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SPECTRAL SUBSPACES OF OPERATOR-VALUED FUNCTIONS

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We give a generalization of the notion of spectral maximal subspaces, for which some of the main results are still valid. We give an application of this theory on a class of operators, defined on some reflexive Banach space.

1. Introduction

Let X be a non-trivial complex Banach space and $B(X)$ the algebra of all bounded operators on X. For any $T \in B(X)$ the operator-valued function

$$R(\lambda) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} T^k ,$$

defined at least for $\lambda \in C\Delta_{r(T)}$, where we have denoted by $r(T)$ the spectral radius of T and by Δ_a the closed disc of radius $a \geq 0$ with centre at the origin of the complex plane. It is clear that $R(\lambda)$ is analytic for $\lambda \in C\Delta_{r(T)}$ and that it commutes with every $A \in B(X)$ which commutes with T. It is well-known that the maximal domain of analyticity of this function is unique and equal to the complement of the union of the spectrum of T and the bounded components of the resolvent set. The

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function $R(\lambda)$ is the key to the definitions of spectra, local spectra, spectral subspaces, and so on. Our aim is to give generalizations of some of these notions, when the function $R(\lambda)$ is replaced with some other analytic operator-valued function. Note that with this change we lose a lot of the fertile theory of spectralness. One of the main losses lies in the following fact. We were in a position to "recognize" the resolvent function $R(\lambda)$ by the condition $(\lambda-T)R(\lambda) = R(\lambda)(\lambda-T) = I$ on the subsets of the plane, disconnected with the maximal analytic domain containing $C\Delta_{r(T)}$. Taking arbitrary operator-valued functions, no such recognition is possible any more, and we must confine ourselves to the case the domain of the function is connected. Furthermore, the maximal domain of analyticity of such a function need not be unique any more.

It is somewhat surprising that in spite of these losses, some of the most important results still hold. Although (the analogues to) the spectrum(or, the local spectrum) need not be uniquely defined, the set of all $x \in X$ which have at least one of its (analogues of) local spectra contained in a fixed, closed subset (with connected complement) of the complex plane, is a (not necessarily closed) linear subspace of X , invariant under every operator, commuting with the function (Proposition 2.4). If this subspace is closed, it has a property, similar to spectral maximality (Theorem 3.2). On the other hand, every subspace with this property is invariant under every operator, commuting with the function (Proposition 3.1).

It is maybe even more surprising that we are in a position to give non-trivial examples of analytic operator-valued functions which have a lot of subspaces with the above mentioned property, similar to spectral maximality. These examples are obtained, when studying a class of operators on some reflexive (abstract) Banach space, which are in a sense close to the Volterra operator.

2. The spectrum of operator-valued function

Let D_A be some connected open subset of the complex plane and let $A : D_A \to B(X)$ be a non-trivial analytic function. It could happen that D_A is a maximal domain of this function. That means: if for some

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connected $D_B \supset D_A$ and analytic $B : D_B \to B(X)$ we have $A(\lambda) = B(\lambda)$ for $\lambda \in D_A$; then $D_B = D_A$. If D_A itself is not maximal, there always exists a connected $\overline{D}_A \supset D_A$ and a function $\overline{A}$ on $\overline{D}_A$ with $\overline{A}(\lambda) = A(\lambda)$ for $\lambda \in D_A$ such that $\overline{D}_A$ is a maximal domain of the function $\overline{A}$. The complement of some maximal domain $\overline{D}_A$ will be called (a possible) spectrum of the analytic function A .

PROPOSITION 2.1. *For any operator* $T \in B(X)$ *the following conditions are equivalent:*

(a) T *commutes with* $A(\lambda)$ *for every* $\lambda \in D_A$ *;*

(b) T *commutes with* $A(\lambda)$ *for every* λ *in some open set, contained in* D_A *;*

(c) T *commutes with every operator* A_k *,* $k = 0, 1, \ldots$ *, where* A_k *are the coefficients of the Taylor expansion*

$$A(\lambda) = \sum_{k=0}^{\infty} A_k (\lambda-\lambda_0)^k$$

of the function A *at some point* $\lambda_0 \in D_A$ *.*

Proof. *(a)* $\Rightarrow$ *(b)*. Clear.

(b) $\Rightarrow$ *(c)*. Let G be such that T commutes with $A(\lambda)$ for every $\lambda \in G$. Then T commutes with $\{A(\lambda)-A(\mu)\}/(\lambda-\mu)$ for any $\lambda, \mu \in G$, $\lambda \neq \mu$. Hence, T commutes with every $A'(\lambda)$, $\lambda \in G$, and by induction T commutes with $A^{(k)}(\lambda)$ for any $\lambda \in G$.

(c) $\Rightarrow$ *(a)*. Let G be the set of points $\lambda_0 \in D_A$ for which *(c)* holds. Since *(c)* is valid, G is non-void. It is clear that G is open. Take now some $\lambda \in D_A$ such that there exists a sequence $\lambda_k \in G$ with $\lambda_k \to \lambda$. As $A(\lambda)$ is analytic, $A^{(j)}(\lambda_k) \to A^{(j)}(\lambda)$ and consequently $\lambda \in G$. The proof now follows by connectedness of D_A .

If an operator $T \in B(X)$ satisfies any one of the three equivalent conditions of Proposition 2.1 we shall say that T commutes with A . The

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set of all $T \in B(X)$, commuting with A, will be denoted by $C(A)$. Note that $C(A)$ is a closed subalgebra of $B(X)$.

EXAMPLE 2.2 (a trivial one). Let $T \in B(X)$ be arbitrary and denote by A the restriction of the resolvent $(\lambda - T)^{-1}$ on some of the components of the resolvent set $\rho(T)$. Then A has the desired properties. Moreover, an operator commutes with T if and only if it is in $C(A)$.

EXAMPLE 2.3. Let $\mathcal{T}$ be a bounded, linear operator on $B(X)$ and denote by $\mathcal{J}$ the identity operator on $B(X)$. Then $\mathcal{R}(\lambda) = (\lambda\mathcal{J}-\mathcal{T})^{-1}$ is an analytic function with values in $B(B(X))$. Choose $T \in B(X)$ and denote $A(\lambda) = \mathcal{R}(\lambda)T$ for λ from the unbounded component of $\rho(\mathcal{T})$. Note that $S \in C(A)$ if and only if S commutes with all $\mathcal{T}^n T$, for $n = 0, 1, \ldots$. To see it, apply Proposition 2.1 to Laurent's expansion

$$A(\lambda) = (\lambda\mathcal{J}-\mathcal{T})^{-1}T = \sum_{n=0}^{\infty} \lambda^{-(n+1)}\mathcal{T}^n T ,$$

valid for all $\lambda \in \mathbb{C}$ with $|\lambda|$ large enough.

For any $x \in X$ we can extend the vector-valued analytic function $A(\lambda)x$ to some function $\hat{x}(\lambda)$, analytic on a maximal connected domain. The complement of this domain will be denoted by $\sigma_A(x)$ and called (a possible) local spectrum of $x \in X$ under A.

For any $F \subset \mathbb{C}$ denote by $X_A(F)$ the set of all $x \in X$ for which there is some $\sigma_A(x) \subset F$.

PROPOSITION 2.4. *If F is closed and has connected complement, then $X_A(F)$ is a linear (not necessarily closed) subspace of X, invariant under every $T \in C(A)$.*

Proof. Choose $x, y \in X_A(F)$ and $\alpha, \beta \in \mathbb{C}$. By the definition we can find possible local spectra $\sigma_A(x) \subset F$ and $\sigma_A(y) \subset F$. Hence there are functions $\hat{x}$ and $\hat{y}$, defined and analytic at least on CF such that $\hat{x}(\lambda) = A(\lambda)x$ and $\hat{y}(\lambda) = A(\lambda)y$ for every $\lambda \in D_A$. Since CF is connected, the function $\alpha\hat{x}(\lambda) + \beta\hat{y}(\lambda)$, $\lambda \in CF$, is an analytic continuation of $A(\lambda)(\alpha x+\beta y)$, $\lambda \in D_A$. Thus there is a possible spectrum

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$\sigma_A(\alpha x+\beta y) \subset F$ and $\alpha x + \beta y \in X_A(F)$. To prove the second assertion, choose $T \in C(A)$, $x \in X_A(F)$ and $\hat{x}$, analytic on CF , with $\hat{x}(\lambda) = A(\lambda)x$ for $\lambda \in D_A$. The function $T\hat{x}$ is then analytic on CF , with $T\hat{x}(\lambda) = A(\lambda)Tx$ for $\lambda \in D_A$, so $Tx \in X_A(F)$.

3. Spectral maximal subspaces

Let Y be some closed subspace of X . For any operator $T \in B(X)$, the restriction T/Y is a bounded operator from the Banach space Y to X . The space of all such operators will be denoted by $B(Y, X)$. Take now any operator-valued analytic function A with domain D_A . The restriction A/Y of A to Y will be defined by $(A/Y)(\lambda) = A(\lambda)/Y$, for $\lambda \in D_A$. Note that A/Y maps D_A analytically into $B(Y, X)$. The complement of any maximal domain of analyticity of A/Y will be denoted by $\sigma(A/Y)$ and called (a possible) spectrum of the restriction A/Y . Note that every spectrum of the restriction is always contained in some spectrum of the function and that to every spectrum of the function there is a spectrum of the restriction, contained in it.

A closed subspace Y of X is a spectral maximal subspace under the function A , if there is some fixed spectrum $\sigma(A/Y)$ of the restriction A/Y such that for every closed subspace Z of X , the validity of inclusion $\sigma(A/Z) \subset \sigma(A/Y)$ at least for one spectrum $\sigma(A/Z)$ of the restriction A/Z , implies $Z \subset Y$.

Note that Y is spectral maximal with respect to some fixed spectrum of the restriction A/Y and this condition need not be satisfied any more, when this particular spectrum is replaced by some other.

PROPOSITION 3.1. *Every spectral maximal subspace is invariant under every* $T \in C(A)$.

Proof. Fix a spectral maximal subspace Y of A , whose spectral maximality is recognized by a fixed spectrum $\sigma(A/Y)$ of the restriction, and choose some $T \in C(A)$. With no loss of generality we may and will assume that T has bounded, everywhere defined inverse. The space $Z = TY$ is then closed by the closed mapping theorem. Let B be the function, analytic on the complement of some fixed spectrum $\sigma(A/Y)$ of the

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restriction, with values in $B(Y, X)$ and such that $B(\lambda) = (A/Y)(\lambda)$ for $\lambda \in D_A$. Then $C(\lambda) = TB(\lambda)(T^{-1}/Z)$ is analytic for $\lambda \in D_A$, with values in $B(Z, X)$. But for $\lambda \in D_A$ and $z \in Z$ we have

$$C(\lambda)z = TB(\lambda)T^{-1}z = TA(\lambda)T^{-1}z = A(\lambda)z ,$$

since $T^{-1}z \in Y$ and $A(\lambda)$ commutes with T. Hence $C(\lambda)$ is a continuation of A/Z and there must be a spectrum $\sigma(A/Z)$ which is contained in the given spectrum $\sigma(A/Y)$ of the restriction. Consequently, $Z \subset Y$ and Y is invariant under T.

THEOREM 3.2. *Let $F \subset \mathbb{C}$ be a closed set with connected complement. If $Y = X_A(F)$ is closed, then there is some spectrum of the restriction A/Y such that $\sigma(A/Y)$ is contained in F, and Y is spectral maximal with respect to every spectrum with this property.*

Proof. Suppose for the moment that we have already found a spectrum $\sigma(A/Y) \subset F$ and take any closed subspace Z of X for which we can find a spectrum $\sigma(A/Z) \subset \sigma(A/Y)$. Let $C(\lambda)$ be the continuation of A/Z, analytic on the complement of $\sigma(A/Z)$. Then for every $z \in Z$, the function $\hat{z}(\lambda) = C(\lambda)z$ is vector-valued analytic on $C\sigma(A/Z) \supset CF$, with $\hat{z}(\lambda) = A(\lambda)z$, for $\lambda \in D_A$. Hence there is some $\sigma_A(z) \subset F$ and $z \in Y$.

It remains to show that there really exists a spectrum of the restriction A/Y, contained in F. For every $y \in Y$, there is by definition a continuation $\hat{y}(\lambda)$, $\lambda \in CF$, of the function $A(\lambda)y$, $\lambda \in D_A$. As CF is connected, this function is unique on CF. For every $\lambda \in CF$ and $k = 0, 1, \ldots$, define a mapping $B^{(k)}(\lambda)$ from Y into X by $B^{(k)}(\lambda)y = \hat{y}^{(k)}(\lambda)$, $y \in Y$, where we have denoted by $\hat{y}^{(k)}$ the kth derivative of the function $\hat{y}$. It is a simple consequence of connectedness that the mappings $B^{(k)}(\lambda)$ are linear.

Now let G be the set of those $\lambda \in CF$ for which $B^{(k)}(\lambda)$ are bounded for every $k = 0, 1, \ldots$. Note that G contains D_A and is necessarily non-void. Taking account of the connectedness of CF it suffices to show that G is open and closed in the relative topology of

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CF, to get $G = CF$. It will then follows that $B^{(0)}(\lambda)$ is a continuation of A/Y, analytic on CF and finally, we shall get the existence of some $\sigma(A/Y) \subset F$.

To show that G is closed relatively in CF, let $\lambda_n \in G$ converges to some $\lambda_0 \in CF$ and fix some $k = 0, 1, \ldots$. For every $y \in Y$ the sequence $B^{(k)}(\lambda_n)y$ converges by the analyticity of $\hat{y}$ to $\hat{y}^{(k)}(\lambda_0) = B^{(k)}(\lambda_0)y$. By the principle of uniform boundedness, the operators $B^{(k)}(\lambda_0)$ are bounded and $\lambda_0 \in G$.

To see that G is open, take any $\lambda_0 \in G$ and choose $r > 0$ such that the closed disc of radius r with centre at λ_0 lies in CF. The sequence of bounded operators $(k!)^{-1}r^kB^{(k)}(\lambda_0)$ then converges strongly to zero. Applying again the principle of uniform boundedness, we see that this sequence of operators has uniformly bounded norms. Hence the series

$$C(\lambda) = \sum_{k=0}^{\infty} \frac{(\lambda-\lambda_0)^k}{k!} B^{(k)}(\lambda_0)$$

converges at least for $\lambda \in \Delta_r(\lambda_0)$, the open disc of radius r with centre at λ_0. Since evidently $C(\lambda) = B(\lambda)$ for $\lambda \in \Delta_r(\lambda_0)$, we have $\Delta_r(\lambda_0) \subset G$ and the theorem holds.

In the next example we shall use the following notation and results. For $T \in B(X)$ denote by $\mathcal{D}_T \in B(B(X))$ the inner derivative $\mathcal{D}_T S = TS - ST$, $S \in B(X)$. An operator $V \in B(X)$ is called T-Volterra element of $B(X)$ (see [*1*]), if $\mathcal{D}_T V = V^2$. For any T-Volterra V, it is true that $\mathcal{D}_T^n V = n!V^{n+1}$. It is also well-known that $\sigma(\mathcal{D}_T) = \sigma(T) - \sigma(T)$ (see [*2*]).

EXAMPLE 3.3. Let V be some T-Volterra element of $B(X)$, and $A(\lambda) = (\lambda J - \mathcal{D}_T)^{-1}V$, for $\lambda \in \rho(\mathcal{D}_T)$. Then

(a) for λ, such that $|\lambda| > r(\mathcal{D}_T)$,

$$(\lambda J-\mathcal{D}_T)^{-1}V = \sum_{k=0}^{\infty} \lambda^{-(k+1)}k!V^{k+1} ,$$

(b) $S \in C(A)$ if and only if S commutes with V,

(c) if Y is invariant under both T and V, then there exists some $\sigma(A/Y) \subset \Delta_{\mathrm{diam}\sigma(T/Y)}$.

Proof. (a) is clear and (b) is clear by (a).

(c) There exists some $\sigma(A/Y)$ contained in the union of $\sigma(\mathcal{D}_{T/Y})$ and the bounded components of it. Since $\sigma(\mathcal{D}_{T/Y}) = \sigma(T/Y) - \sigma(T/Y)$, we thus have

$$\sup\{|\lambda| ; \lambda \in \sigma(A/Y)\} \le \mathrm{diam}\, \sigma(T/Y) .$$

4. On a class of operators

In this section, let X be some reflexive Banach space, $\mathcal{B}$ some σ-algebra of subsets of some set G, and $E : \mathcal{B} \to B(X)$ a countably additive spectral measure. Furthermore, choose $e \in X$, $e^* \in X^*$, the dual space of X, with $e^*e = 1$. Then $W = ee^*$ is a bounded projection of rank one on X.

LEMMA 4.1. *There exists a unique operator-valued, bounded, countably additive (at least in the weak operator topology) measure* $F : \mathcal{B} \times \mathcal{B} \to B(X)$ *with* $F(B \times A) = E(A)WE(B)$, *for any* $A, B \in \mathcal{B}$.

Proof. For any $x \in X$, $x^* \in X^*$, define a scalar measure $\mu_{x,x^*}(B) = x^*E(B)x$, $B \in \mathcal{B}$. It is well-known that in this situation the total variation satisfies

$$|\mu_{x,x^*}|(G) \le 4K\|x\|\|x^*\| ,$$

where $K = \sup\|E(B)\|$. For $A, B \in \mathcal{B}$ define

$$\text{(1)} \qquad \nu_{x,x^*}(B \times A) = x^*F(B \times A)x = x^*E(A)ee^*E(B)x$$
$$= (\mu_{e,x^*} \times \mu_{x,e^*})(A \times B) .$$

Hence, the product measure $\mu_{e,x^*} \times \mu_{x,e^*}$ is the unique scalar measure on

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$\mathcal{B}\times\mathcal{B}$ for which (1) holds on rectangles. As a simple consequence of uniqueness we get that $\nu_{x,x^*}(D)$ is linear in x and in x^* for every $D\in\mathcal{B}\times\mathcal{B}$. Beside this, the total variation of the product measure is equal to the product of total variations. Hence

$$|\nu_{x,x^*}(D)| \le 16K^2\|x\|\|x^*\|\|e\|\|e^*\| ,$$

and for every $D\in\mathcal{B}\times\mathcal{B}$, there is a unique, bounded, linear operator, which we denote by $F(D)$, such that $x^*F(D)x = \nu_{x,x^*}(D)$. Note that $F(D)$ must be uniformly bounded and countably additive at least in the weak operator topology.

Note that we could formally write $F(D)$ as a double integral $F(D) = \iint_{(s,t)\in D} dE(t)WdE(s)$, which exists in a "weak" sense, precisely described above.

From now on let V be a countable chain of measurable sets in G such that $G=\cup V$. For any $s, t\in G$ define $s\le t$ if and only if for every $A\in V$ from $t\in A$ follows $s\in A$. For any $t\in G$ denote

$$G_t^+ = \cap\{A\in V;\ t\in A\} ,\quad G_t^- = \cup\{A\in V;\ t\notin A\} .$$

Note that $G_t^-\subset G_t^+$ are necessarily measurable. We will suppose that

(2) $$|\mu_{e,e^*}|\left(G_t^+-G_t^-\right) = 0 .$$

For any $t\in G$ define now $\varphi(t) = e^*E\left(G_t^+\right)e = e^*E\left(G_t^-\right)e$.

Note that condition (2), together with $e^*e = 1$ and $E(G) = I$, automatically excludes the trivial possibility $V = \{G\}$.

LEMMA 4.2. *(a) The function $\varphi : G\to\mathbb{C}$ is measurable and bounded.*

(b) The set $D_0 = \{(t, s)\in G\times G;\ s\le t\}$ is $\mathcal{B}\times\mathcal{B}$ measurable.

(c) $G_t^+ = \{s\in G;\ s\le t\}$, $G-G_t^- = \{s\in G;\ t\le s\}$.

Proof. *(a)* Denote $V = \{A_1, A_2, \ldots\}$. For any positive integer n, reorder $A_1, A_2, \ldots, A_n$ into $B_1\subset B_2\subset\ldots\subset B_n$ and set $B_0=\emptyset$, $B_{n+1}=G$. For $t\in B_{k+1}-B_k$ define $\varphi_n(t) = \mu_{e,e^*}\left(B_k\right)$, for

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$k = 0, 1, \ldots, n$. Let A_{n_k} be some decreasing and A_{m_k} some increasing subsequence of A_k with

$$G_t^+ = \cap A_{n_k}\ , \quad G_t^- = \cup A_{m_k}\ .$$

Then for every $n > \max(n_k, m_k)$,

$$|\varphi_n(t)-\mu_{e,e^*}(A_{m_k})| \le |\mu_{e,e^*}|(A_{n_k}-A_{m_k})\ .$$

By (2) the right-hand side of this inequality converges to zero; hence $\varphi_n(t)$ converges to $\mu_{e,e^*}(G_t^-) = \varphi(t)$.

(b) For every n set $B_0 \subset B_1 \subset \ldots \subset B_n \subset B_{n+1}$ as above and define

$$D_n = \bigcup_{j=1}^{n+1} (B_j-B_{j-1}) \times B_j\ .$$

At first, assume $(t, s) \in D_0$, and $t \in B_j - B_{j-1}$. By the definition of the relation $s \le t$ we have $s \in B_j$, hence $(t, s) \in D_n$, and consequently $D_0 \subset \cap D_n$. Suppose now that $(t, s) \notin D_0$. By the definition there is some $A \in V$ with $t \in A$ and $s \notin A$. For n large enough we can find some B_j with $t \in B_j - B_{j-1}$, and $s \notin B_j$. Thus $(t, s) \notin D_n$ and consequently $D_0 = \cap D_n$.

The proof of *(c)* is pure verification and will be omitted.

Define now

$$T = \int \varphi(t)dE(t)$$

and

$$V = F(D_0)\ .$$

LEMMA 4.3. *(a) V is T-Volterra.*

(b) $WV = W(I-T)$.

(c) $W(I-T)^n = n!WV^n$.

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Proof. *(a)* Note that

$$V = \iint_{s \le t} dE(t)WdE(s) = \int dE(t)WE\left(G_t^+\right)$$
$$= \int E\left(G-G_s^-\right)WdE(s) \ .$$

The first of the two iterated integrals can be understood, when multiplied by some $x^* \in X^*$ from the left, as an integral of some bounded, measurable, X^*-valued function by the scalar measure μ_{e,x^*}, while the second one, when multiplied by some $x \in X$ from the right, is an integral of some X-valued function by the scalar measure μ_{x,e^*}.

Hence the following computations are valid, if we interpret the integrals in the obvious sense:

$$V^2 = \iint dE(t)WE\left(G_t^+\right)E\left(G-G_s^-\right)WdE(s)$$
$$= \iint_{s \le t} dE(t)ee^*E\left(G_t^+-G_s^-\right)ee^*dE(s)$$
$$= \iint_{s \le t} [\varphi(t)-\varphi(s)]dE(t)WdE(s) = TV - VT \ .$$

(b) Since $E(G) = I$ and $e^*e = 1$, we have

$$WV = e\int \left(1-\varphi(s)\right)e^*dE(s) = W(I-T) \ .$$

(c) By the assertion *(b)*, *(c)* is valid for $n = 1$. Suppose it holds for some n. Then

$$W(I-T)^{n+1} = n!WV^n(I-T) = n!W(I-T)V^n + n!W\left(TV^n-V^nT\right)$$
$$= n!\left(WV^{n+1}+nWV^{n+1}\right) = (n+1)!WV^{n+1} \ .$$

From now on assume that the measure μ_{e,e^*} is positive. As in Example 3.3, define $\mathcal{D}_T : B(X) \to B(X)$ by $\mathcal{D}_T : S \mapsto TS - ST$ and $A(\lambda) = \left(\lambda J-\mathcal{D}_T\right)^{-1}V$.

THEOREM 4.4. *For every* $a \in [0, 1]$,

$$X_A\left(\Delta_a\right) = \cap \mathrm{Ker}\left(e^*E\left(G_t^+\right)V^2\right) ,$$

where the intersection on the right-hand side is taken over all $t \in G$

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with $\varphi(t) < 1 - a$.

Proof. Assume that for some $y \in X$ we have $e^*E(G_t^+)y = 0$, for any $t \in G$ with $\varphi(t) < 1 - a$, then

$$\begin{aligned} n!V^{n+1}y &= \left(\mathcal{D}_T^n V\right)y \\ &= \iint_{D_0} \left(\varphi(t)-\varphi(s)\right)^n \chi_a(s)\,dE(t)W\,dE(s)y \ , \end{aligned}$$

where we have denoted

$$\chi_a(s) = \begin{cases} 1 \ , & \varphi(s) \ge 1 - a \ , \\ 0 \ , & \varphi(s) < 1 - a \ . \end{cases}$$

Hence, for some constant K , independent of n and y , we have $\|n!V^{n+1}y\| \le Ka^n\|y\|$. If $x \in \operatorname{Ker}\left[e^*E(G_t^+)V^2\right]$ for every $t \in G$ with $\varphi(t) < 1 - a$, then $y = V^2x$ satisfies the above assumptions, and $\|(n+2)!V^{n+3}x\| \le (n+2)(n+1)K\|V^2\|a^n\|x\|$. Note Laurent's expansion of $A(\lambda)$, as given in Example 3.3 (a) to see that there is a possible local spectrum of $x \in X$ under A , contained in Δ_a .

On the other hand suppose that for some $x \in X$, $A(\lambda)x$ has an analytic continuation to $C\Delta_a$. Note that for λ with $|\lambda|$ large enough $(\lambda-T)^{-1} = -\sum_{k=0}^{\infty} (1-\lambda)^{-(k+1)}(I-T)^k$; hence by Lemma 4.3 *(c)* and Example 3.3 (a) again

$$\begin{aligned} W(\lambda-T)^{-1}V &= -\sum_{k=0}^{\infty} (1-\lambda)^{-(k+1)}W(I-T)^kV \\ &= -WA(1-\lambda) \ . \end{aligned}$$

Thus, for $y = Vx$, the vector-valued function

$$\begin{aligned} \hat{x}(\lambda) &= -WA(1-\lambda)x = W(\lambda-T)^{-1}y \\ &= ee^*\int \left(\lambda-\varphi(t)\right)^{-1}dE(t)y = e\int \left(\lambda-\varphi(t)\right)^{-1}d\mu_{y,e^*}(t) \end{aligned}$$

has an analytic continuation to any Δ_r with $r < 1 - a$.

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Choose $t \in G$ such that $\varphi(t) = r < 1 - a$ and let Γ be the circular path of radius r with centre at the origin, surrounding the origin in the positive direction. By Fubini's theorem and Cauchy's formula, we have

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\int_\Gamma \big(\lambda-\varphi(t)\big)\hat{x}(\lambda)d\lambda \\
&= e\int d\mu_{y,e^*}(s)\ \frac{1}{2\pi i}\int \frac{\lambda-\varphi(t)}{\lambda-\varphi(s)}\,d\lambda \\
&= -e\int_{s\le t} \big(\varphi(t)-\varphi(s)\big)d\mu_{y,e^*}(s) \\
&= -ee^*E\big(G_t^+\big)\int E\big(G-G_s^-\big)WdE(s)y \\
&= -ee^*E\big(G_t^+\big)Vy = -ee^*E\big(G_t^+\big)V^2x\ .
\end{aligned}$$

COROLLARY 4.5. *For every* $a \in [0, 1]$ *the subspace* $X_A\big(\Delta_a\big)$ *is closed and hence it is spectral maximal.*

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