of the sonic point. In [1] I have given a method for constructing such 'trans-sonic' solutions, with full numerical detail for one of them, here called '$\mathrm{R1}\,\Omega_2$'. From the nozzle flow specified by this solution an infinity of others can be obtained by superposing other solutions (which in themselves may be devoid of physical interest). In this paper I survey the solutions which are readily available as the raw material for the superposition, and give specific examples, calculated for γ (the adiabatic index) equal to 1.4. For application to conventional nozzle design the object of such constructions would be to obtain a nozzle flow-field for which the throat region is suitably shaped and which is carried with adequate precision some distance into the supersonic region; the designer can then prolong the flow field through 'simple wave' regions so as to merge it into a uniform supersonic flow. In § 7, however, I show the possibility of avoiding the simple wave regions, and by hodograph procedures arriving at nozzle shapes for which the flow is everywhere analytic and ultimately uniform.

2. Field Equations

As fundamental dependent variable we take the Legendre potential Ω, whose field equation is

(1) $$q^2(1-q^2)\Omega_{qq} + (1-q^2/q_s^2)(q\Omega_q + \Omega_{\theta\theta}) = 0;$$

here the unit for q is so chosen that $q = 1$ gives the limiting or cavitation speed, while $q_s = \sqrt{(\gamma-1)/(\gamma+1)}$ gives the sonic speed. From any solution of (1) the position coordinates are obtained, in non-dimensional measure, by

(2) $$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta,$$

where

(3) $$X = \Omega_q, \; qY = \Omega_\theta,$$

and the stream function is determined from the consistent equations

(4) $$\psi_q = (1-q^2)^\beta (Y - X_\theta), \; \psi_\theta = -q(1-q^2)^\beta (X + Y_\theta),$$

where $\beta = 1/(\gamma-1)$. In any practical case the inversion to get q, θ as functions of x, y has to be performed numerically; the first step is to determine loci $\psi =$ constant by interpolation in a table of $\psi(q, \theta)$, and then successions of points (x, y) on such loci are found via tables of $X(q, \theta)$ and $Y(q, \theta)$. The arithmetic is elementary, and on modern standards not too burdensome. An overall check can be made from the fact that the slope of a streamline as found from its x, y coordinates must equal θ.

3. Standard Solutions

I. *The Chaplygin set* of potentials $C(\nu)$ of the form

(5) $$\Omega = C(\nu) = q^\nu F_\nu(q^2)\cos\nu\theta \quad (\nu \text{ constant}),$$

where F_ν is in general a hypergeometric function*, is obtained from (1) by separating the variables. This represents a flow having the line $\theta = 0$ as an axis of symmetry, but it runs into a limit-line at the sonic point on

* When ν is a negative integer F_ν is a logarithmically modified hypergeometric function. The case $\nu = -1$ is notable in that F_{-1} is an elementary function (Ringleb [2]).

the axis so it is useful only in superposition. Tables (Ferguson and Lighthill [3], Huckel [4], Petschacker [5]) are available whereby, for $\gamma = 1.4$, the Ψ, X, Y corresponding to $C(\nu)$ can easily be found for a wide set of values of ν, q. In [3] are tabulated functions $\psi_\nu(\tau)$, $\psi'_\nu(\tau)$, where $\tau = q^2$, and in terms of these

$$(6)\quad \begin{cases} X_{C(\nu)} = \dfrac{\nu}{\nu+1}\,\dfrac{\psi_\nu(\tau) + 2\tau\psi'_\nu(\tau)}{q(1-q^2)^\beta}\cos\nu\theta, \\ \qquad\qquad Y_{C(\nu)} = -\dfrac{\nu^2\psi_\nu(\tau) + 2\tau\psi'_\nu(\tau)}{q(1-q^2)^\beta}\sin\nu\theta, \\ \psi_{C(\nu)} = (\nu - 1)\psi_\nu(\tau)\sin\nu\theta. \end{cases}$$

[3] also gives the hypergeometric function $Y_\nu(\tau) = q^{-\nu}\psi_\nu(\tau)$ and its derivative, and these are the functions tabulated in [4]. [5] gives the $q^\nu F_\nu(\tau)$ and $F_\nu(\tau)$ of (5), but not their derivatives.

II. *Radial flow.* The solution

$$(7)\quad \Omega = R = \int \frac{dq}{q(1-q^2)^\beta},\quad X_R = \frac{1}{q(1-q^2)^\beta},\quad Y_R = 0,\quad \Psi_R = -\theta$$

represents purely radial flow, and is closely related to the vacuous case $\nu = 0$ of (5). Here also there is a limit line $q = q_s$, and the solution is useful only in superposition.

III. *A trans-sonic set**. To obtain a flow unencumbered by limit lines the method of [1] is to change the variables from q, θ to q, ϕ, where

$$(8)\qquad \theta = \phi - 2\alpha \arctan\frac{q\sin\phi}{1 - q\cos\phi},\quad 2\alpha(1+\alpha) = \beta,\quad \alpha > 0.$$

The singular locus $\partial\theta/\partial\phi = 0$ for this transformation is

$$(9)\qquad D \equiv 1 - 2(1+\alpha)q\cos\phi + (1+2\alpha)q^2 = 0,$$

or in terms of q, θ

$$(10)\qquad \theta = \pm\left[\frac{1}{q_s}\arctan\sqrt{\frac{q^2-q_s^2}{1-q^2}} - \arctan\sqrt{\frac{q^2/q_s^2-1}{1-q^2}}\right] = \pm\omega(q)$$

The key fact is that this locus is characteristic for (1); hence it is that there are potentials $\Omega(q, \phi)$ which are single-valued across the locus and which in terms of q, θ are triple valued, as required for a nozzle-flow. The result of transforming (1) by (8) is **

* Of the solutions Ω_ν to be discussed only one represents a smooth trans-sonic nozzle-flow, but I name the whole set from this one.

** It is to be noted that in the $q\phi$-context the suffix q indicates a q-derivative for ϕ constant, which is unequal to the q-derivative for θ constant implied by the suffix in (1), (3).

$$D\left(q^2\Omega_{qq}+q\Omega_q+\Omega_{\phi\phi}-\frac{2\beta q^3\Omega_q}{1-q^2}\right)+4\alpha q^2\sin\phi\left(\Omega_{q\phi}-\frac{\beta q\Omega_\phi}{1-q^2}\right)$$
$$+\frac{4\alpha q(\cos\phi-2q+q^2\cos\phi)}{1-q^2}\Omega_{\phi\phi}=0,$$

and in [1] a set of solutions Ω_0, Ω_1, $\cdots$ of this equation has been defined, with full numerical detail as regards Ω_0, Ω_1 and Ω_2. They have the general form

(11) $$\Omega_p = H_p(q,\ \phi)+G_p(q^2,\ qe^{i\phi}),$$

where H_p is an explicit elementary function and G_p is a power series in q^2, $qe^{i\phi}$ which converges for $0\leqq q^2<1$ and $|qe^{i\phi}|<1$; the convergence is fairly rapid for q as large as 0.75. It is hence practicable to use these solutions for $\mathrm{Im}\,\phi>0$ as well as for ϕ real, and an illustration of this will be given in § 7. Regarding the elementary functions H_p it is sufficient to remark that H_1, H_2, $\cdots$ are regular on the singular locus $D=0$, and indeed for all q, ϕ that come into question; but H_0 is singular on this locus:

(12) $$H_0=\Omega_0\ (\text{princ. part})=\frac{(1-q^2)^{1-\beta/2}}{(1+\alpha)D}.$$

The potential Ω_p is i-times the θ-derivative of Ω_{p+1}: in terms of q, ϕ

(13) $$D\Omega_p = i(1-2q\cos\phi+q^2)(\Omega_{p+1})_\phi.$$

Since the coefficients in (1) are real, the real and unreal parts of Ω_p (as functions of q, θ) will separately satisfy (1), provided in the case where ϕ is unreal that (8) gives a real θ. $\mathrm{Rl}\,\Omega_2$ gives a trans-sonic nozzle flow, plotted in figure 3.

For plotting the flow-field corresponding to the real or unreal part of Ω_p we take ϕ as an auxiliary parameter in the calculation of X, Y, ψ. The values X_p, Y_p, ψ_p corresponding to Ω_p are found, via (3), (4), (8) and (13), to be

(14) $$\begin{cases} X_p=(\Omega_p)_q-\dfrac{2i\alpha\Omega_{p-1}\sin\phi}{1-2q\cos\phi+q^2}, \quad Y_p=-\dfrac{i\Omega_{p-1}}{q}, \\ \psi_p=-q(1-q^2)^\beta(iX_{p+1}+Y_p), \end{cases}$$

and corresponding to $\mathrm{Rl}\ \Omega_p$ we take the real parts of these.

4. Symmetrization

To obtain a nozzle-flow having $\theta=0$ as an axis of symmetry we require a potential Ω which is an even function of θ; for then X, x are even functions and Y, y, ψ odd. Now if $\Omega=f(q,\ \theta)$ is *any* solution of (1), $f(q,\ \theta_0+\theta)$ and

$f(q, \theta_0 - \theta)$ also are solutions, and by superposition we obtain $\Omega = f(q, \theta_0 + \theta) + f(q, \theta_0 - \theta)$ as an even solution.

This symmetrization process gives nothing new when applied to the potentials $C(\nu)$ or R, but as regards Ω_p it gives different flows for each value of θ_0. Starting from the potential $\Omega_p(q, \phi)$, we obtain $f(q, \theta_0 + \theta) = f'(q, \theta)$ say, and $f(q, \theta_0 - \theta) = f''(q, \theta)$ by applying in place of (8) the transformations

$$\theta_0 \pm \theta = \phi - 2\alpha \arctan \frac{q \sin \phi}{1 - q \cos \phi}. \tag{15}$$

For the upper sign in (15) the formulae (14) remain valid, but for the lower sign we have to reverse in (14) the signs of Y_p, ψ_p. Hence if, for a given θ, we call ϕ', ϕ'' the values of ϕ that satisfy (15) with the upper and lower signs respectively; if X'_p, Y'_p, ψ'_p are the values given by (14) for $\phi = \phi'$; and if X''_p, Y''_p, ψ''_p are the values given again by (14) for $\phi = \phi''$: then the symmetrized potential $f' + f''$ leads to the values

$$\begin{gathered} X = X'_p + X''_p, \quad Y = Y'_p - Y''_p, \\ \psi = -q(1-q^2)^\beta (iX'_{p+1} - iX''_{p+1} + Y'_p - Y''_p), \end{gathered} \tag{16}$$

and thence the position coordinates x, y follow from (2).

We denote by $S_p(\theta_0)$ the potential thus derived from Ω_p.

5. The Axial qx-Relation

When the speed q of a nozzle flow on the axis of symmetry Ox is known or prescribed the whole flow pattern is determinate, and an approximation to it is furnished by the 'hydraulic' approximation that $yq(1 - q^2)^\beta$ is constant on any streamline. A rough comparison of two flow patterns may accordingly be inferred from their axial qx-relations. On the hodograph method, inversely, we consider x as a function of q, and by (2) $x = X$ when $\theta = 0$, so by (3) the axial relation is $x = \partial\Omega/\partial q$.

This remark is significant in relation to the superposition of standard solutions; the functions X attaching to them are primary numerical data, so the axial xq-relation for a given superposition can be tabulated by trivial labour.

In figure 1 are shown the axial xq-relations for the potentials $\mathrm{R1}\,\Omega_2$ and R. The former shows a trans-sonic flow originating at a stagnation point (x finite for $q = 0$). For the latter, the minimum at $q = q_s$ indicates a flow with the limit-line mentioned in § 3 II. The figure shows also the axial relations for the potential

$$\mathrm{R1}\,\Omega_2 - AR \tag{17}$$

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for certain values of A. The leading feature here is that for $q \sim 0$ the term $-AR$ is dominant, so on the subsonic side we get a flow coming from infinity, approximately radially. Another feature, which however, for

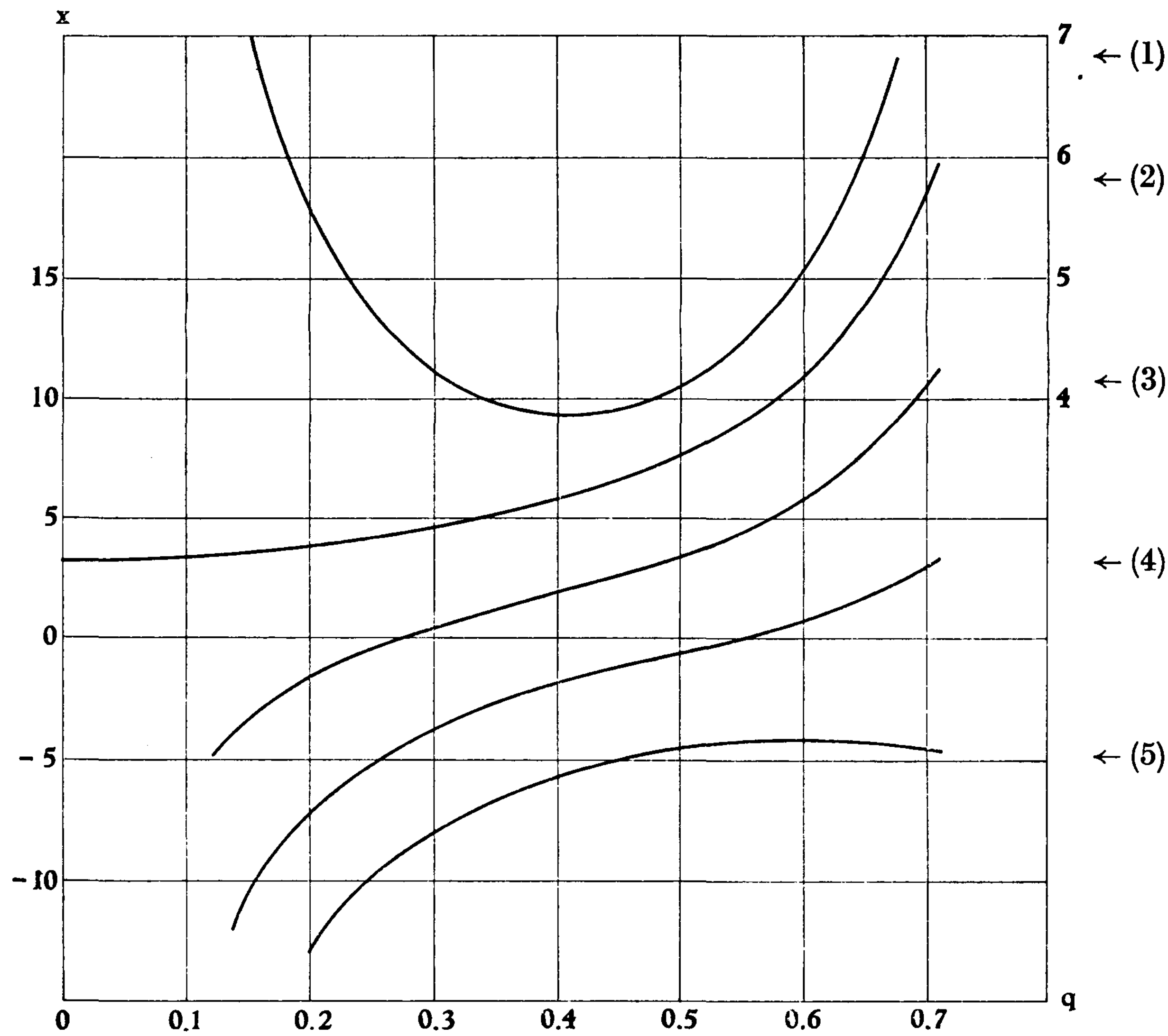

Fig. 1. Axial xq-relations for the potentials: Curve 1: R : (x-scale on right). Curves 2, 3, 4, 5: $\mathrm{Rl}\,\Omega_2 - AR$, for $A = 0, 1, 2, 3$, (x-scale on left).

practical purposes is likely to be less important, is that for $q \sim 1$ the R-curve is steeper than that for $\mathrm{Rl}\,\Omega_2$, so (17) must give $\mathrm{d}x/\mathrm{d}q = 0$ for some supersonic q depending on A, and specifies a nozzle-flow terminating in a limit line in the far supersonic region*.

For the potential $C(\nu)$ the axial relation is $x = \mathrm{d}(q^\nu F_\nu)/\mathrm{d}q$. For $0 < \nu < 1$ its graph is qualitatively like that for R, the significant distinction being that for $q \sim 0$, $x \sim \nu q^{\nu-1}$. For a certain negative range $\nu_0 < \nu < 0$ also the

* The behaviour for $q \sim 1$ is

for $\mathrm{Rl}\,\Omega_\nu$, $x \sim A_\nu(1-q)^{1-\beta/2}$

for R $x \sim (1-q)^{-\beta}$

for $C(\nu)$ $x \sim A_\nu(1-q)^{-\beta}$,

where A_ν changes sign at transitional values of ν.

graph is like that for R, but the other way up; $\nu_0 = (13-\sqrt{889})/10 = -1.68$ for $\gamma = 1.4$. As ν decreases from ν_0 the graph developes turning points additional to that at q_s, while in transitional cases x remains finite at $q = 1$; for example F_ν is a quadratic in q^2 when $\nu = \nu_0$.

From these facts the general march of the axial xq-relations for $C(\nu)$ superposed on Rl Ω_2 can be inferred; for example for $\nu = \nu_0$ the relation is monotonic over the whole range $0 < q < 1$ and the superposition gives a nozzle unencumbered by limit lines anywhere near its axis. In general, for $q \sim 0$ the flow is dominated by the component $C(\nu)$ and the streamlines are approximately $x = (\text{const.})\, y^{1-\nu}$.

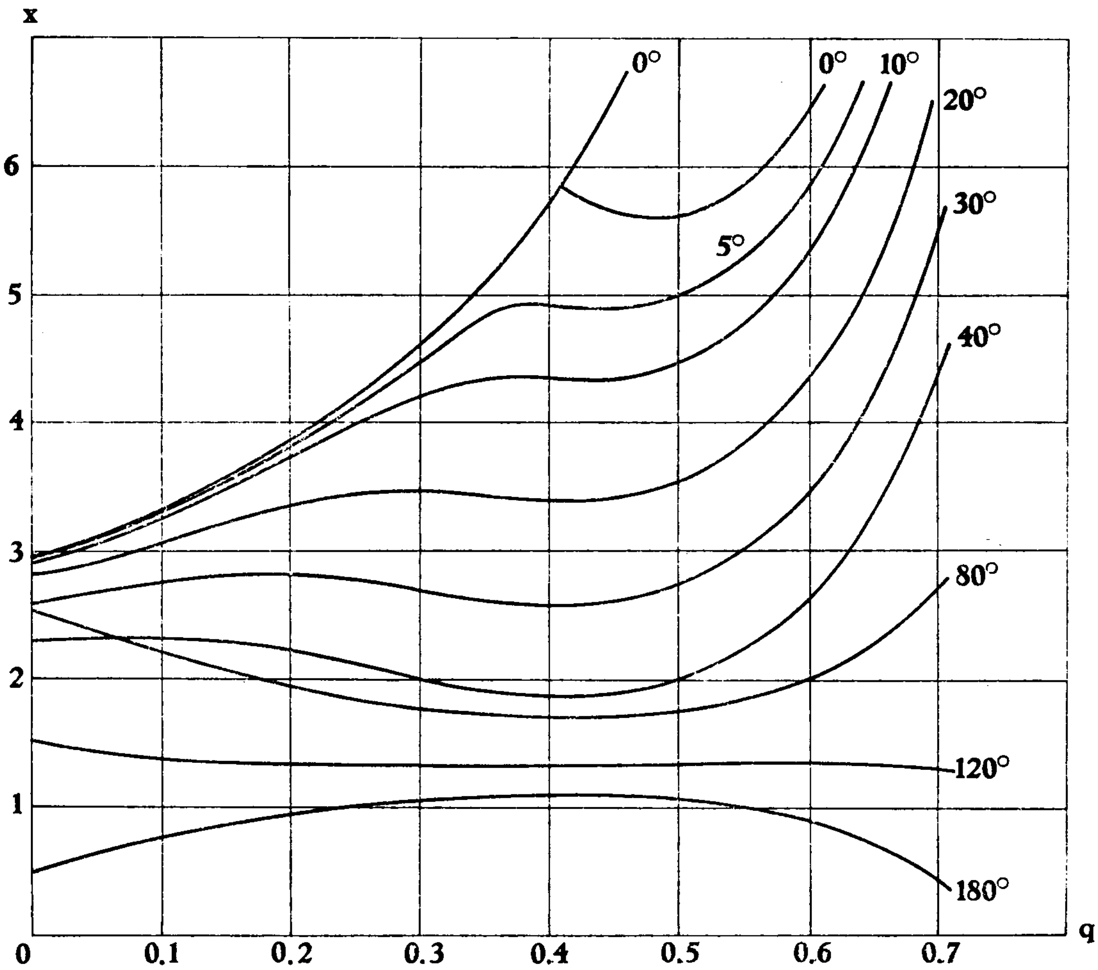

Fig. 2. Axial xq-relations derived from Rl Ω_2 by symmetrization, for $\theta_0 = 0°, 5°, 10°, 20°, 30°, 40°, 80°\ 120°, 180°$. (For compactness the curves for 80°, 120°, 180° have been displaced upwards 2, 3, 3·5 units respectively).

In fig. 2 are shown the axial xq-relations derived from the potential Rl Ω_2 by symmetrization (§ 4) for various values of the parameter θ_0. For $\theta_0 = 0$ the curve branches into two at $q = q_s$; this is because the inversion of (8) gives ϕ as a multiple valued function of θ. The smooth branch of this curve belongs to a smooth trans-sonic flow, but all the others have $dx/dq=0$ for $q = q_s$. For θ_0 around 120° the relation is remarkable in that x is almost constant over a long range of q.

An arbitrary xq-relation can be fitted in the Fourier manner by a series of the X's derived from a suitably chosen sequence of potentials, at least provided the relation is analytic. I shall not here enter into any general theory on this matter, but I give an example of a fitting arrived at by judicious trial. By finite difference methods Emmons [6] approximated to the trans-sonic solution of the non-linear physical flow equations (for $\gamma = 1.4$) for the region bounded by a hyperbola and its minor axis. His paper gives a graph of the axial xq-relation which emerged from this solution. An engineering friend converted the graph into a table, by machine readings recorded (rather speculatively) 'correct' to 0.001 inch, and I fitted this table by superposing four of the standard solutions, viz.

(18) $$1.27\, x(\text{Emmons}) \sim \mathrm{Rl}\, X_2 - 1.091\, X_R - 0.331\, X_{S(22°)} - 8.35 X_{C(10)}.$$

The fitting was made at 19 points ranging from $q = 0.176$ $x = -3.995$ to $q = 0.648$, $x = 3.108$, (specimens are shown in the table below) and the maximum residual was 0.004, a proportional accuracy of about 1 in 1000.

q	$\mathrm{Rl}\, X_2$	$-1{\cdot}091 X_R$	$-0{\cdot}331 X_{S(22°)}$	$-8{\cdot}35 X_{C(10)}$	sum	sum $\div$ 1·270	x_E
0.176	3.737	−6.703	−2.110	0.000	−5.076	−3.997	−3.995
0.400	5.756	−4.218	−2.138	−0.015	−0.615	−0.484	−0.483
0.566	9.594	−5.058	−2.512	+0.040	2.064	1.625	1.627
0.648	13.828	−6.571	−3.296	−0.010	3.951	3.111	3.108

The streamline calculated thence from the hodograph formulae, with a view to fitting Emmons's hyperbolic boundary, is however discrepant by about 1 in 30 at its subsonic end. The explanation, in view of the close axial fitting, is perhaps that Emmons's numerical solution was more accurate — as *a* solution of the flow equations — near the axis than at remote points.

6. Flow-Fields Calculated by Superposition

For the potential $\mathrm{Rl}\, \Omega_2 - AR$ the flow-field is shown in figure 3 for $A = 0$, and in figure 4 for $A = 1.012$, 1.840. In figure 3 the whole flow-field, up to the bounding limit-lines, is indicated. In figure 4 the fields are carried far enough transversely to show the breakdown of regular flow by the appearance of limit-lines in the throat region; the outermost regular streamline has infinite curvature at the point of appearance (cusp) of the limit-lines, and this point is upstream from the geometrical throat for $A = 1.012$, but about at the throat for $A = 1.84$.

Figure 5 shows (i) one streamline for the superposition of four standard potentials used to imitate Emmons's solution as in (18), in comparison with

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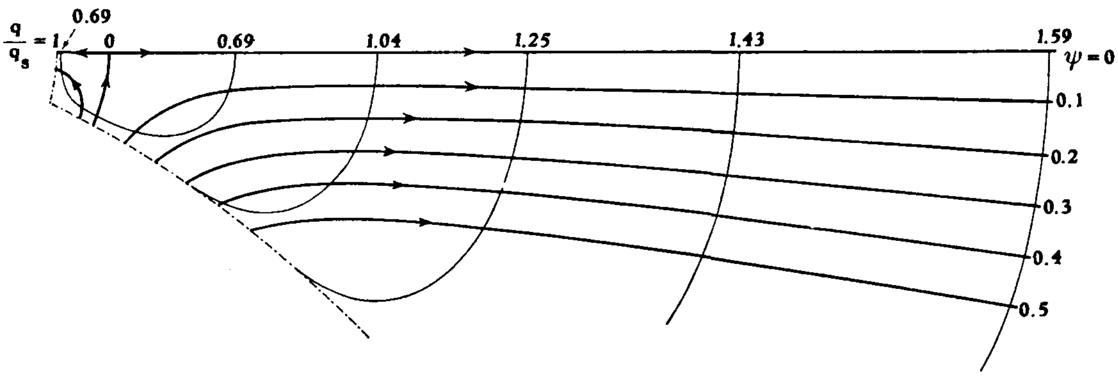

Fig. 3. Flow-field for potential Rl Ω_2. Streamlines: thick. Isovels: thin. Limit lines: chain dot.

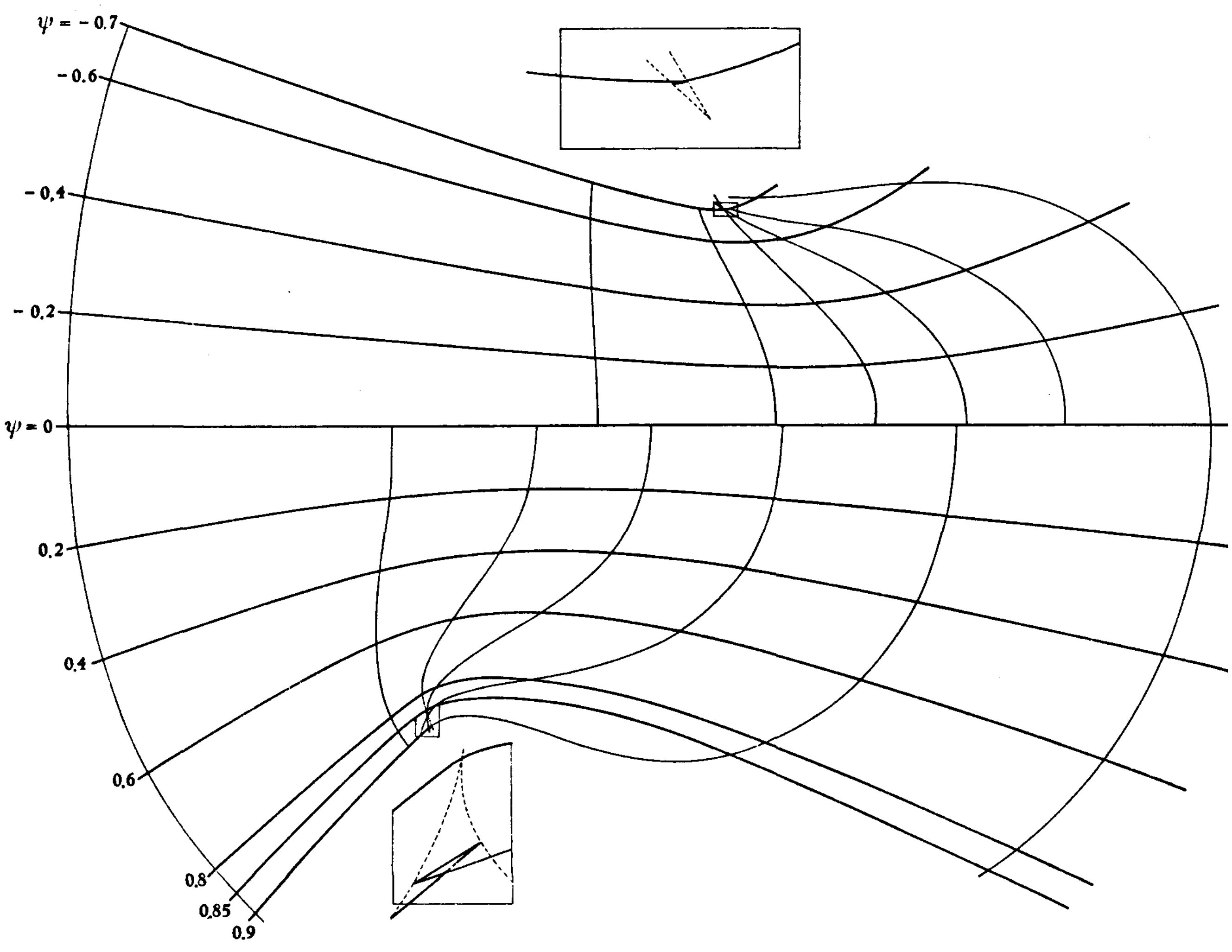

Fig. 4. Flow-fields for potentials Rl $\Omega_2 - AR$. $A = 1{\cdot}012$: lower half. $A = 1{\cdot}840$: upper half. In each case the isovels are, from the left, $q/q_s = 0{\cdot}34$, $0{\cdot}69$, $0{\cdot}98$, $1{\cdot}20$, $1{\cdot}39$, $1{\cdot}55$, $1{\cdot}70$ i.e. $M = 0{\cdot}32$, $0{\cdot}66$, $0{\cdot}98$, $1{\cdot}26$, $1{\cdot}53$, $1{\cdot}83$, $2{\cdot}15$. Enlargements show incipient limit lines (dotted).

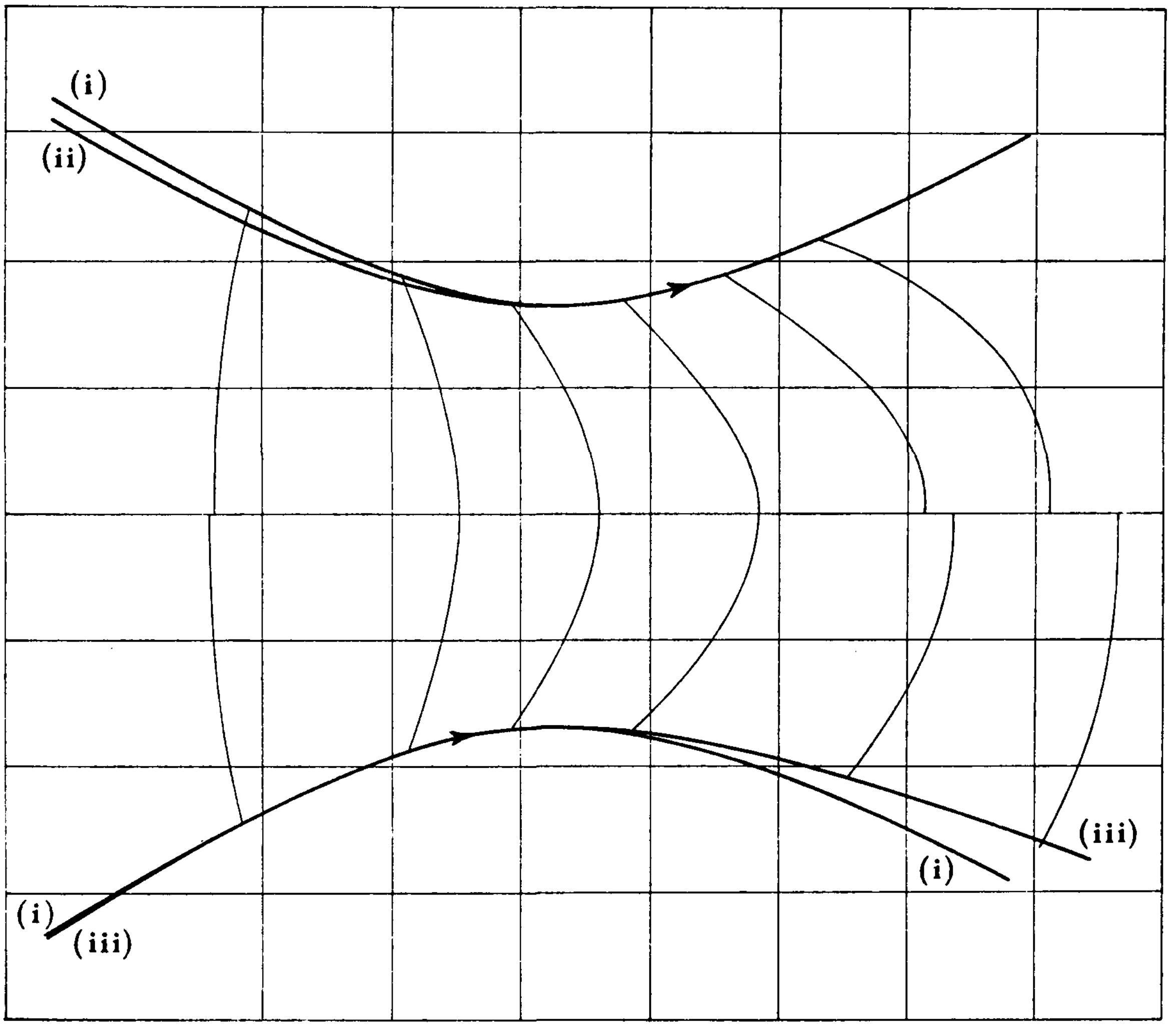

Fig. 5. Flows close to that of Emmons (see text). Isovels for $M = 0{\cdot}46$, $0{\cdot}75$, $1{\cdot}0$, $1{\cdot}32$, $1{\cdot}68$, $1{\cdot}90$ are shown fro solution (i) above and solution (iii) below.

(ii) Emmons's hyperbolic boundary and (iii) a streamline for a superposition of three standard potentials $\mathrm{Rl}\,\Omega_2 - 1.012R + 4.86C(5)$. The comparison between (i) and (iii) is of interest because their axial xq-relations agree to about 1 in 1000 over the range $0.3 \leqq q \leqq 0.52$.

7. Nozzles in Which the Supersonic Flow is Ultimately Uniform

A supersonic wind tunnel has to have a test-section in which the velocity is uniform. In practice this section must be not too far downstream from the throat, and the transition to uniform flow from the non-uniform flow in the throat requires that there be at least two surfaces which in the analytical sense are singular for the flow, in that certain velocity-derivatives are there discontinuous. However, there are flows without singularities for which the velocity *tends* to uniformity at infinity downstream, and such a flow might be a good enough approximation to what is practically required.

Such asymptotically uniform nozzle flows may be constructed by the

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hodograph method. The ideas underlying what follows are (1) that because of the possibility of symmetrization (§ 4) we can in the first instance disregard the requirement of symmetry; (2) that to require $q \to q_0$ as $x \to \infty$ is in hodograph terms to require $x \to \infty$ as $q \to q_0$, and by (12), (14) we get this infinity on the singular locus $D = 0$ for the flow determined by the potential Ω_1. By reference to (10) we see then that, for a given limiting q_0, the θ_0 to use in the symmetrization process will be $\theta_0 = \omega(q_0)$.

Working now with the variables q, ϕ, (15) gives on the axis of the proposed nozzle

$$\theta_0 = \phi - 2\alpha \arctan \frac{q \sin \phi}{1 - q \cos \phi} = f(q, \phi), \quad \text{say,} \tag{19}$$

which is to be satisfied by a point (q_0, ϕ_0) on the singular locus $D = 0$.

For neighbouring points $(q_0 + \delta q, \phi_0 + \delta\phi)$ on the axis the relation between δq, $\delta\phi$ is found from a Taylor expansion. We have

$$\frac{\partial f}{\partial \phi} = \frac{D}{1 - 2q \cos \phi + q^2}, \quad \frac{\partial f}{\partial q} = -\frac{2\alpha \sin \phi}{1 - 2q \cos \phi + q^2},$$

and when $D = 0$,

$$\frac{\partial^2 f}{\partial \phi^2} = \frac{2(1 + \alpha) q \sin \phi}{1 - 2q \cos \phi + q^2}.$$

Hence the Taylor series for points on the axis is

$$0 = -\left.\frac{2\alpha \sin \phi}{1 - 2q \cos \phi + q^2}\right|_0 \delta q + \left.\frac{(1 + \alpha) q \sin \phi}{1 - 2q \cos \phi + q^2}\right|_0 (\delta\phi)^2 + \cdots, \tag{20}$$

and since for a *trans-sonic* nozzle the relevant values of δq are negative, $\delta\phi$ must be to a first approximation pure-imaginary. Hence, for real θ_0, q, it is an *unreal* solution ϕ of (19) that is relevant; and since the non-elementary parts of the potentials Ω_0, Ω_1, Ω_2 are power series in q^2, $qe^{i\phi}$, computation will be easier if we take ϕ to have a positive imaginary part.

We have now to show that the complete ranges of q, θ that can occur in a nozzle such as is proposed are furnished by unreal solutions of (15).

If $\phi = \sigma + i\zeta$, the condition that (15) gives $\theta_0 \pm \theta$ real is

$$\cos \sigma = \frac{\sinh (\zeta/\alpha) + q^2 \sinh (2\zeta + \zeta/\alpha)}{2q \sinh (\zeta + \zeta/\alpha)}, \tag{21}$$

and then

$$\theta_0 \pm \theta = \sigma - \alpha \arctan \frac{2q \cosh \zeta \sin \sigma - q^2 \sin 2\sigma}{1 - 2q \cosh \zeta \cos \sigma + q^2 \cos 2\sigma}. \tag{22}$$

For a given ζ, (21) gives a real σ between $\pm\frac{1}{2}\pi$ if

(23)
$$\frac{\sinh (\zeta/2\alpha)}{\sinh (\zeta + \zeta/2\alpha)} \leqq q \leqq \frac{\cosh (\zeta/2\alpha)}{\cosh (\zeta + \zeta/2\alpha)},$$

and it is easily shown that as q runs over this range (so that σ makes an excursion from 0 towards $\pm\frac{1}{2}\pi$ and then back to 0), the inverse tangent in (22) runs from 0 to $\pm\pi$, so that $\theta_0 \pm \theta$ runs from 0 to $\mp\pi\alpha$. By continuation, values of $\theta_0 \pm \theta$ outside $(-\pi\alpha, \pi\alpha)$ are obtained by further excursions of q over the range (23), so all values of θ are attainable. Finally, in the limit $\zeta = 0$, (21) gives $\cos\phi = \cos\sigma = \{1 + (1+2\alpha)q^2\}/2(1+\alpha)q$, so that (q, ϕ) is on the singular locus $D = 0$; and as ζ increases to ∞ the q-range (23) contracts continuously towards $q = 0$. Hence all relevant values of q, θ are covered; it may be noted that for $\theta = \alpha\pi$, (10) gives $q = 1$, so any supersonic speed in the ultimate test-section can be secured by suitably choosing θ_0 between 0 and $\alpha\pi$.

The potential Ω_1 is unreal, and we shall show that a flow having the desired character when (q, ϕ) is near the singular locus is obtained by symmetrizing its real part. In the desired neighbourhood ζ is small, and (2) gives (suppressing till the end the zero suffix from q_0, σ_0)

$$\cos\sigma = \frac{1 + (1 + 2\alpha)q^2}{2(1+\alpha)q} + O(\zeta^2),$$

and thence

$$\begin{aligned} D &= 1 + (1 + 2\alpha)q^2 - 2(1+\alpha)q\cos\sigma\cosh\zeta + 2i(1+\alpha)q\sin\sigma\sinh\zeta \\ &= 2i(1+\alpha)q\zeta\sin\sigma + O(\zeta^2). \end{aligned}$$

Hence (11) and (12) give

$$i\Omega_0 = \frac{(1 - q^2)^{1-\beta/2}}{2(1+\alpha)^2 q\zeta\sin\sigma} + O(1).$$

The principal parts of X_1, Y_1, ψ_1 now follow from (14), viz.

(24)
$$\begin{cases} X_1 = -\dfrac{\alpha(1 - q^2)^{1-\beta/2}}{(1+\alpha)^2 q(1 - 2q\cos\sigma + q^2)\zeta} + O(1), \\ Y_1 = -\dfrac{(1 - q^2)^{1-\beta/2}}{2(1+\alpha)^2 q^2 \sin\sigma \cdot \zeta} + O(1) \\ \psi_1 = -q(1 - q^2)^{\beta} Y_1 + O(1), \end{cases}$$

and taking real parts leaves these unaffected.

For the symmetrization we must form combinations, as in (16), of the values (24) that belong to the two transformations (22). At the point (q, σ) on the singular locus with which we are concerned the right hand side of (22) (with $\zeta = 0$) reduces to θ_0. For a neighbouring point $(q - Q, \sigma + \delta\phi)$, we obtain as a generalization of (20)

$$(25) \qquad \pm\theta = \frac{2\alpha \sin\sigma \cdot Q}{1 - 2q\cos\sigma + q^2} + \frac{(1+\alpha)Q\sin\sigma}{1 - 2q\cos\sigma + q^2}(\delta\phi)^2 + \cdots,$$

and for θ small this gives

$$(26) \qquad \zeta = -i\delta\phi = \sqrt{\frac{2\alpha Q}{(1+\alpha)q}}\sqrt{\left(1 \mp \frac{1 - 2q\cos\sigma + q^2}{2\alpha\sin\sigma}\frac{\theta}{Q}\right)}.$$

Distinguishing as ζ', ζ'' the values of ζ belonging to the two signs, the principal part of ψ becomes

$$(27) \qquad -q(1-q^2)^{\beta}(Y_1' - Y_1'') = \frac{q(1-q^2)^{1+\beta/2}}{2(1+\alpha)^2 q^2 \sin\sigma}\left(\frac{1}{\zeta'} - \frac{1}{\zeta''}\right).$$

Now near the throat of the proposed nozzle all the formulae are regular and it will be sufficient to restrict ψ to be bounded. Since ψ is constant along a streamline we may therefore restrict θ so that in (27) $1/\zeta' - 1/\zeta''$ is bounded, and by (26) the condition for this is

$$(28) \qquad \theta = O(Q^{3/2}).$$

Hence the values of $1/\zeta'$, $1/\zeta''$ derived from (26) may be expressed as binomial series in θ/Q, and for the symmetrized solution we obtain from (24) the principal parts

$$(29) \qquad \begin{cases} X = -\dfrac{\alpha(1-q_0^2)^{1-\beta/2}}{(1+\alpha)^2 q_0(1 - 2q_0\cos\sigma_0 + q_0^2)}\left(\dfrac{1}{\zeta'} + \dfrac{1}{\zeta''}\right) = -\dfrac{A}{\sqrt{Q}}, \\ Y = -\dfrac{(1-q_0^2)^{1-\beta/2}}{2(1+\alpha)^2 q_0^2 \sin\sigma_0}\left(\dfrac{1}{\zeta'} - \dfrac{1}{\zeta''}\right) = -\dfrac{B\theta}{q_0 Q^{3/2}}, \end{cases}$$

where

$$(30) \qquad A = \frac{2(1-q_0^2)^{-\beta/2}}{\sqrt{\{2\alpha(1+\alpha)q_0\}}}, \quad B = \tfrac{1}{2}A\left\{\frac{1-q_0^2}{2(1+\alpha)\sin\sigma_0}\right\}^2;$$

in arriving at these values of A, B we have used the relation

$$(1+\alpha)(1 - 2q_0\cos\sigma_0 + q_0^2) = \alpha(1-q_0^2),$$

which says that (q_0, σ_0) is on the singular locus $D = 0$. By (2) the relations between position and velocity coordinates are accordingly, to leading order,

$$(31) \qquad x = -\frac{A}{\sqrt{(q_0 - q)}}, \quad y = -\frac{B\theta}{q_0(q_0-q)^{3/2}}.$$

For the leading approximation to ψ we require by (16) the principal part of $i(X_2' - X_2'')$, which like $Y_1' - Y_1''$ is in general $O(1)$. This is

$$(32) \quad iX_2(q, \phi + \delta\phi') - iX_2(q, \phi + \delta\phi'') \sim i(\delta\phi' - \delta\phi'')X_{2\phi}\left(q, \phi + \frac{\delta\phi' + \delta\phi''}{2}\right),$$

where $i(\delta\phi' - \delta\phi'') = \zeta'' - \zeta'$ and by (26), $\frac{1}{2}(\delta\phi' + \delta\phi'') = i\{2\alpha Q/(1+\alpha)q\}^{1/2}$. Now from (13) and (14) we find

$$iX_{2\phi} = \frac{DX_1}{1 - 2q\cos\phi + q^2}, \tag{33}$$

and thence (32) is found to reduce to

$$i(X_2' - X_2'') = \theta X,$$

to leading order. Thus by (2), (16) becomes

$$\psi = -q(1-q^2)^\beta(\theta X + Y) = -q(1-q^2)^\beta y, \tag{34}$$

which could indeed have been asserted ab initio as the hydraulic approximation. However, by suitably judged work with expansions the approximations such as (32) can be refined; the result is

$$\begin{aligned}\psi &= -q(1-q^2)\,\beta \mathrm{Rl}\,(iX_2' - iX_2'' + Y_1' - Y_1'') \\ &= -q(1-q^2)\beta\{Y + \theta X - \tfrac{1}{2}Y\theta^2/Q + O(\theta^3 Q^{-3/2})\},\end{aligned} \tag{35}$$

where X, Y are given by (29) with errors $O(1)$, $O(\theta Q^{-1/2})$ respectively. Also by (2), to the same approximation as (35), $Y + \theta X = y$, and

$$q(1-q^2)^\beta = q_0(1-q_0^2)^\beta(1 + CQ),$$

where $C = (q_0^2 - q_s^2)/q_0 q_s^2(1-q_0^2)$. Thus finally we find, on substituting for Q and θ from (29)

$$\psi = -q_0(1-q_0^2)^\beta y\left\{1 + \frac{a^2}{x^2}\left(1 - \frac{\lambda y^2}{x^2}\right)\right\} + O\left(\frac{y^3}{x^6}\right), \tag{36}$$

with

$$a^2 = \frac{4(q_0^2/q_s^2 - 1)}{\beta q_0^2(1-q_0^2)^{1+\beta}}, \quad \lambda = \frac{2(q_0^2/q_s^2 - 1)}{q_0(1-q_0^2)}. \tag{37}$$

The approximation (36) is of course valid only when x is large, but it suggests a nozzle with a throat, as in figure 6; the flow is in the direction of x decreasing because of the negative sign in (29).

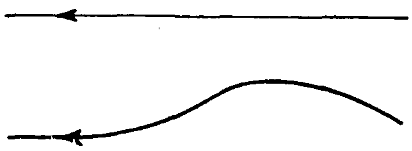

Fig. 6. Nozzle with ultimately uniform section.

So much for the approximation at the supersonic end $q \sim q_0$. At the subsonic end $q \sim 0$, and as we have seen $\phi \sim i\infty$. More precisely, the leading approximation to the transformation (8) is here

$$1 - qe^{-i\phi} \sim (qe^{-i\theta})^{1/\alpha},$$

so

$$D = (1+\alpha)(1-qe^{-i\phi})(1-qe^{i\phi}) - \alpha(1-q^2) \sim -\alpha + (1+\alpha)(qe^{-i\theta})^{1/\alpha}.$$

Now in (11) $H_1 = -\log(1-qe^{-i\phi})$ and $G_0 = (1+\alpha)^{-1} + O(qe^{i\phi})$. Hence

$$\Omega_0 \sim -1/\alpha - (qe^{-i\theta})^{1/\alpha}/(1+\alpha)^2,$$

and by (14)

$$X_1 \sim \frac{e^{-i\phi}}{1-qe^{-i\phi}} + \frac{2i\sin\phi\{1+\alpha(qe^{-i\phi})^{1/\alpha}/(1+\alpha)^2\}}{(1-qe^{-i\phi})(1-qe^{i\phi})}$$

$$\sim -\frac{\alpha q^{1/\alpha-1}e^{-i\theta/\alpha}}{(1+\alpha)^2(qe^{-i\theta})^{1/\alpha}},$$

provided $1/\alpha < 2$, which is so if $\beta > \frac{3}{2}$, $\gamma < \frac{5}{3}$. Hence

(38) $$X \sim -2\alpha/(1+\alpha)^2 q \sim -\infty.$$

Since also by (29) $X \sim -\infty$ for $q \sim q_0$, the axial xq-relation for the symmetrized flow Rl $(\Omega_1(q, \theta_0+\theta) + \Omega_1(q, \theta_0-\theta))$ must have a maximum — which in fact occurs for $q = q_s$ — and the flow has a limit-line. To arrive at a smooth trans-sonic nozzle flow we therefore superpose the potential $-A$ Rl Ω_2, with $A > 0$. The maximum of the axial relation is thereby displaced to a subsonic q, say at (x_1, q_1), and the corresponding singularity of the flow ([1], Appendix 1) is that the axial streamline is prolonged by two non-axial branches meeting at a cusp at x_1. Finally this singularity can be removed, if desired, by superposing also a suitable multiple of the potential R.

References

[1] Cherry, T. M., A transformation of the hodograph equation, etc. *Phil. Trans. Roy. Soc. London Ser.* A 904 (1953), 583–626.

[2] Ringleb, F., Exakte Lösungen der Differentialgleichung einer adiabatischen Gasströmung, *Z. Angew. Math. Mech.* 20 (1940), 185–198.

[3] Ferguson, D. F. and Lighthill, M. J., The hodograph transformation in trans-sonic flow. IV Tables. *Proc. Roy. Soc. London. Ser.* A 192 (1947), 135–142.

[4] Huckel, Vera, Tables of hypergeometric functions for use in compressible flow theory. *N.A.C.A. Tech. Note No. 1716, Washington*, October 1948.

[5] Petschacher, Martha, Tabelle di Funzioni Ipergeometriche. *Rend. di Mat. e. Appl.* V 9 (1950), 389–420.

University of Melbourne

https://doi.org/10.1017/S144678870002509X Published online by Cambridge University Press