Astin Bulletin (2024), **54**, pp. 626–651
doi:10.1017/asb.2024.23

International Actuarial Association
Association Actuarielle Internationale

RESEARCH ARTICLE

Optimal annuitization under stochastic interest rates

Yannick Dillschneider[1], Raimond Maurer[2] and Peter Schober[2]

[1]Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, Netherlands
[2]Finance Department, Goethe University, Theodor-W.-Adorno-Platz 3, 60323 Frankfurt, Germany
Corresponding author: Yannick Dillschneider; Email: y.dillschneider@uva.nl

Received: 24 July 2023; **Revised:** 25 April 2024; **Accepted:** 6 June 2024

Keywords: Annuity risk; interest rate risk; deferred life annuities; dynamic portfolio choice

JEL codes: D15; G11; G51; G52; J26

Abstract
The decision about when and how much to annuitize is an important element of the retirement planning of most individuals. Optimal annuitization strategies depend on the individual's exposure to annuity risk, meaning the possibility of meeting unfavorable personal and market conditions at the time of the annuitization decision. This article studies optimal annuitization strategies within a life-cycle consumption and portfolio choice model, focusing on stochastic interest rates as an important source of annuity risk. Closing a gap in the existing literature, our numerical results across different model variants reveal several typical structural effects of interest rate risk on the annuitization decision, which may however vary depending on preference specifications and alternative investment opportunities: When allowing for gradual annuitization, annuity risk is temporally diversified by spreading annuity purchases over the whole pre-retirement period, with annuity market participation starting earlier in the life cycle and becoming more extensive with increasing interest rate risk. Ruling out this temporal diversification possibility, as embedded in many institutional settings, incurs significant welfare losses, which are increasing with higher interest rate risk, together with larger overall demand for annuitization.

1. Introduction

A payout life annuity is an insurance product that promises the annuitant a regular stream of life-long income in exchange for a nonrefundable premium payable at inception of the contract. The survival-contingent annuity payments offered by the contract may start immediately after the purchase or be delayed for some deferral period. Due to the contingency of their payments, life annuities allow individuals to allocate accumulated wealth exclusively to states in which they are alive, thereby offering protection against outliving retirement assets and the ability to earn the mortality credit. Yet, the allocation of retirement assets via annuitization is usually irreversible because in practice, acquired life annuities cannot be sold back to the issuing insurance company.

The decision about when and how much to annuitize is an important element of retirement planning and accordingly has received significant attention in actuarial and financial research (see Alexandrova and Gatzert, 2019 for an overview). When allocating wealth in a stochastic environment, the to-be annuitant is exposed to the risk of meeting unfavorable personal and market conditions at the time of the annuitization decision, referred to as *annuity risk*. As a particular component of annuity risk, the focus of this article is on interest rate risk and its impact on the structure and timing of the demand for life annuities. From the perspective of an individual, stochastic interest rates have two major consequences that are relevant to the annuitization demand. First, they lead to fluctuating premia for life annuities, as the insurance company adjusts its pricing in response to changes in the relevant discount factors. Second, they may affect the returns on other investments such as bonds and stocks that are available to households for allocating their savings. In consequence, stochastic interest rates lead to random variation in the relative attractiveness of annuitization compared to other financial investments.

Theoretical investigations of the optimal life annuity demand of rational individuals with uncertain lifespans are based on consumption and portfolio choice models over the life cycle. The aim of this article is to extend the existing literature by studying the effects of stochastic interest rates on the annuitization demand within such a life-cycle model. In our model, the individual is exposed to deterministic mortality risk, receives risky labor income during the working life, and decides each period how to allocate financial wealth between consumption and investments in risky stocks, bonds, and life annuities. Our main contribution to the existing literature thereby is twofold.

First, we determine and analyze the optimal gradual annuitization strategy under stochastic interest rates, that is, allowing the individual to annuitize arbitrary fractions of wealth at any time before retirement. In the presence of annuity risk, unconstrained optimal annuitization decisions are expected to be spread over the complete life cycle, opportunistically exploiting favorable personal and market conditions as well as mortality credit constellations if they sufficiently outweigh the illiquidity costs due to irreversibility. By distributing annuity purchases over the whole life cycle instead of concentrating them to a single date, a significant temporal diversification of annuity risk may potentially be achieved. Our numerical results across various model variants suggest that the optimal gradual annuitization strategy indeed strongly employs temporal diversification of annuity risk by spreading annuity purchases virtually over the entire life cycle. In most cases, this effect becomes significantly more important with increasing levels of interest rate risk. Specifically, we find that increasing interest rate risk leads to annuity demand that is both higher and occurs much earlier in the life cycle, such that the individual more opportunistically exploits market states in which annuities are relatively cheap. Compared to the case of constant interest rates, where annuity demand picks up around the mid-life cycle, the average individual starts to annuitize about fifteen years earlier and acquires approximately 30% more annuities in our baseline model with stochastic interest rates.

Second, accounting for certain institutional constraints that are frequently reflected in the academic research on optimal annuitization, we determine the optimal one-time annuitization strategy under stochastic interest rates, that is, allowing the individual to annuitize only at the retirement date. With this restriction imposed, the to-be annuitant is particularly exposed to the risk of meeting unfavorable personal and market conditions at the decision date. In our analysis, we quantify the economic costs for the individual associated to enforcing one-time annuitization relative to the optimal gradual annuitization strategy identified before. The economic costs are driven by consumption sacrifices prior to retirement to build up additional precautionary buffer holdings against annuity risk when its temporal diversification is infeasible. Typically, these extra savings eventually even lead to more one-time annuitization compared to cumulated gradual annuitization. Consistent with our previous insights, our numerical results for most model variants suggest a sizable welfare loss associated to one-time annuitization. Reflecting the structure of the optimal gradual annuity demand, the magnitude of the welfare loss due to the timing constraint is generally positively related to the prevailing level of interest rate risk. Under our baseline interest rate risk parameterization, we find a welfare loss of almost 40% of initial wealth and slightly above 2% of annual certainty-equivalent consumption when one-time annuitization is enforced. In comparison, without interest rate risk, the welfare loss amounts to merely around 5% of initial wealth and 0.3% of annual certainty-equivalent consumption. These welfare losses over the whole life cycle are opposed by welfare gains over the retirement period, accompanied by relative increases in annuitization.

Essentially, by optimally choosing allocations to stocks, bonds, and life annuities, the investor in our model replicates a retirement product that is optimal with respect to the subjective life-cycle preferences. From a general perspective, our results thus provide insights into the optimal structuring of retirement products with both investment and annuitization features. These comprise variable annuities, which may involve other complex features (see Feng *et al*., 2022 for a recent overview), as well as various types of retirement accounts. Our normative results highlight the importance of gradual annuitization options within such retirement products.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

Our normative investigations are not only of relevance from the perspective of potential annuitants and their advisors but also for insurance companies and other providers of annuity contracts. Specifically, such normative insights may be used to guide their product offerings to optimally cater the needs of pension insurance customers. In this regard, the strong gradual component of the optimal annuity demand within our models is at odds with the one-time nature of decisions embedded in many real-life retirement products involving annuitization features. As our analysis reveals, enforcing such one-time decisions generally is inefficient and economically costly from the perspective of the annuitant. Moreover, it typically leads to a more volatile annuity demand, sensitive to the particular realization of annuity risk at the time of the annuitization decision.

Normative investigations are also the basis to assess empirically observed annuity demand, which typically exhibits a sizable discrepancy relative to theoretical predictions, referred to as the annuity puzzle. Beyond the normative sphere, our work also contributes insights that may be valuable for the measurement and explanation of the empirical annuity puzzle. Indeed, our findings could indicate an additional dimension of the annuity puzzle, which is commonly measured over the retirement period. Our numerical results in this article instead strongly suggest accounting for the gradual structure of annuity demand over the complete life cycle. The pronounced differences to one-time annuitization might prompt a critical reflection of the common practice, as temporal restrictions of the annuitization decision tend to elevate the theoretically expected annuity demand.

With our research, we contribute to the rich actuarial and financial literature studying optimal retirement portfolio choice over the life cycle in the presence of an uncertain lifetime. In essence, this literature aims at quantifying the optimal demand for retirement and pension insurance products (e.g., Aase, 2015, 2016). A long-standing question in this literature concerns the amount and timing of the demand for life annuities (see Alexandrova and Gatzert, 2019 for a recent review), reflecting not only the possibility of direct investments into life annuity contracts offered by insurance companies but also annuitization decisions that are embedded in certain personal pension plans. Early work by Yaari (1965) and more recently by Davidoff *et al*. (2005) show that, given frictionless complete annuity markets and fair prices, a rational individual with an uncertain lifetime and no bequest motive optimally chooses to fully annuitize accumulated wealth at retirement. Even if annuity markets are incomplete or if the individual has a bequest motive, partial annuitization of the retirement assets is optimal in many settings (see Davidoff *et al*., 2005; Hainaut and Devolder, 2006). This literature subsequently expanded by embedding life annuities into consumption and portfolio choice models over the complete life cycle with incomplete financial markets. The employed models are extensions of the seminal work by Cocco *et al*. (2005), Gomes and Michaelides (2005), who develop a realistically calibrated life-cycle model that takes into account stock return risk, income risk during the working life and in retirement, and realistic frictions (no short selling, borrowing constraints on labor and pension income). One strand of this literature extends this classical model by allowing the individual to allocate savings into different types of fixed (Horneff *et al*., 2008b, 2008c, 2010b; Milevsky and Young, 2007) and variable (Horneff *et al*., 2009, 2010a, 2015; Maurer *et al*., 2013) annuities. Other contributions address the institutional framework of including life annuities in tax-qualified defined-contribution retirement accounts (Horneff *et al*., 2020), investigate other than CRRA preferences (Huang and Milevsky, 2008), study the impact of systematic mortality risk (Maurer *et al*., 2013), and analyze health risk (Pang and Warshawsky, 2010; Reichling and Smetters, 2015; Peijnenburg *et al*., 2017). All mentioned studies corroborate a sizable demand for annuity products while allowing for various annuitization strategies, but only consider a constant interest rate setting.

Only a handful of articles investigate the impact of stochastic interest rates on the demand for life annuities within a life-cycle model, each imposing certain restrictive assumptions. Koijen *et al*. (2011) determine the optimal allocation between nominal, inflation-linked, and variable annuities at retirement in a life-cycle model with real interest rate and inflation risk, but the annuitization decision is restricted to the retirement date. Also, they abstract from idiosyncratic labor income risk in order to solve analytically for the optimal hedging policies before retirement. Huang et al. (2017) study the optimal purchasing strategy that maximizes expected utility of income from deferred annuities at a fixed date under mean-reverting annuity payout yields, but their work abstracts from portfolio considerations, labor income risk, and does not allow for gradual annuitization. Recently, Owadally *et al*. (2021) study optimal annuitization under stochastic interest rates in a dynamic portfolio choice model without labor income risk. They restrict the attention to CRRA utility and a coarse five-year decision horizon starting in the mid-life cycle, without directly assessing the impact of the level of interest rate risk. These existing studies emphasize the dependence of optimal annuitization strategies on interest rate states which, when ignored, can be structurally misleading and economically costly. Our work differs from this literature by analyzing both gradual and one-time annuitization within a dynamic life-cycle model, focusing particularly on the impact of the level of interest rate risk.

The remainder of this article is organized as follows. In Section 2, we describe the setup of our life-cycle model. Section 3 determines and analyzes the optimal gradual annuitization strategy. Subsequently, Section 4 considers the optimal one-time annuitization strategy and quantifies the associated welfare losses. Finally, Section 5 concludes the article. The Supplementary Material contains an online appendix with additional details and results.

2. Life-cycle model

To study optimal annuitization strategies in a stochastic interest rate setting, we set up a dynamic portfolio choice model in discrete time for a nonrepresentative household. By assumption, the household consists of a single individual who has an annual decision horizon and is subject to deterministic mortality risk. In this regard, we use $\pi_{t,s}$ to denote the individual's constant conditional probability to survive from time t to time s. The life cycle is partitioned into two stages: a working life, during which the individual earns an exogenous stochastic labor income, and a retirement period starting at a fixed retirement date, during which the individual is entitled to a retirement income. Concretely, the individual's life cycle ranges from age $T_l = 20$ to at most age $T_u = 100$, with a prespecified retirement age $T_{\text{ret}} = 65$. The household has access to a financial market to transfer wealth through time, which in particular allows investments in a money market account, a stock, and deferred life annuities.

In the remainder of this section, we describe the formal setup of our life-cycle model. Section 2.1 specifies the financial market. Section 2.2 then presents the household's labor income dynamics. These components are aggregated through the individual's preferences into the associated dynamic optimization problem in Section 2.3. Finally, Section 2.4 presents the concrete calibration of our model. Supplementary details are provided in the online appendix, on the model in section OA as well as on the numerical solution and simulation methods in section OB.

2.1. Financial assets

Three types of financial assets are available to the household in the base case: a short-term interest-bearing asset in the form of a money market account and a stock, representing the most common asset types held by households, as well as a life annuity whose payouts are deferred until retirement. Further investment opportunities such as longer-term bonds and life insurance contracts will be incorporated in some of the alternative model variants.

For sufficient realism, we introduce certain constraints on the timing of investments and on whether or not investment decisions may be reversed. Table 1 provides an overview of the timing constraints imposed in different financial market specifications that will be used in this article. Investments in the money market account and stock are perfectly liquid: both assets can be bought and sold freely throughout the whole life cycle. Unlike investments in these liquid assets, purchases of life annuities are restricted to a set of eligible purchase dates and irreversible in that once acquired annuity claims cannot be sold again, reflecting prevalent institutional constraints. For the eligible annuity purchase

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

Table 1. *Eligible purchase ages for financial assets.*

Asset	Market specification $\mathcal{B}$	$\mathcal{A}^I$	$\mathcal{A}^B$
Stock	20–99	20–99	20–99
Money market	20–99	20–99	20–99
Life annuity	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
Life insurance		20–99	
Longer-term bond			20–99

ages $\mathcal{T}$, we consider two alternatives: gradual annuitization over the whole pre-retirement life cycle ($\mathcal{T} = \mathcal{T}_{20-64} = \{20, \ldots, 64\}$) and one-time annuitization at age 64 ($\mathcal{T} = \mathcal{T}_{64} = \{64\}$). Stocks, bonds, and life annuities are the only assets available in our base-case market specification ($\mathcal{B}$). In alternative market specifications, we further add a life insurance product ($\mathcal{A}^I$) and a longer-term bond ($\mathcal{A}^B$), which both can be acquired (and the latter also sold) throughout the entire life cycle.

Prices of all assets are determined exogenously and modeled consistently through no-arbitrage pricing considerations. To preserve formal consistency of our financial market model, we use a continuous-time formulation, which can be exactly converted to a Gaussian discrete-time model. This feature is exploited in the estimation of model parameters as well as in the solution of our dynamic model for the optimal policies and a subsequent simulation study. Further supplementary details of the model are discussed in section OA of the online appendix.

To describe the continuous-time dynamics of real interest rates, we use the Vasicek (1977) mean-reverting model for the short rate r_t, analogous to, for example, Munk and Sørensen (2010). The log stock price ς_t is modeled consistently to the short rate through a dependence in the drift, leading to the joint dynamics

$$\begin{pmatrix} \mathrm{d}r_t \\ \mathrm{d}\varsigma_t \end{pmatrix} = \begin{pmatrix} \kappa(\theta - r_t) \\ r_t + \mu - \frac{1}{2}\sigma_\varsigma^2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_\varsigma \end{pmatrix} \begin{pmatrix} \mathrm{d}Z_t^r \\ \mathrm{d}Z_t^\varsigma \end{pmatrix}. \tag{2.1}$$

Here, κ is the speed of mean reversion, θ is the long-term mean, and σ_r is the instantaneous volatility of the short rate. Moreover, μ denotes the constant risk premium of the stock, and σ_ς is the log stock price volatility. Finally, Z_t^r and Z_t^ς are independent standard Brownian motions.

Derived from the continuous-time specification in Equation (2.1), we obtain the joint discrete-time dynamics

$$\begin{pmatrix} r_{t+1} - r_t \\ \varsigma_{t+1} - \varsigma_t \end{pmatrix} = \begin{pmatrix} \hat{\kappa}(\hat{\theta} - r_t) \\ a_\delta + b_\delta r_t + \mu - \frac{1}{2}\sigma_\varsigma^2 \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_r & 0 \\ \hat{\sigma}_\delta & \sigma_\varsigma \end{pmatrix} \begin{pmatrix} \hat{Z}_{t+1}^r \\ \hat{Z}_{t+1}^\varsigma \end{pmatrix}, \tag{2.2}$$

in terms of interest rate and stock price parameters

$$\hat{\kappa} = 1 - e^{-\kappa\tau}, \quad \hat{\theta} = \theta, \quad \hat{\sigma}_r^2 = \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa\tau}),$$
$$a_\delta = \theta\tau - \frac{\theta}{\kappa}(1 - e^{-\kappa\tau}), \quad b_\delta = \frac{1}{\kappa}(1 - e^{-\kappa\tau}), \quad \hat{\sigma}_\delta^2 = \frac{\sigma_r^2}{2\kappa^3}(4e^{-\kappa\tau} - e^{-2\kappa\tau} + 2\kappa\tau - 3).$$

Moreover, $\hat{Z}_{t+1}^r, \hat{Z}_{t+1}^\varsigma \sim N(0, 1)$ and independent over time as well as with respect to each other. In Equation (2.2), the term $\mu - \frac{1}{2}\sigma_\varsigma^2 + \sigma_\varsigma \hat{Z}_{t+1}^\varsigma$ corresponds to the usual discretization of a geometric Brownian motion, while the remainder $\int_t^{t+1} r_s\,\mathrm{d}s = a_\delta + b_\delta r_t + \hat{\sigma}_\delta \hat{Z}_{t+1}^r$ accounts for the dependence of the stock price on the short rate through its drift. Even though we do not incorporate a diffusive correlation in our specification (2.1), we nevertheless obtain a correlation of short rate and stock price in discrete time. Eventually, the gross return $R_{t+1}^S = V_{t+1}^S / V_t^S$ of the stock price $V_t^S = \exp(\varsigma_t)$ is log-normally distributed conditional on r_t.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

It is well known that the Vasicek (1977) short rate process allows for closed-form bond prices. Specifically, we can write the price of a τ-period zero-coupon bond (with unit nominal value) as an exponentially affine function of the short rate,

$$V_t^B(\tau) = \exp(a_B(\tau) + b_B(\tau)\, r_t)\,, \tag{2.3}$$

where $a_B(\tau)$ and $b_B(\tau)$ are coefficients depending on τ, defined in terms of the market price of interest rate risk parameter λ by

$$a_B(\tau) = -\frac{\sigma_r^2 b_B(\tau)^2}{4(\kappa+\lambda\sigma_r)} - (\tau + b_B(\tau))\left(\frac{\theta\kappa}{\kappa+\lambda\sigma_r} - \frac{\sigma_r^2}{2(\kappa+\lambda\sigma_r)^2}\right) \tag{2.4a}$$

$$b_B(\tau) = \frac{e^{-(\kappa+\lambda\sigma_r)\tau} - 1}{\kappa+\lambda\sigma_r}\,. \tag{2.4b}$$

As a special case, the gross return on the money market account over a unit time horizon from t to $t+1$ can be written as $R_{t+1}^B = 1/V_t^B(1)$. Similarly, the gross return on a τ-period bond over the same horizon equals $V_{t+1}^B(\tau-1)/V_t^B(\tau)$.

As a mortality-linked product, the pricing of a life annuity not only depends on the short rate dynamics, but also incorporates risk-adjusted conditional survival probabilities $\tilde{\pi}_{t,s}$. These may generically adjust for adverse selection problems, mortality risk premia, and general actuarial loadings. Predominantly, however, we will be concerned with the actuarially fair setting in which $\tilde{\pi}_{t,s} = \pi_{t,s}$ aligns with actual mortality beliefs of the household. By the assumed independence of mortality and interest rate risk, the annuity price is computed by weighting each cash flow with the conditional probability $\tilde{\pi}_{t,t+\tau}$ to be alive at the payment date $t+\tau$ and discounting with the current term structure of interest rates reflected in the τ-period bond price in Equation (2.3). Specifically, the price $V_t^A(d)$ of a life annuity at time t, with unit annual payments deferred until time d, can be expressed as

$$V_t^A(d) = \sum_{\tau=d-t}^{T_u-t} \tilde{\pi}_{t,t+\tau}\, V_t^B(\tau)\,, \tag{2.5}$$

with bond price $V_t^B(\tau)$ given by Equation (2.3). The default case we will treat in this article uses deferral until the retirement date T_{ret}, for which we denote the corresponding price $V_t^A = V_t^A(T_{\text{ret}})$ as in Equation (2.5).

Similarly, life insurance also constitutes a mortality-linked product. To ensure computational tractability within our model, we only consider a one-period life insurance contract. The single premium $V_t^I(1)$ of this contract with unit benefit payable at time $t+1$ in the event of death is determined from the one-period mortality probability $1-\tilde{\pi}_{t,t+1}$ and the one-period bond price $V_t^B(1)$. Concretely, the premium equals $V_t^I(1) = (1-\tilde{\pi}_{t,t+1})\, V_t^B(1)$.

2.2. Labor income

Throughout the working life, the individual earns labor income that we allow to be stochastic, which ultimately converts into retirement income. Our labor income process follows the one defined by Cocco *et al*. (2005), who build on Gourinchas and Parker (2002), Zeldes (1989). Formally, we define the labor income Y_t earned at time t through

$$\log Y_t := \begin{cases} \log G_t + \log P_t + \vartheta_t & , \text{ if } t \le T_{\text{ret}} \\ \log o_{\text{rep}} + \log G_{T_{\text{ret}}} + \log P_{T_{\text{ret}}} & , \text{ if } t > T_{\text{ret}}\,. \end{cases} \tag{2.6}$$

Labor income according to Equation (2.6) consists of several components, which differ in their composition before and after the retirement age $T_{\text{ret}} = 65$.

During the working life (i.e., at $t \le T_{\text{ret}}$), labor income is determined by the age-dependent average income G_t. The household effectively earns a stochastic multiple of this deterministic average income,

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

where the multiplier corresponds to the permanent income component P_t and a transitory shock ϑ_t. Specifically, the permanent income component in Equation (2.6) is modeled as

$$\log P_t = \log P_{t-1} + \varepsilon_t\,, \tag{2.7}$$

with permanent and transitory shocks $\varepsilon_t = -\frac{1}{2}\sigma_\varepsilon^2 + \sigma_\varepsilon\,\hat{Z}_t^\varepsilon$ and $\vartheta_t = -\frac{1}{2}\sigma_\vartheta^2 + \sigma_\vartheta\,\hat{Z}_t^\vartheta$, respectively, for independent noise terms $\hat{Z}_t^\varepsilon \sim N(0, 1)$ and $\hat{Z}_t^\vartheta \sim N(0, 1)$. Following a common approach, we thereby assume all components of the income process to be uncorrelated with the financial market.

After retirement (i.e., at $t > T_{\text{ret}}$), labor income is determined by a deterministic replacement rate o_{rep} applied to the permanent income $G_{T_{\text{ret}}}P_{T_{\text{ret}}}$ realized at the retirement age. Determining retirement income in this way is a common and implementation-friendly approach followed in the life-cycle literature to approximate actual retirement income calculation rules.

2.3. Preferences and dynamic optimization

By assumption, the household is equipped with recursive preferences of Epstein and Zin (1989) type with a standard constant relative risk aversion (CRRA) utility function. As in the specification of Córdoba and Ripoll (2017), we define a utility index $\mathcal{V}_t$ that aggregates today's consumption C_t as well as (uncertain) future utility $\mathcal{V}_{t+1}$ and liquid financial wealth W_{t+1}, accounting for mortality risk. Conditional on being alive at time t, the utility index $\mathcal{V}_t$ is expressed in consumption units as

$$\mathcal{V}_t = \left(C_t^{1-\psi} + \rho\ \left(\pi_{t,t+1}\mathbb{E}_t[\mathcal{V}_{t+1}^{1-\gamma}] + (1-\pi_{t,t+1})\, b\, \mathbb{E}_t[W_{t+1}^{1-\gamma}]\right)^{\frac{1-\psi}{1-\gamma}} \right)^{\frac{1}{1-\psi}} \tag{2.8a}$$

$$\mathcal{V}_{T_u} = C_{T_u}\,, \tag{2.8b}$$

with elasticity of intertemporal substitution (EIS) $1/\psi > 0$, relative risk aversion (RRA) $\gamma \neq 1$, subjective time discount factor $\rho < 1$, and a bequest motive with strength $b \geq 0$. As presented in Equation (2.8), for $t < T_u$, the utility index $\mathcal{V}_t$ aggregates immediate consumption C_t and discounted expected future utility in both possible survival states at time $t + 1$. The latter derives from $\mathcal{V}_{t+1}$ if the individual is still alive at time $t + 1$ or from bequest of the liquid financial wealth W_{t+1} if the individual dies between t and $t + 1$. Accordingly, future utilities are aggregated by weighting with the survival probability $\pi_{t,t+1}$ and mortality probability $1 - \pi_{t,t+1}$, respectively. Provided that the individual is alive at the terminal point in time T_u, the utility index then simply equals the immediate consumption.

Recursive utility as in Equation (2.8) allows to disentangle intertemporal substitution from risk aversion, that is, consumption smoothing over time and states, respectively, which has proven useful when matching moments from model simulations to microeconomic data of annuity demand (Inkmann *et al.*, 2011). In addition, it facilitates the comparison between the recursive and the standard time-separable CRRA utility, which can be obtained as a special case of the recursive specification (2.8) when setting $\psi = \gamma$.

The household faces the problem of maximizing the utility index $\mathcal{V}_{T_l}$ in Equation (2.8) by choice of an optimal policy that determines consumption and the investment in financial assets. At every point in time $t = T_l, \ldots, T_u$, the individual may spend money for consumption (C_t), purchase stocks (S_t) that yield the stochastic return R_{t+1}^S, or save in the money market account (B_t) for one period with conditionally deterministic return R_{t+1}^B. At certain eligible ages $\mathcal{T}$ before the retirement age T_{ret}, the individual can also irreversibly acquire deferred life annuity claims (A_t) for a nonrefundable premium V_t^A. Each such investment contributes to the inventory of cumulated life annuity claims (L_t), whose yearly payments start at retirement and are contingent on the annuitant's survival. Investments in further assets are possible in alternative model specifications. Consumption spending and investments are financed from liquid financial wealth (W_t), including labor and retirement income (Y_t), as well as payments from life annuity claims during retirement. Accordingly, the following budget constraint is enforced in the base case:

$$C_t + S_t + B_t + A_t = W_t + \mathbb{1}_{\{t \geq T_{\text{ret}}\}} L_t\,, \tag{2.9}$$

in addition to intertemporal consistency constraints on the evolution of W_t and L_t.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

Table 2. *Base-case model parameters.*

Preferences		Labor income		Short rate		Stock price	
γ	5	σ_ε	0.1030	κ	0.1667	μ	0.0358
ψ	5	σ_ϑ	0.2717	θ	0.0182	σ_ς	0.1876
ρ	0.97	o_{rep}	0.6821	σ_r	0.0105		
b	0			λ	−9.5941		

Formally, this optimization problem just described can be formulated recursively in terms of a value function $J_t(W_t, L_t, r_t, P_t)$ by the Bellman principle of optimality (Bellman, 1954). Conditional on being alive at time t, the value function J_t corresponds to the utility index $\mathcal{V}_t$ in Equation (2.8) given optimal choices for policies C_τ, S_τ, B_τ, and A_τ for all $\tau \geq t$, subject to appropriate budget constraints. Since the optimization problem cannot be solved analytically, we employ numerical solution methods. Overall, we need to track as state variables the current time t itself as well as the contemporaneous liquid financial wealth (W_t), cumulated life annuity claims (L_t), short rate (r_t), and permanent income (P_t). In this respect, we note that the problem can be normalized with respect to the permanent income component due to homogeneity, which offers significant computational advantages by saving one input dimension of the value function J_t. Despite posing substantial computational challenges that need to be addressed, the general solution procedure is fairly standard and, therefore, its details are deferred to section OB of the online appendix.

To analyze the optimal policies obtained from the numerical solution methodology, we simulate the model using a Monte Carlo simulation with $N_{\text{MC}} = 10,000$ paths starting from age $T_l = 20$. Each such path corresponds to one possible life cycle of the individual, for which in every simulated year we evaluate the optimal policies as obtained from solving the problem numerically. All paths are simulated conditional on survival of the individual until age $T_u = 100$. On each path, the individual starts with the average income of age 19 as its liquid wealth, and we assume that it has not acquired any annuity claims yet. The short rate is initialized at its mean. As a result from the simulation, we obtain a distribution of state variables and optimal policies for every age.

2.4. Model calibration

For our purposes, a model consists of (i) a financial market specification from Table 1 together with a set of eligible annuity purchase ages $\mathcal{T}$; (ii) a set of parameters that determine the individual's preferences and labor income evolution as well as the dynamics of interest rates and the stock price; and (iii) a time-consistent collection of survival probabilities $\pi_{t,s}$ and $\tilde{\pi}_{t,s}$. Regarding the latter, we obtain single-period survival probabilities $\pi_{t,t+1}$ from the 2019 US female population mortality table (Arias and Xu, 2022), which yields multiperiod survival rates by $\pi_{t,s} = \prod_{\tau=t}^{s-1} \pi_{\tau,\tau+1}$. Unless explicitly noted otherwise, we set $\tilde{\pi}_{t,s} = \pi_{t,s}$ for pricing purposes. Apart from these mortality specifications, the default parameterization of our models is shown in Table 2, with all quantities being denominated in real terms.

For the utility index in Equation (2.8), our base case employs CRRA preferences by setting $\gamma = \psi = 5$ with a time discount rate $\rho = 0.97$, which is in line with typical parameterizations of life-cycle models (e.g., Gomes, 2020). Our supplementary analysis in the online appendix (see sections OD and OE) will also cover alternative Epstein–Zin parameterizations $\gamma \neq \psi$ beyond the CRRA case. The baseline parameterization abstracts from a bequest motive, which we will incorporate in our analysis. For the income process in Equation (2.6), our base case is the same as the high school case (Tables 1, 2, and 4) of Cocco *et al*. (2005) with retirement age fixed at $T_{\text{ret}} = 65$.

The parameters for the short rate and the stock price in Table 2 are estimated using empirical data over the period from January 2000 to December 2016. The real stock price is measured as the S&P 500

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

total return index deflated by the CPI, which we obtain from Datastream. For real interest rates, we use zero-coupon TIPS yields available through the Fed. Details of the estimation procedure are described in section OA of the online appendix. For the mean short rate, we obtain a reasonable value of $\theta = 0.0182$ with speed of mean reversion $\kappa = 0.1667$ and an (instantaneous) volatility of $\sigma_r = 0.0105$ within the continuous-time specification. With regard to the discrete-time dynamics, this implies parameters $\hat{\theta} = 0.0182$, $\hat{\kappa} = 0.1535$, and $\hat{\sigma}_r = 0.0097$. To determine bond prices as in Equation (2.3), the associated risk-neutral mean short rate is $\tilde{\theta} = \theta\kappa/\tilde{\kappa} = 0.0461$, as typical (e.g., p. 722 in Piazzesi, 2010) higher than its real-world counterpart θ, with speed of mean reversion $\tilde{\kappa} = \kappa + \lambda\sigma_r = 0.0659$ and the same volatility σ_r, connected through a market price of interest rate risk parameter λ. For the stock price, we have the additional parameters $a_\delta = 0.0014$, $b_\delta = 0.9211$, and $\hat{\sigma}_\delta = 0.0057$, together with the excess return $\mu = 0.0358$ and volatility parameter $\sigma_\varsigma = 0.1876$. Combined, this yields an expected stock return of $\log \mathbb{E}_t[R^S_{t+1}] = 0.0540$ when $r_t = \theta$.

3. Optimal gradual annuitization over the life cycle

This section is concerned with analyzing the optimal gradual annuitization strategy over the life cycle. Thereby, we consider an ideal situation that provides the individual unrestricted access to deferred life annuities during the entire pre-retirement period ($\mathcal{T}_{20-64}$). In our base case, we solve the household's optimization problem with the parameterization presented in Table 2.

A particular emphasis is then placed on investigating the effects of changes in the level of interest rate risk, which we measure by the short rate volatility σ_r. The case of gradual annuitization under constant interest rates treated in the existing literature (e.g., Horneff *et al*., 2010b) will arise as a limiting case when reducing interest rate risk to zero. According to our agenda, we start in Section 3.1 by presenting the optimal gradual annuitization strategy for different levels of short rate volatility. In Section 3.2, we then dig deeper into understanding the optimal annuitization policy over time as a function of the contemporaneous short rate. Equipped with these insights, we analyze the effects of bequest motives and life insurance in Section 3.3 as well as longer-term bonds in Section 3.4, followed by an investigation of optimal annuitization within retirement accounts in Section 3.5. Supplementary to these results, the online appendix investigates the effects of labor income risk in section OD.1, alternative Epstein-Zin preference parameterizations in section OD.2, and actuarial loadings in section OD.3.

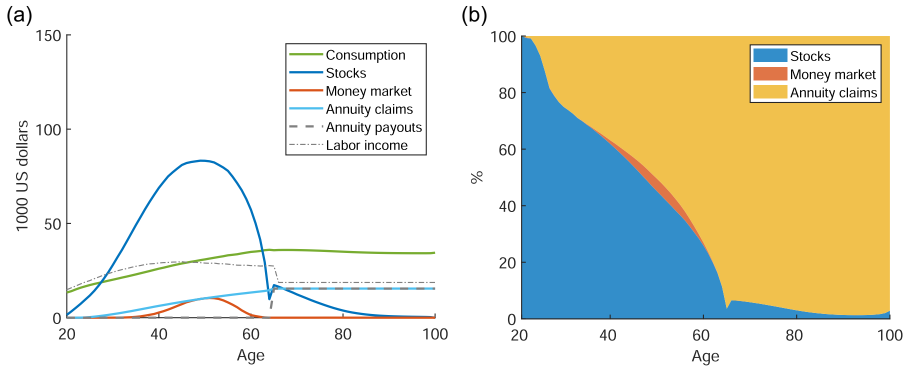
3.1. Optimal life-cycle demand and asset allocation

We start by analyzing the optimal life-cycle demand and asset allocation for gradual annuitization ($\mathcal{T}_{20-64}$) in the base-case market $\mathcal{B}$ with model parameters taken from Table 2. To do so, we compute averages over the optimal policies on all simulated paths to generate average life-cycle profiles of consumption, stock purchases, money market investments, and cumulated annuity claims. Moreover, we compute the average portfolio holdings of stocks, bonds, and annuity claims.

Figure 1 shows the resulting average profiles. Specifically, Figure 1(a) depicts the life-cycle profiles and Figure 1(b) the respective asset allocation profiles.

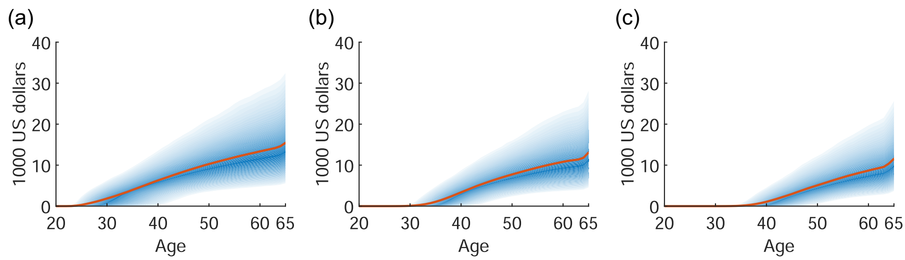
Both reflect many typical features observed in common life-cycle models, which we will not discuss here in much detail, but instead refer the reader to the comprehensive overviews provided by, for example, Cocco *et al*. (2005), Gomes (2020). Consistent with these common outcomes, the individual in our model starts with a strong exposure to stocks to harvest the equity premium and build up financial wealth, which over the course of the life cycle is shifted toward interest rate investments as human capital diminishes and retirement approaches. What is particularly remarkable about the profiles in Figure 1 is the substantial demand for life annuities that emerges already from young ages of the individual. Indeed, when allowing for gradual annuitization, annuity claims are starting to accumulate as early as age 23 for the average household. Toward the mid-life cycle at age 40, the present value of annuity claims takes a significant share of about 37.5% in the asset allocation. At age 64, the final eligible date

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press



Life-cycle asset demand | Life-cycle asset allocation

Figure 1. *Average life-cycle profiles (panel a) and asset allocation as a percentage of financial wealth (panel b) for gradual annuitization ($\mathcal{T}_{20-64}$) with $\sigma_r = 1.05\%$ in market $\mathcal{B}$.*



$\sigma_r = 1.05\,\%$ | $\sigma_r = 0.53\,\%$ | $\sigma_r = 0.00\,\%$

Figure 2. *Distribution of cumulated annuity claims L_t^{opt} (shaded area) and average cumulated annuity claims $\bar{L}_t^{opt}$ (red line) at age t for different levels of short rate volatility σ_r in market $\mathcal{B}$. The shaded area corresponds to the range between the 5th and 95th percentile of the distribution at a given t; darker colors indicate higher density.*

for annuitization immediately preceding retirement, the average portfolio consists to roughly 97% of annuity claims. A major fraction of this gradual annuity demand over the life cycle turns out to be due to a precautionary savings motive induced by labor income risk, as the individual builds up buffer savings to absorb retirement income risks (see section OD.1 of the online appendix).

The observed gradual annuitization timing is much earlier compared to what is reported by the reference literature with constant interest rates (e.g., Horneff *et al*., 2008b, 2010b). To gain some further insight into the dependence of the annuity demand on the level of interest rate risk, Figure 2 shows the accumulation of annuity claims over the life cycle for different choices of σ_r.

For the base-case parameterization with $\sigma_r = 1.05\%$, Figure 2(a) reiterates the evidence on the early demand for annuity claims and establishes a sizable cross-sectional variation across different simulation paths. As a general pattern, annuitization is delayed and cross-sectional variation is decreased when reducing the level of interest rate risk that the individual is exposed to in Figure 2(b) and (c). For the latter constant interest rate case, the inception of annuitization is postponed to almost age 40. The lower the level of interest rate risk, the higher is also on average the fraction of annuity claims acquired at the terminal eligible purchase date just before retirement. However, with our parameterization, the observed spike in annuity demand at age 64 is far less prominent compared to the one reported by Horneff *et al*. (2010b). Such a spike may suggest that the individual would like to optimally further postpone annuitization until after retirement (cf. Horneff *et al*., 2008a; Milevsky and Young, 2007). Indeed, supplementary

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

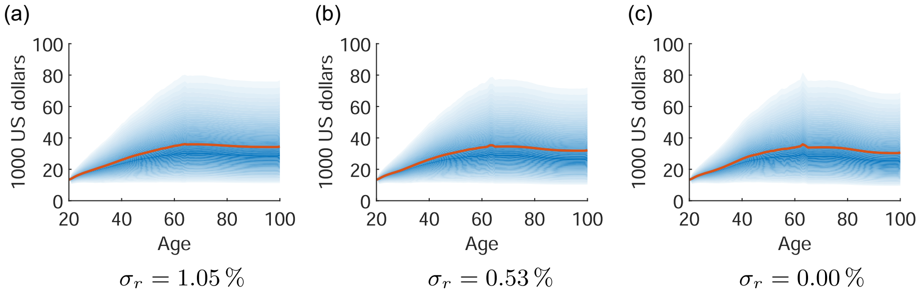


Figure 3. *Distribution of consumption C_t^{opt} (shaded area) and average consumption $\bar{C}_t^{opt}$ (red line) at age t for different levels of short rate volatility σ_r in market $\mathcal{B}$. The shaded area corresponds to the range between the 5th and 95th percentile of the distribution at a given t; darker colors indicate higher density.*

results (not reported) suggest that the spike in annuity demand actually disappears once allowing for unconstrained annuitization throughout the whole life cycle, even after retirement (i.e., $\mathcal{T}_{20-99}$).

According to the life-cycle profiles obtained for each interest rate scenario, payouts of acquired annuity claims make up a substantial fraction of the income available to finance consumption spending in retirement. Annuitization is not only delayed but also significantly fewer claims are acquired over the life cycle when interest rate risk is lower. For our base-case parameterization with $\sigma_r = 1.05\%$, average cumulated annuity claims amount to 15,443 US dollars, while these reduce by roughly one-quarter to 11,563 US dollars when interest rates are constant.

A natural question, which will be addressed in more detail in Section 4, is in how far this significantly lower annuity income translates into lower consumption during retirement. Even though annuity payments constitute an important part of income during retirement, the individual also has alternative income sources that may buffer or amplify the effects on consumption spending. As it turns out, much of the decrease in annuity income is actually compensated by significantly larger stock holdings accumulated before retirement, whose decumulation in retirement partially absorbs the lower annuity payouts. Figure 3 shows the resulting distribution of consumption over the life cycle for different choices of σ_r.

The economic mechanism leading to these effects can be quite involved, as lower interest rate risk affects all asset return characteristics. For long-term investments, lower interest rate risk implies that stock returns become less risky (by Equation (2.2)), while at the same time annuities increasingly forfeit their ability to lock in attractive long-term yields. We next establish some evidence that supports this latter effect as an important driver of annuity demand.

3.2. Optimal annuitization policy

Our goal is to determine the optimal annuitization policy for each age as a function of the contemporaneous short rate. To obtain a robust characterization of the dependence of the annuity demand on the short rate alone, we evaluate the optimal annuitization policy for a broad range of short rates on the average paths of liquid wealth and acquired annuity claims for all eligible purchase ages. Figure 4 shows the resulting optimal gradual annuitization policy.

Noting that the short rate domain is chosen to cover the same probability mass in Figure 4(a)–(d), we observe two major effects in the optimal policies for varying levels of interest rate risk. First, before the terminal eligible purchase date, the optimal annuitization policy becomes flatter as a function of the short rate and more centered to the mid-life cycle where human capital starts depreciating. In fact, for our base case with $\sigma_r = 1.05\%$, we observe a strongly increasing demand for annuities towards the upper boundary of the short rate in Figure 4(a), starting already at quite young ages. Virtually no demand is observed for short rates at or below the long-run mean $\theta = 1.82\%$. Hence, the individual opportunistically exploits favorable interest rate environments to acquire annuity claims when they are comparably

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

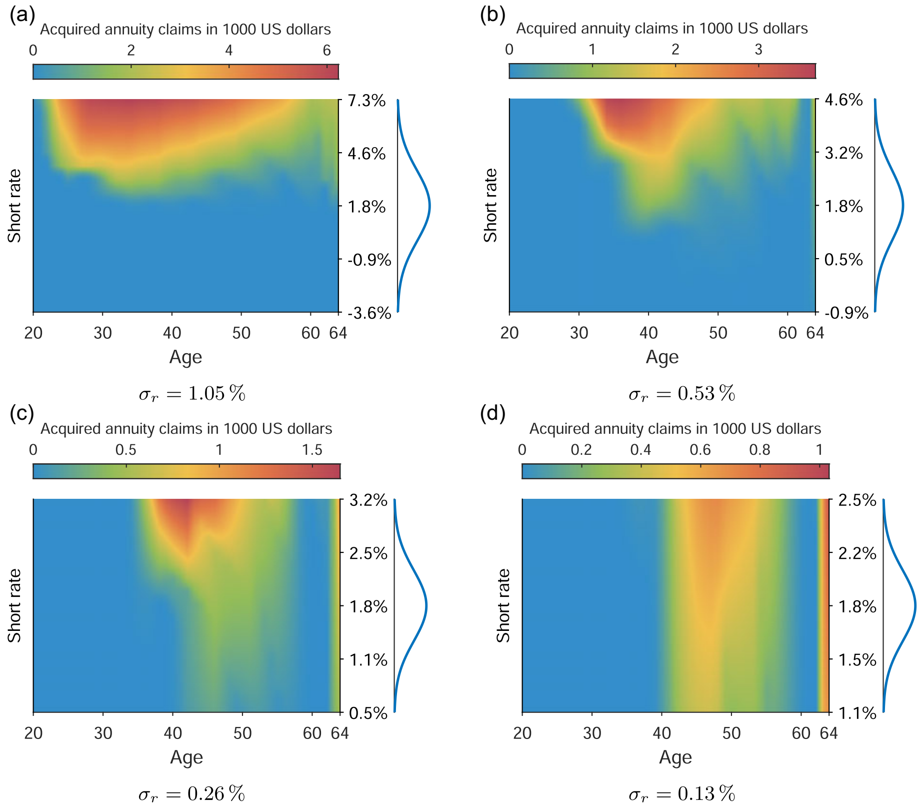


Figure 4. *Optimal gradual annuitization policy for the base-case short rate volatility σ_r from Table 2 (panel a), $\sigma_r/2$ (panel b), $\sigma_r/4$ (panel c), and $\sigma_r/8$ (panel d) in market $\mathcal{B}$. The policy is obtained by evaluating r_t-values on a short rate grid along the average path of wealth $\bar{W}_t^{opt}$ and acquired annuity claims $\bar{L}_t^{opt}$ from the simulation sample over time. The age-64 distribution of r_t is shown on the right of each subfigure; the range covers three standard deviations around the mean.*

cheap and allow to lock in attractive long-term yields. A similar annuitization strategy is suggested by Huang et al. (2017), who consider annuities on a stand-alone basis. When decreasing the level of interest rate risk, the annuity demand in our model becomes significantly less pronounced and shifts to older ages, as the individual effectively postpones annuitization until later in the life cycle to benefit from higher mortality credits and avoid the illiquidity costs of annuities. While being high in the early life cycle with lower financial wealth and labor income, the subjective (marginal) illiquidity costs associated to annuities indeed decrease towards the mid-life cycle with accumulating financial wealth and income, as the capacity for irreversible investments increases. Accordingly, for $\sigma_r = 0.53\%$, demand can only be observed after age 30 in Figure 4(b), while for $\sigma_r = 0.26\%$ it is already shifted almost to age 40 in Figure 4(c). In addition to shifting in time, the annuity demand also exhibits lower sensitivity with respect to the short rate, so that annuitization is less opportunistic and optimal even for lower interest rate levels. For $\sigma_r = 0.13\%$, Figure 4(d) reflects annuity demand that is postponed completely to the mid-life cycle between ages 40 and 60, almost irrespective of the level of the short rate.

Second, an increasing fraction of the annuity demand is acquired at the terminal eligible purchase date immediately before retirement when interest rate risk decreases. For the base case with $\sigma_r = 1.05\%$ or even the case with $\sigma_r = 0.53\%$, we can hardly observe any substantial annuity demand at that date in Figure 4(a) and (b), respectively. However, for the lower levels of interest rate risk with $\sigma_r = 0.26\%$ and $\sigma_r = 0.13\%$ in Figure 4(c) and (d), respectively, a prominent concentration of purchases at the last

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

opportunity to claim the mortality credit emerges. As mentioned before, we understand this spike in annuity demand to reflect a binding constraint, such that the individual would actually prefer to further postpone annuitization beyond entry into retirement.

In favorable states of liquid wealth above its average, the structural patterns of the optimal annuitization policy in Figure 4 shift further to the center of the short rate distribution. Temporal diversification of annuity risk, while already relevant in the absence of interest rate risk (Horneff *et al*., 2010b), becomes even more important in the presence of interest rate risk as optimal annuitization is spread essentially over the whole life cycle. In the latter case, deferred life annuities allow access to the longer end of a nontrivial term structure of interest rates in addition to offering a mortality credit, which outweighs the downside of illiquidity and the deferral of the income stream much earlier in the life cycle. With this structure, the optimal annuitization policy explains the distribution of cumulated annuity claims in Figure 2.

3.3. Impact of bequest motives and life insurance

The baseline parameterization of our life-cycle model abstracts from bequest motives by setting $b = 0$. As a consequence, the individual is concerned with reallocating wealth to states in which it is alive. This favors life annuities, which specifically serve this purpose with regard to the retirement period. When bequest motives are introduced, some wealth is also preserved for states in which the individual is dead, which is eventually bequeathed to the heirs. In general, a bequest motive can be expected to increase overall savings and, as annuity payments cease in the event of death, imply a relative reallocation to liquid financial assets that are independent of the individual's survival (e.g., Bommier and Grand, 2014; Lockwood, 2012, 2018). It is thus not surprising that Inkmann *et al*. (2011) identify bequest motives, alongside the EIS and RRA, as an important driver of annuity demand.

To quantify the impact of bequest motives in our life-cycle model, we modify our base-case parameterization in Table 2 to incorporate a nonzero bequest strength b. Consistent with bequest strengths suggested in the literature (e.g., Polkovnichenko, 2007), we consider two cases with $b = 2^{\gamma} = 32$ and $b = 5^{\gamma} = 3125$, the latter being rather at the high end of reasonable values.

The presence of a bequest motive also induces demand for life insurance, which as a mirrored counterpart of life annuities allows to allocate wealth exclusively to states in which the individual is dead. Similar to annuitization options, life insurance features may also be embedded in various retirement products (e.g., in variable annuities in the form of guaranteed minimum death benefits). To account for the effects of life insurance demand on the annuitization decision, we further extend our base-case model by also including a one-period term life insurance contract, similar to Hubener *et al*. (2014), where the restriction to a single period is for computational reasons. At any point in time t, the individual may enter a term life insurance contract, covering the period up to time $t + 1$. If the individual dies in the respective period, the acquired benefits are bequeathed to the heirs. When determining the premium $V_t^I(1)$ at which the individual can purchase term life insurance per unit benefit, we maintain that $\tilde{\pi}_{t,t+1} = \pi_{t,t+1}$.

For the chosen bequest strengths, we thus investigate the market specifications $\mathcal{B}$ and $\mathcal{A}^I$. At higher bequest strengths, we find a significant increase in the overall level of precautionary savings throughout the life cycle, especially toward the end of the life cycle when mortality rates increase. The increase is weaker when term life insurance is available, as the individual is able to more effectively allocate wealth to different mortality-related states. Figure 5, moreover, shows the effects on cumulated annuity claims for different levels of interest rate risk.

Interestingly, compared to the effect on precautionary savings, we find a rather moderate impact on the nominal level of cumulated annuity claims for the cases in Figure 5(a)–(c) when additional life insurance is not available. Accumulation even increases slightly in some cases during the earlier part of the life cycle. Closer to retirement with increasing mortality rates, eventually fewer annuity claims are purchased relative to our base case, leading to a decline in the average cumulated annuity claims held at retirement. The effects become more pronounced and set in earlier at lower levels of interest rate

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

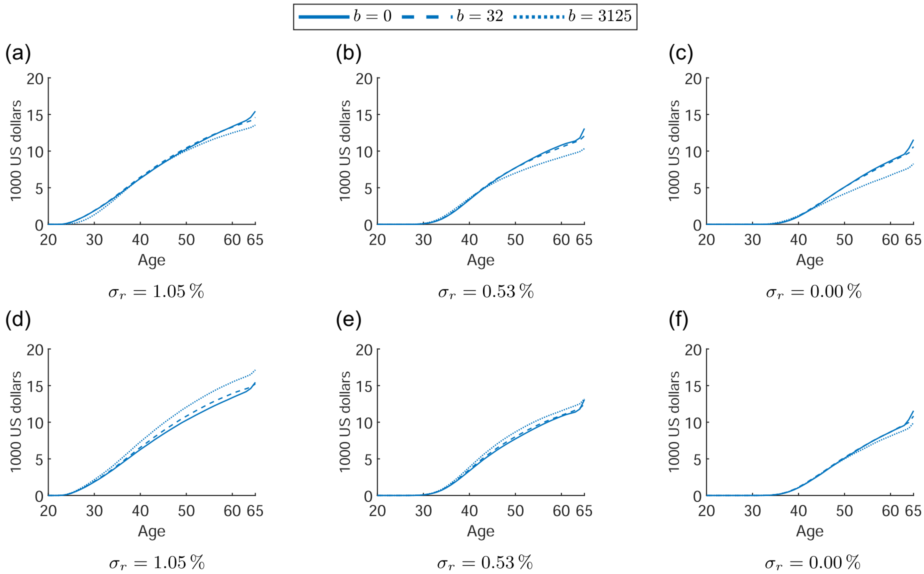


Figure 5. *Average cumulated annuity claims $\bar{L}_t^{opt}$ at age t for different levels of short rate volatility σ_r and bequest strengths b, without life insurance in market $\mathcal{B}$ (top row) and with life insurance in market $\mathcal{A}^I$ (bottom row).*

risk. For our baseline level with $\sigma_r = 1.05\%$, we observe a drop of average cumulated annuity claims to 13,550 US dollars (−12.26%) for the large bequest strength $b = 3125$. Without interest rate risk, average cumulated annuity claims at retirement amount to only 8266 US dollars (−28.51%) for the same bequest strength. Hence, in the presence of bequest motives, the individual holds significantly more precautionary savings by substantially increasing liquid asset holdings, but at the same time only moderately adjusts nominal annuity purchases. Overall, this preserves much of the temporal structure of the gradual annuity demand observed in our baseline cases without a bequest motive. As demand closer to retirement declines, especially under lower levels of interest rate risk, the spike in annuity purchases at the last eligible purchase date is flattened out, which reflects the less binding character of the timing constraint when bequest motives are present.

Except for the case without interest rate risk, the effects are generally reversed for the cases in Figure 5(d)–(f) when additionally incorporating term life insurance. For higher levels of interest rate risk, the introduction of a bequest motive even has an increasing effect on the accumulation of life annuity claims over the life cycle. In the baseline case with $\sigma_r = 1.05\%$, average cumulated annuity claims at retirement increase to 17,137 US dollars (+10.97%) for the large bequest strength $b = 3125$. Intuitively, this increase results due to the overall higher level of precautionary savings and since the individual may serve the bequest motive more accurately and at a lower cost through life insurance rather than through buffer savings in non-mortality-related financial assets (stocks and bonds), which frees up wealth that can instead be used for annuitization.

3.4. Impact of longer-term bonds

When interest rates are stochastic in our model, longer-term bonds cannot be replicated using rolling short-term investments into the money market account. In our baseline model, life annuities are the only way for the individual to obtain exposure to longer-term yields. Thereby, the demand for life annuities may absorb any demand for longer-term bonds, as the individual tries to duration-match consumption expenses later in the life cycle. If available, longer-term bonds may also partially crowd out demand for

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

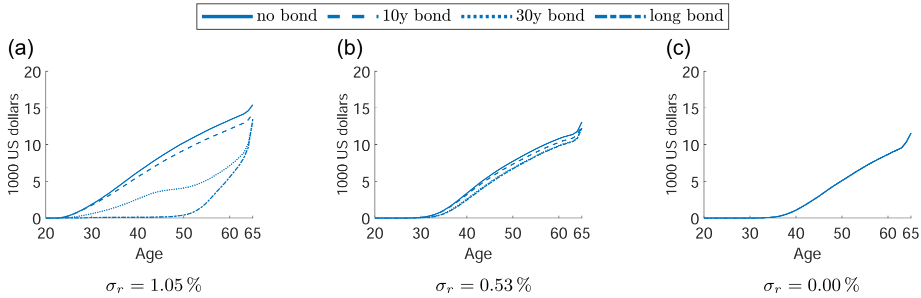


Figure 6. *Average cumulated annuity claims $\bar{L}_t^{opt}$ at age t for different levels of short rate volatility σ_r in market $\mathcal{B}$ and with different longer-term bonds in market $\mathcal{A}^B$.*

life annuities, depending on the liquidity advantage of bonds relative to the mortality credit offered by life annuities.

In order to quantify the impact of longer-term bonds within our model, we would ideally want to offer the individual a menu of longer-term bonds having different maturities to choose from. For computational reasons, this is however infeasible within our setup. Instead, we include only one longer-term bond with a certain maturity as an additional financial asset. To investigate maturity effects, we consider as realistic options a 10-year bond and a 30-year bond, corresponding to maturities that are typically available in real-world markets. Moreover, we consider the idealized long bond that matures at time $T_u + 1$, which at age 20 would correspond to an 81-year maturity. All bonds are priced according to Equation (2.3).

The ability to effectively obtain exposure to longer-term interest rate risk through bonds decreases the overall level of precautionary savings in the presence of interest rate risk. Figure 6 investigates the impact of the availability of longer-term bonds on cumulated annuity claims for different levels of interest rate risk.

Not surprisingly, more pronounced effects are observed for higher levels of interest rate risk. Without interest rate risk, the introduction of longer-term bonds naturally is inconsequential, as they may be replicated using rolling investments into the money market account. Increasing interest rate risk to $\sigma_r = 0.53\%$ only has a moderate effect under any of the considered bond maturities. On average, the accumulation of annuity claims is slightly delayed and reduced, with cumulated annuity claims at retirement on average amounting to 12,270 US dollars (−6.25%) even when a long bond is available. For our baseline case with $\sigma_r = 1.05\%$, we observe the strongest effects by far. If a long bond is available, annuitization is substantially delayed on average, with the accumulation of annuity claims picking up speed only at age 50. However, given the sizable accumulation over the remaining eligible dates, total cumulated annuity claims at retirement drop only to 13,505 US dollars (−12.55%). These effects are dampened significantly when decreasing the maturity of the longer-term bond. Given a 30-year bond, annuity claims are on average accumulated starting early in the life cycle, initially at a slower speed compared to the base case, but more quickly shortly before retirement. For a 10-year bond, the accumulation of annuity claims over the whole life cycle is only mildly affected. Under sufficiently high levels of interest rate risk, the availability of a bond with long maturity therefore postpones, but eventually only moderately reduces, the accumulation of annuity claims over the life cycle. A substantial gradual component survives in the demand for annuitization. Nevertheless, the availability of longer-term bonds breaks the usual pattern of increasing gradual demand when the level of interest rate risk increases. However, we expect these results to be considerably sensitive to the relative pricing of life annuities and longer-term bonds, in particular when taking into account that life annuities in our model do not carry an illiquidity discount that rationally reflects the irreversibility of the annuitization decision. A further investigation in this direction would require an equilibrium-based pricing approach that explicitly takes into account both liquidity and mortality risk, which goes beyond the scope of the present article.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

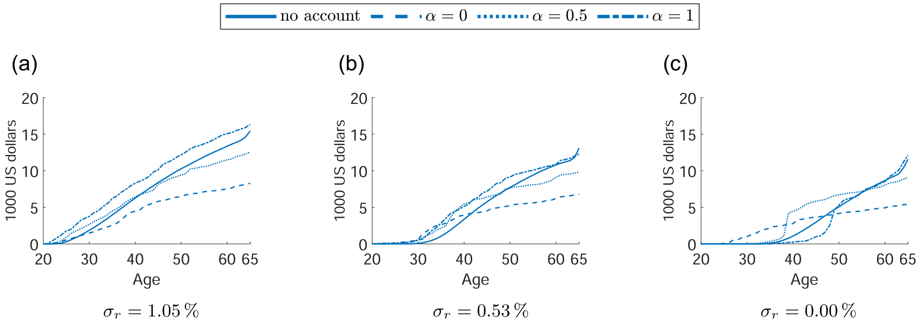


Figure 7. *Average cumulated annuity claims $\bar{L}_t^{opt}$ at age t for different levels of short rate volatility σ_r in market $\mathcal{B}$ and for different matching rates α in market $\widetilde{\mathcal{B}}$.*

3.5. Impact of retirement accounts

In essence, our base-case market $\mathcal{B}$ allows the individual to replicate a retirement product, including explicit gradual annuitization decisions, optimally engineered with respect to the subjective life-cycle preferences. Thereby, the investor in our model effectively constructs and manages a particular type of variable annuity (see, e.g., Feng *et al*., 2022 for an overview). This approach reflects the increasing accessibility of life annuities for private investors, as the market for direct life annuity investments has been growing significantly. In contrast, the related literature typically investigates certain predefined structures of variable annuities (e.g., Horneff *et al*., 2009, 2010a, 2015; Maurer *et al*., 2013). Beyond that, annuitization options are also embedded in various retirement products that otherwise give the individual either no or only limited control over the relevant investment decisions. For this reason, we now study optimal annuitization decisions within a stylized retirement account.

In the resulting market $\widetilde{\mathcal{B}}$, the individual is still able to directly invest into stocks and the money market, as before in market $\mathcal{B}$. However, life annuity purchases are now only possible through the separate retirement account. Each year before retirement, a certain fraction of the individual's income is contributed to the account, which we set equal to a constant 10% of permanent income P_t. In addition, we also allow for external contributions to the retirement account, for example, through an employer matching scheme or government subsidies. For each dollar that the individual contributes, the account therefore increases by $1+\alpha$ dollars for some given $\alpha \geq 0$. Our analysis considers three cases: $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$, covering empirically reasonable matching rates. In total, nominal annual contributions to the retirement account thus amount to $10\%(1+\alpha)P_t$.

The holdings in the retirement account may be used to gradually buy life annuities at the eligible annuity purchase ages $\mathcal{T}_{20-64}$. Any holdings remaining in the account after annuitization are periodically invested into a portfolio consisting of stocks and the money market account. For our purposes, we use a constant allocation with 60% stock fraction, rebalanced annually. At retirement, the account holdings that have not been annuitized are transferred with a one-time benefit payment to the liquid financial wealth of the individual.

By its construction, the retirement account with its regular contributions forces the individual to make savings for retirement throughout the whole working life. While investments into liquid financial assets are externally managed within the account, generally at suboptimal allocations, the individual is still entirely in control over the annuitization strategy. In this setting, Figure 7 shows the optimal accumulation of annuity claims for different levels of interest rate risk in market $\widetilde{\mathcal{B}}$, relative to the base-case market $\mathcal{B}$.

Also within the retirement account in market $\widetilde{\mathcal{B}}$, a gradual annuitization strategy remains optimal. Similar to our prior observations, the annuitization decision is typically deferred until later in the life cycle under lower levels of interest rate risk. Not surprisingly, the level of accumulated annuity claims generally increases with higher matching rates α. A higher matching rate directly increases the account

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

holdings available for annuitization and further induces a wealth effect, as the individual becomes richer overall. Typically, with contributions to the retirement account starting at the beginning of the life cycle, the individual starts annuitizing even earlier compared to market $\mathcal{B}$ that offers complete flexibility over all investment and annuitization decisions. In our baseline interest rate risk case with $\sigma_r = 1.05\%$ in market $\widetilde{\mathcal{B}}$, the individual on average ends up with acquired annuity claims of 8287 US dollars (-46.34%) when $\alpha = 0$. In contrast, cumulated annuity claims rise to 16,332 US dollars ($+5.76\%$) with $\alpha = 1$, while the accumulation of annuity claims now on average almost sets in right at the beginning of the life cycle. Overall, despite the exogenously reduced flexibility of annuitization within a retirement account, the general structural patterns of the optimal annuitization strategy that have been identified before are mostly preserved.

4. Optimal one-time annuitization and its welfare costs

Complementing the analysis in Section 3, this section investigates the optimal one-time annuitization strategy. Hence, we are here considering a situation motivated by prevalent institutional constraints, by which the annuitization decision is temporally restricted to a single point in time just before retirement. We account for this timing constraint by enforcing one-time annuitization at age 64. In our base case, we thus solve the household's optimization problem for eligible annuity purchase ages $\mathcal{T}_{64}$ with the parameterization presented in Table 2.

The results of Section 3 suggest that this timing constraint is binding and exposes the household to additional annuity risk, by which the annuitization decision depends on the individual's personal income and wealth situation as well as the state of the overall financial market at a particular point in time. Substituting the gradual demand for annuities, the individual now is forced to build up excess precautionary buffer savings in liquid assets to protect against annuity risk and the retirement income risk associated with it. Consequently, we expect to observe an impairing effect of the annuitization timing constraint on the individual's economic welfare, as consumption has to be sacrificed for additional savings. It may thereby happen that these larger precautionary savings ultimately also lead to a larger accumulation of annuity claims relative to the gradual case. To ascertain the degree to which these effects are driven by interest rate risk, we analyze our model again for different levels of short rate volatility σ_r.

We begin our analysis by presenting the optimal one-time annuitization strategy for different levels of short rate volatility in Section 4.1. Subsequently, we investigate different parameterizations and their effects on annuity demand and welfare. We consider bequest motives and life insurance in Section 4.2, longer-term bonds in Section 4.3, and retirement accounts in Section 4.4. Supplementary to these results, the online appendix investigates different levels of labor income risk in section OE.1, alternative preference parameterizations in section OE.2, and actuarial loadings in section OE.3.

4.1. Optimal life-cycle demand and asset allocation

We first analyze the optimal life-cycle demand and asset allocation for one-time annuitization ($\mathcal{T}_{64}$) in market $\mathcal{B}$ under the default parameterization in Table 2 and compare it to the gradual annuitization results discussed in Section 3.1. The required quantities are again obtained from computing averages of optimal policies over the simulated paths.

The resulting average profiles are depicted in Figure 8. In particular, Figure 8(a) shows the life-cycle profiles and Figure 8(b) the corresponding asset allocation profiles.

Not surprisingly, the profiles look very different from the ones obtained in Figure 1, as the large gradual demand for annuities is now infeasible. Compensating for that, we observe a higher average stock demand and a strong rise in average investment in the money market. As the individual can only acquire annuities at a single date and is thereby exposed to the prevailing personal and market conditions at that particular point in time, the increase in liquid asset holdings can be characterized as hedge demand for diversifying this annuity risk at the eligible purchase date, in line with Koijen *et al*. (2011).

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

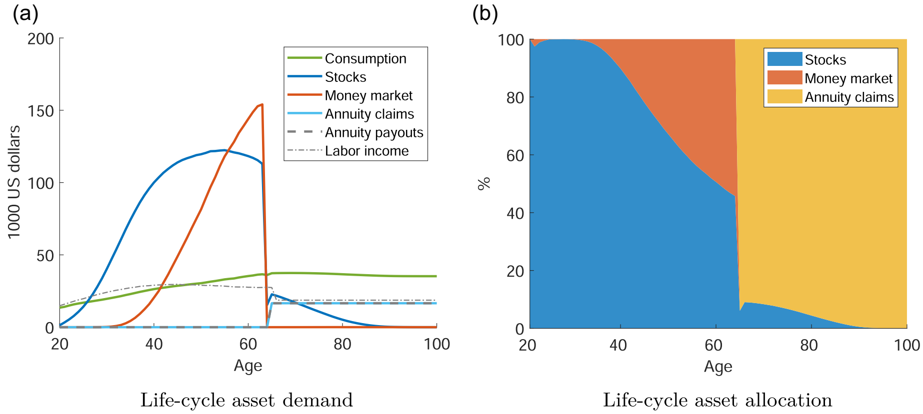


Figure 8. *Average life-cycle profiles (panel a) and asset allocation as a percentage of financial wealth (panel b) for one-time annuitization ($\mathcal{T}_{64}$) with $\sigma_r = 1.05\%$ in market $\mathcal{B}$.*

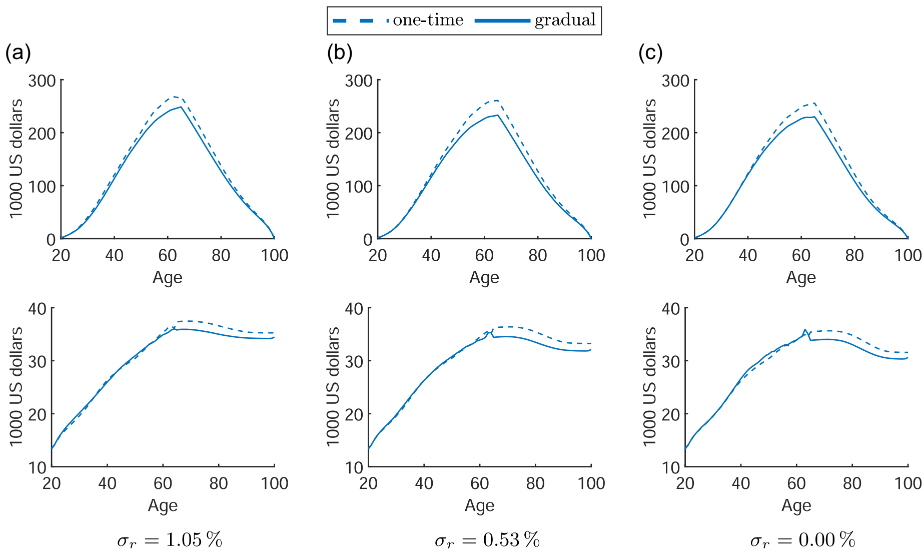


Figure 9. *Average precautionary savings (top row) and average consumption (bottom row) for different levels of short rate volatility σ_r, comparing one-time annuitization ($\mathcal{T}_{64}$) and gradual annuitization ($\mathcal{T}_{20-64}$) in market $\mathcal{B}$.*

Consistent with this interpretation, the average money market holdings are completely converted to annuity claims at the eligible purchase date immediately before retirement and the average stock holdings are reduced by 90%.

It is well established that access to annuities reduces the need to accumulate wealth that is disinvested in retirement to finance consumption of the retiree (Horneff *et al.*, 2008b, 2010b). Beyond that, our results suggest that gradual access to annuities over the whole life cycle, as compared to a merely a one-time opportunity to annuitize, significantly flattens the humped shape that is present before retirement for average liquid wealth, that is, money market and stock holdings. Additionally, Figure 9 contrasts the average precautionary savings, including acquired annuity claims, that are held over the life cycle under different levels of interest rate risk and the associated average consumption profiles.

In each of the cases in Figure 9(a)–(c), the restriction to one-time annuitization goes along with higher precautionary savings demand in total. Hence, the individual has to forgo consumption throughout the pre-retirement life cycle in order to self-insure against annuity risk. The timing of this reduction depends on the level of interest rate risk. For our base-case parameterization with $\sigma_r = 1.05\%$ in Figure 9(a), we find that additional precautionary savings begin to noticeably accumulate already before age 30. With lower levels of interest rate risk at $\sigma_r = 0.53\%$ in Figure 9(b) and $\sigma_r = 0.00\%$ in Figure 9(c), excess precautionary saving are delayed until later in the life cycle and become higher overall, especially around the retirement date. Interestingly, the excess precautionary savings are not fully annuitized at the eligible purchase date, but rather a residual part of liquid precautionary savings is decumulated over the whole retirement period, overall leading to higher retirement consumption.

Clearly, the economic cost to the individual of following the one-time annuitization strategy must ultimately outweigh its benefits, since one-time annuitization would have been feasible as a nested strategy when allowing for gradual annuitization (i.e., $\mathcal{T}_{64} \subset \mathcal{T}_{20-64}$). However, our analysis in Section 3 reveals that the optimal gradual annuitization strategy is distinctively different from one-time annuitization. It remains to quantify the cost in terms of forgone consumption over the life cycle, resulting from the excess precautionary savings and consumption postponement observed in Figure 9. For this purpose, following Horneff *et al*. (2010a), we determine the required one-time increase in initial liquid wealth that makes the individual indifferent between one-time and gradual annuitization. Differences in the resulting indifference wealth levels $\hat{W}_{20}$, normalized to the calibrated initial wealth under gradual annuitization, will be interpreted as monetary measures of the welfare loss to the individual due to timing constraints on the annuitization strategy. In addition, similar to Cocco and Gomes (2012), we convert the stochastic consumption streams obtained by solving and simulating our model into a constant certainty-equivalent consumption stream $\hat{C}_t$, such that both yield the same utility index in Equation (2.8) at a particular reference age t. Any decline in certainty-equivalent consumption may also be interpreted as a measure of the welfare loss to the individual. Both welfare loss measures are related, in that the difference in indifference wealth levels corresponds to the (subjective) value of an immediate life annuity that pays out the difference in annual certainty-equivalent consumption.

Table 3 reports indifference wealth levels, certainty-equivalent consumption streams at age 20 (i.e., covering the whole life cycle) and age 65 (i.e., covering retirement only), as well as summary statistics of cumulated annuity claims at the retirement date, for gradual and one-time annuitization and varying levels of interest rate risk.

For all different interest rate risk scenarios, the individual requires a substantial one-time increase in initial wealth to compensate for timing constraints on the annuitization decision. For the base-case interest rate risk level $\sigma_r = 1.05\%$, the wealth welfare loss measured by the required wealth increase corresponds to 36.99% (5489 US dollars) under one-time annuitization relative to gradual annuitization. Consistent with this, the certainty-equivalent consumption stream is larger with gradual annuitization compared to one-time annuitization. The consumption welfare loss over the entire life cycle amounts to 2.30% (357 US dollars) of certainty-equivalent consumption per year with gradual annuitization at $\sigma_r = 1.05\%$ and gradually decreases with the reduction of interest rate risk. This lower life-cycle consumption, due to more precautionary savings throughout the working life, is compensated to some degree by larger average certainty-equivalent consumption during retirement. Over the retirement period in isolation, the incurred welfare gain at $\sigma_r = 1.05\%$ is 4.26% (1508 US dollars), which slightly increases when reducing interest rate risk. The sacrificing of consumption before retirement to build up additional precautionary buffer savings for one-time annuitization on average leads to a significant reallocation of consumption from the working life to the retirement period. As life annuity purchases cannot be effectively substituted by the other available financial assets in our base-case model, restricting the annuitization decision to a specific point in time thus implies significant welfare losses over the whole life cycle compared to gradual annuitization, such that losses are larger the higher the level of interest rate risk. Supplementary results (not reported) additionally suggest that considering $\mathcal{T}_{20-64}$ instead of $\mathcal{T}_{20-99}$ only implies a relatively

Table 3. *Indifference wealth levels $\hat{W}_{20}$ at age 20, certainty-equivalent consumption $\hat{C}_{20}$ (over the whole life cycle) and $\hat{C}_{65}$ (over the retirement period), as well as distribution statistics of cumulated annuity claims L_{65}^{opt} at age 65 with one-time annuitization ($\mathcal{T}_{64}$) and gradual annuitization ($\mathcal{T}_{20-64}$) for different levels of short rate volatility σ_r in market $\mathcal{B}$. Absolute values are denominated in US dollars. Relative values are expressed using the gradual values as denominator.*

σ_r		$\hat{W}_{20}$	$\hat{C}_{20}$	$\hat{C}_{65}$	Avg. L_{65}^{opt}	Med. L_{65}^{opt}	Std. L_{65}^{opt}
	$\mathcal{T}_{64}$	20,329	15,128	36,954	16,570	13,633	11,617
1.05%	$\mathcal{T}_{20-64}$	14,840	15,484	35,446	15,443	13,234	9050
	Δ_{abs}	5489	−357	1508	1127	399	2568
	Δ_{rel}	36.99%	−2.30%	4.26%	7.30%	3.01%	28.37%
	$\mathcal{T}_{64}$	16,907	15,131	35,690	14,562	12,110	9867
0.53%	$\mathcal{T}_{20-64}$	14,840	15,279	33,866	13,088	11,171	7819
	Δ_{abs}	2067	−148	1825	1474	939	2048
	Δ_{rel}	13.93%	−0.97%	5.39%	11.27%	8.41%	26.20%
	$\mathcal{T}_{64}$	15,488	15,103	34,766	12,870	10,728	8849
0.00%	$\mathcal{T}_{20-64}$	14,840	15,151	33,151	11,563	9704	7369
	Δ_{abs}	648	−48	1615	13,307	1024	1480
	Δ_{rel}	4.36%	−0.32%	4.87%	11.30%	10.55%	20.08%

mild life-cycle welfare loss for each of the interest rate risk scenarios considered. In that sense, the main gradual demand for annuitization arises before retirement.

As demonstrated by Figure 8, for the case of one-time annuitization, most of the accumulated liquid wealth is eventually converted to annuity claims at the eligible purchase date. Table 3 also allows to quantify the extent to which this one-time annuitization occurs and allows to contrast it to gradual annuitization. What is remarkable is that, in the case of one-time annuitization, the annuitant acquires *more* annuity claims on average and in the median relative to gradual annuitization, albeit being generally exposed to higher annuity risk. Intuitively, the risk-averse individual accumulates precautionary buffer savings for more adverse conditions than those realizing on average, leaving it typically with some excess savings that are additionally annuitized. This leads to a higher average annuity income in retirement under each of the different interest rate scenarios, consistent with the larger certainty-equivalent consumption during that period. Specifically, the difference in the median of acquired annuity claims increases when the short rate volatility σ_r falls. The same holds true for the average cumulated annuity claims, henceforth denoted by $\bar{L}_{65}^{\text{opt}}$, whose relative difference rises with falling interest rate risk. Comparing the full distributions of cumulated annuity claims moreover reveals a larger downside potential, that is, a higher frequency of no or only few acquired annuity claims, for the case of one-time annuitization. Intuitively, this occurs when precautionary buffer savings eventually turn out to be insufficient when particularly adverse conditions realize. This effect diminishes with decreasing levels of interest rate risks, as the corresponding component of annuity risk vanishes and only the personal component remains.

4.2. Impact of bequest motives and life insurance

As in Section 3.3, we investigate the impact of bequest motives and life insurance on our results. For quantifying the effects on welfare losses due to one-time annuitization and on cumulated annuity claims, we maintain the parameterization suggested before. Hence, we consider bequest strengths $b = 2^\gamma = 32$ and $b = 5^\gamma = 3125$ without and with one-period term life insurance.

Table 4 reports the indifference wealth levels and average cumulated annuity claims for these cases; certainty-equivalent consumption streams are investigated in section OE.4 of the online appendix.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

Table 4. *Indifference wealth levels $\hat{W}_{20}$ at age 20 and average cumulated annuity claims $\bar{L}_{65}^{opt}$ at age 65 for different levels of short rate volatility σ_r and bequest strengths b, without and with life insurance in markets $\mathcal{B}$ and $\mathcal{A}^I$, respectively. Absolute values are denominated in US dollars. Relative values are expressed using the gradual values as denominator.*

		Without life insurance				With life insurance			
		$b = 32$		$b = 3125$		$b = 32$		$b = 3125$	
σ_r		$\hat{W}_{20}$	$\bar{L}_{65}^{opt}$	$\hat{W}_{20}$	$\bar{L}_{65}^{opt}$	$\hat{W}_{20}$	$\bar{L}_{65}^{opt}$	$\hat{W}_{20}$	$\bar{L}_{65}^{opt}$
1.05%	$\mathcal{T}_{64}$	20,499	14,847	18,056	12,407	20,815	15,493	21,827	17,362
	$\mathcal{T}_{20-64}$	14,840	14,601	14,840	13,550	14,840	15,240	14,840	17,137
	Δ_{abs}	5659	245	3215	−1143	5975	253	6987	226
	Δ_{rel}	38.13%	1.68%	21.67%	−8.44%	40.26%	1.66%	47.08%	1.32%
0.53%	$\mathcal{T}_{64}$	16,965	13,135	16,093	10,438	17,057	13,512	17,285	14,053
	$\mathcal{T}_{20-64}$	14,840	12,088	14,840	10,337	14,840	12,486	14,840	13,182
	Δ_{abs}	2125	1046	1252	101	2216	1026	2445	871
	Δ_{rel}	14.32%	8.66%	8.44%	0.97%	14.93%	8.22%	16.47%	6.60%
0.00%	$\mathcal{T}_{64}$	15,517	11,686	15,252	8840	15,539	11,980	15,545	11,086
	$\mathcal{T}_{20-64}$	14,840	10,611	14,840	8266	14,840	10,840	14,840	9918
	Δ_{abs}	676	1075	412	574	699	1141	704	1169
	Δ_{rel}	4.56%	10.13%	2.77%	6.95%	4.71%	10.52%	4.74%	11.78%

For $b = 32$, we observe quantitatively similar relative welfare declines over the whole life cycle as in the cases in Table 3. Our baseline interest rate risk scenario with $\sigma_r = 1.05$ % yields a relative wealth welfare loss of 38.13% (5659 US dollars) in the case without life insurance and 40.26% (5975 US dollars) in the case with life insurance. When increasing the bequest strength to $b = 3125$ as life insurance is unavailable, life-cycle welfare losses decrease. In the baseline interest rate risk scenario, the wealth welfare loss now amounts to 21.67% (3215 US dollars). The higher overall demand for precautionary savings identified in Section 3.3 on average leads to lower consumption levels compared to the base case, which is further substantiated in section OE.4 of the online appendix. Yet, the lower gradual demand for annuities reduces the welfare losses implied by one-time annuitization when the bequest strength is large enough. However, when also incorporating life insurance, the larger bequest strength of $b = 3125$ slightly increases life-cycle welfare losses. In the baseline interest rate risk scenario, the wealth welfare loss now amounts to 47.08% (6987 US dollars). These latter effects are consistent with the positive impact on annuitization observed in Section 3.3. Despite bequest motives significantly shifting preferences away from just the states in which the individual is alive, life annuities remain an important part of overall precautionary savings, so that the economic cost of constraining temporal diversification of annuity risk is still substantial. As before, the incurred welfare losses continue to be increasing in the level of interest rate risk, while relative differences in average annuity claims tend to decrease. A negative effect is typically observed for the relative change in average cumulated annuity claims when increasing the bequest strength.

4.3. Impact of longer-term bonds

Complementing the analysis in Section 3.4, we also determine in how far the availability of longer-term bonds affects welfare losses due to one-time annuitization and the accumulation of annuity claims. We again consider three different maturities of longer-term bonds to quantify the effects.

For these different cases, Table 5 reports the indifference wealth levels and average cumulated annuity claims; certainty-equivalent consumption streams are discussed in section OE.5 of the online appendix.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

Table 5. *Indifference wealth levels $\hat{W}_{20}$ at age 20 and average cumulated annuity claims $\bar{L}_{65}^{opt}$ at age 65 for different levels of short rate volatility σ_r and different longer-term bonds in market $\mathcal{A}^B$. Absolute values are denominated in US dollars. Relative values are expressed using the gradual values as denominator.*

		10y bond		30y bond		Long bond	
σ_r		$\hat{W}_{20}$	$\bar{L}_{65}^{\text{opt}}$	$\hat{W}_{20}$	$\bar{L}_{65}^{\text{opt}}$	$\hat{W}_{20}$	$\bar{L}_{65}^{\text{opt}}$
1.05%	$\mathcal{T}_{64}$	16,695	15,065	14,969	15,182	14,902	15,484
	$\mathcal{T}_{20-64}$	14,840	14,241	14,840	13,070	14,840	13,505
	Δ_{abs}	1854	824	129	2112	62	1979
	Δ_{rel}	12.50%	5.78%	0.87%	16.16%	0.42%	14.65%
0.53%	$\mathcal{T}_{64}$	15,861	14,243	15,415	14,286	15,377	14,292
	$\mathcal{T}_{20-64}$	14,840	12,647	14,840	12,319	14,840	12,270
	Δ_{abs}	1020	1596	574	1968	537	2022
	Δ_{rel}	6.87%	12.62%	3.87%	15.97%	3.62%	16.48%
0.00%	$\mathcal{T}_{64}$	15,488	10,013	15,488	10,008	15,488	10,310
	$\mathcal{T}_{20-64}$	14,840	11,562	14,840	11,561	14,840	11,559
	Δ_{abs}	648	−1549	648	−1553	648	−1249
	Δ_{rel}	4.36%	−13.39%	4.36%	−13.43%	4.36%	−10.81%

Without interest rate risk, the introduction of longer-term bonds, due to their redundancy irrespective of the concrete maturity, does not have any effect compared to the case reported in Table 3. The situation changes once interest rates are stochastic, as the introduction of a long bond leads to a substantial drop in the welfare loss. In our base case with $\sigma_r = 1.05$ %, the wealth welfare loss now only amounts to 0.42% (62 US dollars). Welfare losses remain somewhat larger when considering shorter (and more realistic) maturities of the bond, but nevertheless drop significantly compared to the baseline model without a longer-term bond. The results for a 30-year bond are generally quite close to those of a long bond, while the 10-year bond incurs larger losses. To reconcile these findings with our prior observations in Section 3.4, we note that even though a substantial gradual demand for annuitization remains optimal, the availability of a longer-term bond makes it easier to substitute this demand if the annuitization decision is temporally restricted. The substitution may only be achieved to a limited degree with a 10-year bond, but much more effectively with a 30-year bond or a long bond. As detailed in section OE.5 of the online appendix, this substitution on average goes along with a significant reallocation of consumption over the life cycle, together with an increase in cumulated annuity claims. In our base case with $\sigma_r = 1.05\%$, the relative change in average cumulated annuity claims increases to 14.65% (1979 US dollars). However, given our insights regarding different Epstein-Zin preference specifications (see section OE.2 of the online appendix), it can be expected that this outcome is sensitive with respect to utility specifications that convey different temporal preferences. Moreover, from what was already mentioned in Section 3.4, we suspect that also the relative pricing of life annuities and longer-term bonds will be highly influential for the quantification of welfare losses.

4.4. Impact of retirement accounts

Considering market $\widetilde{\mathcal{B}}$ according to Section 3.5, life annuity purchases can be made only from the holdings of a separate retirement account. In a one-time annuitization setting, the sole difference is that the annuitization decision is restricted to the eligible age set $\mathcal{T}_{64}$, for which we quantify the associated welfare losses relative to gradual annuitization with eligible age set $\mathcal{T}_{20-64}$.

Our analysis covers the different choices of matching rates α introduced in Section 3.5. Table 6 reports the associated indifference wealth levels and average cumulated annuity claims.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press

Table 6. *Indifference wealth levels $\hat{W}_{20}$ at age 20 and average cumulated annuity claims $\bar{L}_{65}^{opt}$ at age 65 for different levels of short rate volatility σ_r and matching rates α in market $\widetilde{\mathcal{B}}$. Absolute values are denominated in US dollars. Relative values are expressed using the gradual values as denominator.*

		$\alpha = 0$		$\alpha = 0.5$		$\alpha = 1$	
σ_r		$\hat{W}_{20}$	$\bar{L}_{65}^{\text{opt}}$	$\hat{W}_{20}$	$\bar{L}_{65}^{\text{opt}}$	$\hat{W}_{20}$	$\bar{L}_{65}^{\text{opt}}$
1.05%	$\mathcal{T}_{64}$	21,340	8326	20,412	11,232	19,351	13,338
	$\mathcal{T}_{20-64}$	14,840	8287	14,840	12,553	14,840	16,332
	Δ_{abs}	6500	39	5571	−1321	4511	−2993
	Δ_{rel}	43.80%	0.47%	37.54%	−10.52%	30.39%	−18.33%
0.53%	$\mathcal{T}_{64}$	18,846	7472	17,816	10,201	17,457	12,205
	$\mathcal{T}_{20-64}$	14,840	6781	14,840	9836	14,840	12,281
	Δ_{abs}	4006	691	2976	365	2617	−77
	Δ_{rel}	26.99%	10.20%	20.05%	3.71%	17.63%	−0.62%
0.00%	$\mathcal{T}_{64}$	18,209	6843	16,979	9414	16,729	11,274
	$\mathcal{T}_{20-64}$	14,840	5444	14,840	9071	14,840	12,142
	Δ_{abs}	3369	1399	2139	343	1889	−867
	Δ_{rel}	22.70%	25.71%	14.41%	3.78%	12.73%	−7.14%

Within the retirement account in market $\widetilde{\mathcal{B}}$, the restriction of the annuitization decision to a single date also incurs substantial welfare losses to the individual, which is expected given the strong gradual annuity demand observed in Section 3.5. As usual, welfare losses are diminishing at lower levels of interest rate risk. Moreover, welfare losses are also somewhat decreasing with higher matching rates α, due to a wealth effect induced by higher external contributions to the individual's retirement account. In our base case with $\sigma_r = 1.05$ %, the wealth welfare loss is 43.80% (6500 US dollars) when $\alpha = 0$, slightly higher than what is observed for market $\mathcal{B}$ in Table 3. The wealth welfare loss decreases to a still significant 30.39% (4511 US dollars) when $\alpha = 1$. While the former choice $\alpha = 0$ is associated to roughly the same amount of cumulated annuity claims in both one-time and gradual annuitization environments, average cumulated annuity claims drop significantly under one-time relative to gradual annuitization for larger α, an effect that is weakened or even reversed under lower levels of interest rate risk.

5. Conclusion

Building on a life-cycle model with stochastic interest rates, the contribution of this article is twofold. First, we study the optimal gradual annuitization strategy, which prescribes how an individual can diversify annuity risk by spreading life annuity purchases over the life cycle. Our numerical results mostly suggest that higher interest rate risk leads to an earlier and more extensive participation in the annuity market. Essentially, annuitization is optimally spread out over almost the entire life cycle and opportunistically exploits favorable states in which annuities are relatively cheap. Hence, with higher interest rate risk, the to-be annuitant is more engaged in temporal diversification of annuity risk. Second, we study the optimal one-time annuitization strategy and welfare losses implied by such timing constraints on the annuitization decision. Effectively prohibiting temporal diversification of annuity risk, we find that enforcing one-time annuitization at retirement typically leaves the individual with a significant welfare loss compared to gradual annuitization, which increases in magnitude with the level of interest rate risk. The welfare loss for one-time annuitization relative to gradual annuitization is driven by the need to build up extensive precautionary buffer savings to insure against the additional annuity risk, which may ultimately lead to more extensive annuitization.

Exceptions to these general patterns occur especially when a longer-term bond with sufficiently long maturity is available. In that case, our numerical results suggest that the welfare loss due to one-time annuitization becomes negligible when the level of interest rate risk increases, while a sizable but smaller gradual component of annuity demand nevertheless survives. Despite the minor economic cost of one-time annuitization, the optimal gradual annuity demand and life-cycle consumption profile may still substantially differ from what is suggested by one-time annuitization. In this regard, we find that the availability of a longer-term bond under stochastic interest rates even magnifies the one-time annuitization demand relative to the cumulated gradual demand. However, we believe that the understanding of the relative demand for life annuities and longer-term bonds with its complex substitution effects is still rather incomplete. Model-based results are likely sensitive to, among other factors, the relative pricing of the respective assets, whose further investigation would require equilibrium-based considerations that are beyond the scope of the current article, but offer an interesting route for future research.

With our work, we address a gap in the existing literature on optimal annuitization, which typically either neglects interest rate risk or prescribes a one-time annuitization decision at the retirement date. The former case disregards an important driver of the annuitization decision, while the latter case prohibits the temporal diversification of annuity risk and instead only allows to partially hedge the risk of meeting unfavorable market conditions at the purchase date with other (non-annuity) assets. Overall, our results underline the importance of such temporal diversification of annuity risk, which structurally alters the optimal annuitization strategy and life-cycle consumption profile. In markets where the annuitization demand can be less effectively substituted through other assets, temporal diversification may thus contribute significantly to the annuitant's welfare. Our results in this article suggest that temporal diversification of annuity risk is not only relevant to direct life annuity investments but also more generally with regard to annuitization decisions embedded in various retirement products.

From a broader perspective, our results might point to an additional dimension of the annuity puzzle. This is typically understood as a discrepancy between the annuity demand predicted by reasonably parameterized life-cycle models and actually observed annuity demand, commonly measured during retirement and with respect to a one-time (or closely related) annuitization strategy. According to our model, when accounting for gradual annuity demand under stochastic interest rates, the average amount of annuities acquired over the life cycle is typically lower than under one-time annuitization, which thus may reduce the magnitude of the annuity puzzle under the common measurement approach. On the other hand, our model implies annuity demand that is essentially spread over the entire life cycle, a property which is itself probably hard to reconcile with the temporal structure of empirically observed demand. Including our insights into an empirical analysis of the annuity puzzle could thus be an interesting route for future research.

To put our results into perspective, we conclude by briefly discussing some limitations of our analysis in this article. In order to keep our life-cycle model parsimonious and manageable from a computational perspective, we restrict the attention to interest rates and labor income as the main sources of risk. Throughout, we specifically assume that real annuities are traded and mostly maintain the additional assumption that they are priced in an actuarially fair way and promptly react to interest rate changes, which may not adequately reflect reality. For interest rates, we prescribe a Vasicek model that offers a high degree of tractability, while it would be interesting to extend our results to other model classes. The same holds true for stock prices, for which we use a conditionally log-normal specification. Selecting a larger universe of assets could also be insightful, specifically by allowing the household to more flexibly assemble the structure of insurance for future states. Further interesting factors were neglected in our analysis. As a our model is formulated in real terms, we ignore the possible effects of inflation risk, which would affect the prices of consumption goods, asset prices, and labor income. Moreover, we do not consider possible tax effects, which may involve a favorable treatment of life annuities relative to other investments. Employing constant mortality rates, we neglect the effects of stochastic mortality risk or ambiguity and, moreover, the potential impact of changes in health status together with associated out-of-pocket health expenses. Finally, we focus on preference specifications in the Epstein-Zin class, while alternative preference specifications, such as loss aversion, habit formation, and others, might be able to generate some interesting additional insights.

Supplementary material. The supplementary material for this article can be found at https://doi.org/10.1017/asb.2024.23

Acknowledgments. This article was previously circulated under the title "Dynamic Portfolio Choice with Annuities When the Interest Rate Is Stochastic." A major part of this project was conducted while all authors were affiliated with Goethe University Frankfurt. This work was supported by funding from the German Investment and Asset Management Association (BVI). We thank the initiative High Performance Computing in Hessen for granting us computing time at the LOEWE-CSC cluster and the Lichtenberg High Performance Computer. We thank Arwed Ebner, Andreas Hubener, Michel Vellekoop, and seminar participants at Goethe University for helpful comments and discussions. Benjamin Lucht provided excellent research assistance.

References

Aase, K.K. (2015) Life insurance and pension contracts I: The time additive life cycle model. *ASTIN Bulletin*, **45**(1), 1–47.
Aase, K.K. (2016) Life insurance and pension contracts II: The life cycle model with recursive utility. *ASTIN Bulletin*, **46**(1), 71–102.
Alexandrova, M. and Gatzert, N. (2019) What do we know about annuitization decisions? *Risk Management and Insurance Review*, **22**(1), 57–100.
Arias, E. and Xu, J. (2022) United states life tables, 2019. National Vital Statistics Reports 70(19). National Center for Health Statistics.
Bellman, R. (1954) The theory of dynamic programming. *Bulletin of the American Mathematical Society*, **60**(6), 503–515.
Bommier, A. and Grand, F.L. (2014) Too risk averse to purchase insurance? *Journal of Risk and Uncertainty*, **48**(2), 135–166.
Cocco, J.F. and Gomes, F.J. (2012) Longevity risk, retirement savings, and financial innovation. *Journal of Financial Economics*, **103**(3), 507–529.
Cocco, J.F., Gomes, F.J. and Maenhout, P.J. (2005) Consumption and portfolio choice over the life cycle. *Review of Financial Studies*, **18**(2), 491–533.
Córdoba, J.C. and Ripoll, M. (2017) Risk aversion and the value of life. *Review of Economic Studies*, **84**(4), 1472–1509.
Davidoff, T., Brown, J.R. and Diamond, P.A. (2005) Annuities and individual welfare. *American Economic Review*, **95**(5), 1573–1590.
Epstein, L.G. and Zin, S.E. (1989) Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, **57**(4), 937–969.
Feng, R., Gan, G. and Zhang, N. (2022) Variable annuity pricing, valuation, and risk management: A survey. *Scandinavian Actuarial Journal*, **2022**(10), 867–900.
Gomes, F. (2020) Portfolio choice over the life cycle: A survey. *Annual Review of Financial Economics*, **12**(1), 277–304.
Gomes, F. and Michaelides, A. (2005) Optimal life-cycle asset allocation: understanding the empirical evidence. *Journal of Finance*, **60**(2), 869–904.
Gourinchas, P.-O. and Parker, J.A. (2002) Consumption over the life cycle. *Econometrica*, **70**(1), 47–89.
Hainaut, D. and Devolder, P. (2006) Life annuitization: Why and how much? *ASTIN Bulletin*, **36**(2), 629–654.
Horneff, V., Maurer, R. and Mitchell, O.S. (2020) Putting the pension back in 401(k) retirement plans: Optimal versus default deferred longevity income annuities. *Journal of Banking & Finance*, **114**, 105783:1–14.
Horneff, V., Maurer, R., Mitchell, O.S. and Rogalla, R. (2015) Optimal life cycle portfolio choice with variable annuities offering liquidity and investment downside protection. *Insurance: Mathematics and Economics*, **63**, 91–107.
Horneff, W.J., Maurer, R., Mitchell, O.S. and Dus, I. (2008a) Following the rules: Integrating asset allocation and annuitization in retirement portfolios. *Insurance: Mathematics and Economics*, **42**(1), 396–408.
Horneff, W.J., Maurer, R., Mitchell, O.S. and Stamos, M.Z. (2009) Asset allocation and location over the life cycle with investment-linked survival-contingent payouts. *Journal of Banking & Finance*, **33**(9), 1688–1699.
Horneff, W.J., Maurer, R., Mitchell, O.S. and Stamos, M.Z. (2010a) Variable payout annuities and dynamic portfolio choice in retirement. *Journal of Pension Economics & Finance*, **9**(2), 163–183.
Horneff, W.J., Maurer, R. and Rogalla, R. (2010b) Dynamic portfolio choice with deferred annuities. *Journal of Banking & Finance*, **34**(11), 2652–2664.
Horneff, W.J., Maurer, R. and Stamos, M.Z. (2008b) Life-cycle asset allocation with annuity markets. *Journal of Economic Dynamics and Control*, **32**(11), 3590–3612.
Horneff, W.J., Maurer, R. and Stamos, M.Z. (2008c) Optimal gradual annuitization: Quantifying the costs of switching to annuities. *Journal of Risk and Insurance*, **75**(4), 1019–1038.
Huang, H. and Milevsky, M.A. (2008) Portfolio choice and mortality-contingent claims: The general HARA case. *Journal of Banking & Finance*, **32**(11), 2444–2452.
Huang, H., Milevsky, M.A. and Young, V.R. (2017) Optimal purchasing of deferred income annuities when payout yields are mean-reverting. *Review of Finance*, **21**(1), 327–361.
Hubener, A., Maurer, R. and Rogalla, R. (2014) Optimal portfolio choice with annuities and life insurance for retired couples. *Review of Finance*, **18**(1), 147–188.

Inkmann, J., Lopes, P. and Michaelides, A. (2011) How deep is the annuity market participation puzzle? *Review of Financial Studies*, **24**(1), 279–319.

Koijen, R.S., Nijman, T.E. and Werker, B.J. (2011) Optimal annuity risk management. *Review of Finance*, **15**(4), 799–833.

Lockwood, L.M. (2012) Bequest motives and the annuity puzzle. *Review of Economic Dynamics*, **15**(2), 226–243.

Lockwood, L.M. (2018) Incidental bequests and the choice to self-insure late-life risks. *American Economic Review*, **108**(9), 2513–2550.

Maurer, R., Mitchell, O.S., Rogalla, R. and Kartashov, V. (2013) Lifecycle portfolio choice with systematic longevity risk and variable investment-linked deferred annuities. *Journal of Risk and Insurance*, **80**(3), 649–676.

Milevsky, M.A. and Young, V.R. (2007) Annuitization and asset allocation. *Journal of Economic Dynamics and Control*, **31**(9), 3138–3177.

Munk, C. and Sørensen, C. (2010) Dynamic asset allocation with stochastic income and interest rates. *Journal of Financial Economics*, **96**(3), 433–462.

Owadally, I., Jang, C. and Clare, A. (2021) Optimal investment for a retirement plan with deferred annuities. *Insurance: Mathematics and Economics*, **98**, 51–62.

Pang, G. and Warshawsky, M. (2010) Optimizing the equity-bond-annuity portfolio in retirement: The impact of uncertain health expenses. *Insurance: Mathematics and Economics*, **46**(1), 198–209.

Peijnenburg, K., Nijman, T. and Werker, B.J. (2017) Health cost risk: A potential solution to the annuity puzzle. *Economic Journal*, **127**(603), 1598–1625.

Piazzesi, M. (2010) Affine term structure models. In *Handbook of Financial Econometrics: Tools and Techniques*, 691–766. Elsevier.

Polkovnichenko, V. (2007) Life-cycle portfolio choice with additive habit formation preferences and uninsurable labor income risk. *Review of Financial Studies*, **20**(1), 83–124.

Reichling, F. and Smetters, K. (2015) Optimal annuitization with stochastic mortality and correlated medical costs. *American Economic Review*, **105**(11), 3273–3320.

Vasicek, O. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics*, **5**(2), 177–188.

Yaari, M.E. (1965) Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies*, **32**(2), 137–150.

Zeldes, S.P. (1989) Optimal consumption with stochastic income: Deviations from certainty equivalence. *Quarterly Journal of Economics*, **104**(2), 275–298.

https://doi.org/10.1017/asb.2024.23 Published online by Cambridge University Press