Proceedings of the Edinburgh Mathematical Society (1976-77), 20, 187-197 ©

COMPLETELY REDUCIBLE NEAR-RINGS

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(Received 11th March 1975)

To establish our notation N will always denote a (left) near-ring without any type of multiplicative identity (unless the contrary is stated) satisfying $0n = 0$ for each $n \in N$ where 0 is the additive identity of N. A group M, written additively, which admits N as a set of right multipliers is a (right) N-*module* if $a \in M$, n_1, $n_2 \in N$ implies $a(n_1 + n_2) = an_1 + an_2$ and $a(n_1n_2) = (an_1)n_2$. When N has a two-sided identity, 1, we suppose that $a1 = a$ for each $a \in M$. A subgroup X of M is an N-*subgroup* of M if it is an N-module; X is a *submodule* of M if it is a normal subgroup of M and $a \in M, x \in X, n \in N$ implies $(a + x)n - an \in X$. We denote by $SL(M)$ the set of N-subgroups and by $L(M)$ the set of submodules of M. Since N may be regarded as an N-module we can talk about N-subgroups and submodules of N although we usually call the submodules of N *right ideals of* N. Other definitions can be found in (6).

An N-subgroup A of M is *semicomplemented* if there exists $B \in L(M)$ with $A \cap B = (0)$, $A + B = M$; B is called a *semicomplement* of A. If each $A \in SL(M)$ is semicomplemented then $SL(M)$ is said to be semicomplemented. An N-subgroup A of M is *module-essential* if whenever B is a non-zero submodule of M then $A \cap B \neq (0)$.

A submodule A of M is *minimal* if it contains no N-subgroups other than (0) and A. If M is a direct sum of minimal submodules then M is *completely reducible*. The near-ring N is completely reducible if it is completely reducible as an N-module.

In (7; Theorem 3) we proved

Theorem 1. *For an N-module M the following are equivalent:-*
(i) *M is completely reducible;*
(ii) *M has no proper module-essential N-subgroups;*
(iii) *$SL(M)$ is semicomplemented.*

Proposition 1. *If N has a left identity the following are equivalent:–*
(i) *N is completely reducible;*
(ii) *each maximal N-subgroup of N is semicomplemented;*
(iii) *each proper N-subgroup of N is contained in a semicomplemented proper N-subgroup of N.*

Proof. Clearly (i)⇒(ii)⇒(iii). Suppose (iii) and let A be a proper

https://doi.org/10.1017/S0013091500026250 Published online by Cambridge University Press

module-essential N-subgroup of N. Then A is contained in a proper N-subgroup of N which is semicomplemented. This contradiction and Theorem 1 establishes that (iii)⇒(i).

Completely reducible near-rings with left identity clearly have the minimum condition on N-subgroups. Retaining the chain condition but not the left identity we can prove

Proposition 2. *If N has the minimum condition on N-subgroups the following are equivalent:-*

(i) *N is completely reducible;*

(ii) *each non-zero N-subgroup of N contains a non-zero semicomplemented N-subgroup of N.*

Proof. That (i)⇒(ii) is trivial. Suppose (ii) and let X be an N-subgroup of N which is not semicomplemented. Let $T \subset X$ be a non-zero N-subgroup of N semicomplemented by $A \in L(N)$. Then $X = T + X_1$ where $X_1 = A \cap X$ and $T \cap X_1 = (0)$. If X_1 is semicomplemented by $Y \in L(N)$ and if $u \in X \cap (A \cap Y)$ then $u \in (T + X_1) \cap A \cap Y$ so that

$$u = t + x_1 = a \quad (t \in T,\ x_1 \in X_1,\ a \in A \cap Y).$$

Thus $t = a - x_1 \in T \cap A = (0)$ and $u = x_1 \in X_1 \cap A \cap Y = (0)$. Then $X \cap (A \cap Y) = (0)$. Now let $z \in N = T + A$ so that $z = t + a$ $(t \in T,\ a \in A)$. Since $N = X_1 + Y$, $a = x_1 + y$ $(x_1 \in X_1,\ y \in Y)$ and

$$z = t + x_1 + y \in (T + X_1) + (A \cap Y) = X + A \cap Y.$$

It follows that X is semicomplemented if X_1 is semicomplemented. If X_1 is not semicomplemented we can apply the same construction to X_1 to obtain X_2 and then X_3 etc with $\ldots \subsetneq X_n \subsetneq \ldots \subsetneq X_1$ contrary to the minimum condition for N-subgroups. It follows that (ii)⇒(i).

In (7; Theorem 4) we gave a proof of

Proposition 3. *If N is a near-ring with left identity the following are equivalent:-*

(i) *N is completely reducible;*

(ii) *N has no nilpotent N-subgroups and has the minimum condition on N-subgroups.*

Later we will give an alternative proof of this result. A near-ring N is *regular* if for each $r \in N$ there exists $s \in N$ with $r = rsr$. It is easy to see

Lemma 1. *If N is a near-ring with identity the following are equivalent:–*

(i) *N is regular;*

(ii) *for each $a \in N$ there is a non-zero idempotent $e \in N$ with $aN = eN$;*

(iii) *for each $a \in N$ there is a right ideal B of N with $aN \cap B = (0)$ and $aN + B = N$.*

We observe that (i)$\Rightarrow$(ii)$\Rightarrow$(iii) irrespective of whether N has an identity. Furthermore if we assume that N has minimum condition on N-subgroups we can use Proposition 2 to prove

Corollary 1. *If N has minimum condition on N-subgroups and if N is regular then N is completely reducible.*

Later we will consider the converse of this. For the present we observe

Proposition 4. *If N is a near-ring with identity the following are equivalent*:-

(i) *N is completely reducible*;

(ii) *N has the minimum condition on N-subgroups and is regular.*

If R is a ring then R is completely reducible if and only if every R-module is completely reducible. We are unable to prove this for near-rings. However, calling an N-module M *monogenic* if $M = mN$ for some $m \in M$ we have

Proposition 5. *If N is a near-ring with left identity then N is completely reducible if and only if every monogenic N-module is completely reducible.*

Proof. Clearly if every monogenic N-module is completely reducible so is N. For the converse let $M = mN$ with $m \in M$. For $I \in SL(M)$, $T = \{n \in N: mn \in I\} \in SL(N)$ and $I = mT$. Let $P \in L(N)$ with $T \cap P = (0)$, $T + P = N$. Then $M = mT + mP = I + mP$, $I \cap mP = (0)$, $mP \in L(M)$.

An N-subgroup A of a module M is *essential* if $A \cap B \neq (0)$ whenever B is a non-zero N-subgroup of M. Then

Corollary 2. *If N has an identity and is completely reducible and if M is an N-module then M has no essential N-subgroups.*

Proof. Let $A \in SL(M)$ be essential and $x \in M$ with $x \neq 0$. From Proposition 5, xN is completely reducible. Let $K \in L(xN)$ with $xN \cap A \cap K = (0)$, $xN \cap A + K = xN$. But $xN \cap K = K$ so $A \cap K = (0)$ and $K \in SL(M)$ so $K = (0)$. Thus $xN \cap A = xN$ and $x \in A$. But then $M \subseteq A$.

Let M be a completely reducible N-module with $M = \oplus_\Lambda M_\lambda$ where M_λ is a minimal submodule of M and P be any minimal N-subgroup of M. Denote by $\{\Pi_\alpha: M \to M_\alpha\}$ the family of natural projections and by θ_α the restriction of Π_α to P. Clearly $\theta_\alpha = 0$ or θ_α is an N-isomorphism. For each minimal N-subgroup P of M let $H(P)$ denote the sum of all those submodules of M which are isomorphic as N-modules to P. $H(P)$ is the *homogeneous component* of P and clearly

Proposition 6. *If M is completely reducible then $M = \oplus H(P)$ where P ranges over all the minimal N-subgroups of M.*

We notice that we can define homogeneous components for general modules in just the same way. If P is a minimal N-subgroup of M then

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$P \subset H(P)$ if M is completely reducible. However this need not be so if M is not completely reducible; for example the symmetric group S_3 on 3 elements is a $(Z, 1)$-module (notation in Fröhlich (5)), where Z is the set of integers, in which the subset $P = \{e, \alpha\}$ with $\alpha^2 = e$ is a minimal $(Z, 1)$-subgroup for which $H(P) = (0)$.

Lemma 2. *If F is a homogeneous component in a completely reducible near-ring N then F is an ideal.*

Proof. Clearly $\alpha \in \mathrm{Hom}_N(N, N)$ implies $\alpha F \subseteq F$ in a completely reducible near-ring. For $x \in N$ define $\alpha_x \in \mathrm{Hom}_N(N, N)$ by $\alpha_x(n) = xn$.

For an N-module M we denote by $\mathrm{Soc}\,(M)$ the sum of all the minimal submodules of M. As before it is not necessary for $\mathrm{Soc}\,(M)$ to contain all the minimal N-subgroups of M. Trivially $\mathrm{Soc}\,(M) = M$ if and only if M is completely reducible. If M is not completely reducible denote by T the intersection of all the module essential N-subgroups of M.

We shall, on several occasions, use

Lemma 3. *If M is an N-module and $A \in SL(M)$ there exists $B \in L(M)$ with $A \cap B = (0)$ and $A + B$ module-essential in M.*

Proof. The family of submodules of M having trivial intersection with A is non-empty since it contains (0). For any chain $B_1 \subseteq B_2 \subseteq \cdots$ of submodules of M with $A \cap B_i = (0)$ for each i we see that $A \cap (\cup B_i) = (0)$. Hence by Zorn's Lemma there is a maximal submodule B of M with $A \cap B = (0)$. Clearly if $X \in L(M)$ with $(A + B) \cap X = (0)$ then

$$A \cap (B + X) = (0)$$

and since $B + X \in L(M)$ this contradicts the maximality of B.

Proposition 7. *T is completely reducible as an N-module.*

Proof. If $X \in SL(T)$ then $X \in SL(M)$ and by Lemma 3 we can choose $Q \in L(M)$ maximal subject to $X \cap Q = (0)$. Then $X + Q$ is module essential so $T \subseteq X + Q$ and $T = X + T \cap Q$ where $T \cap Q \in L(T)$.

For P a minimal submodule of M we have $P = T \cap P$ so $\mathrm{Soc}\,(M) \subseteq T$.

Proposition 8. *If T is a submodule of M then $\mathrm{Soc}\,(M) = T$.*

Proof. Let $p \in T \backslash \mathrm{Soc}\,(M)$ and $Q \in L(M)$ be maximal subject to the two conditions $\mathrm{Soc}\,(M) \subseteq Q$ and $p \notin Q$. $Q_1 = Q \cap T \in L(M)$. Using Lemma 3 let $A \in L(M)$ with $Q_1 \cap A = (0)$, $Q_1 + A$ module essential in M. Then $T = Q_1 + A_1$ where $A_1 = A \cap T$. If $X \neq (0)$ and $X \in SL(A_1)$ then $X \in SL(M)$ so there exists $B \in L(M)$ with $X \cap B = (0)$, $X + B \cap A_1 = A_1$. A_1, $B \in L(M)$ so $B \cap A_1 = B_1 \in L(M)$. If $X \neq A_1$ then $B_1 \neq (0)$ so for some $C_1 \in L(M)$, $B_1 \cap C_1 = (0)$, $B_1 + C_1 = A_1$. Now $X \neq (0)$ implies $B_1 \neq (0) \neq C_1$. Clearly $B_1 \cap Q = C_1 \cap Q = (0)$ and $p \in (Q + B_1) \cap (Q + C_1)$. Writing $p = q_1 + b = q_2 + c$ then $-q_2 + q_1 \in (B_1 + C_1) \cap Q = A_1 \cap Q = (0)$ and $b = c = 0$

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contrary to $p \not\in Q$. Hence $X = A_1$ and A_1 is a minimal submodule of M from which $A_1 \subseteq \text{Soc}(M) \subseteq Q$ and $A_1 = (0)$ since $Q \cap A_1 = (0)$. Then $T = Q_1 = T \cap Q$ so $T \subset Q$ contrary to $p \not\in Q$. It follows that $T = \text{Soc}(M)$.

Whether T is always a submodule of M is unknown.

Proposition 9. *If T is not a submodule of M then Soc (M) is the largest submodule of M contained in T.*

Proof. Let $Q \in L(M)$ with $\text{Soc}(M) \subsetneq Q \subseteq T$ and $q \in Q \backslash \text{Soc}(M)$. If $A \in L(M)$ is maximal subject to the two conditions $q \not\in A$ and $\text{Soc}(M) \subseteq A$ then $A_1 = A \cap Q \in L(M)$. Let $B \in L(M)$ with $A_1 \cap B = (0)$, $A_1 + B$ module essential in M. Then $B_1 = B \cap Q \in L(M)$. As in the proof of Proposition 8 we can show that B_1 is minimal leading to $B_1 = (0)$ and $Q = \text{Soc}(M)$.

In Proposition 3 we have seen that if N has a left identity then the property of being completely reducible is equivalent to having minimum condition on N-subgroups and no nilpotent N-subgroups. To drop the requirements of a left identity and minimum condition we recall some results on radicals for near-rings.

If Γ is a near-ring module then Γ is

type 2: if Γ has no proper N-subgroups and $\Gamma N \neq (0)$;
type 1: if Γ has no proper submodules, $\Gamma N \neq (0)$ and $\gamma \in \Gamma$ implies $\gamma N = (0)$ or $\gamma N = \Gamma$.
type 0: if Γ has no proper submodules and $\gamma N = \Gamma$ for some $\gamma \in \Gamma$.

We define

$$J_i(N) = \bigcap \{r_N(\Gamma): \Gamma \text{ is a type } i\ N\text{-module}\}$$

where $r_N(\Gamma) = \{n \in N: \Gamma_n = (0)\}$. If Γ has no type i N-modules we put $J_i(N) = N$. A right ideal I of N is *modular* if there exists $a \in N$ with $x - ax \in I$ whenever $x \in N$. $D(N)$ is the intersection of all the modular maximal right ideals of N with $D(N) = N$ if N has no modular maximal right ideals. It is known that $J_0(N) \subseteq D(N) \subseteq J_1(N) \subseteq J_2(N)$. Furthermore $J_0(N)$ contains all the nilpotent ideals of N, $J_2(N)$ all the nilpotent N-subgroups.

If A is a minimal non-nilpotent N-subgroup of N then $A = eN$ for some idempotent $e \in A$. Let $A \in SL(N)$ be non-nilpotent and $A \subseteq J_2(N)$. $N = eN + r(e)$ and $n - en \in r(e)$ for each $n \in N$. Thus $r(e)$ is modular and $N/r(e) \cong eN$. Since eN is type 2 we say that $r(e)$ is 2-primitive. Betsch (**1**; Satz 3.2) proved that $J_2(N)$ is the intersection of the 2-primitive right ideals of N so $J_2(N) \subseteq r(e)$ contrary to $e^2N \neq (0)$.

Theorem 2. *If N is completely reducible then $J_2(N)$ is the sum of all the nilpotent right ideals of N and $J_2(N)^2 = (0)$.*

Proof. $J_2(N)$ is the sum of all the minimal right ideals of N which it contains and we have seen that each of these is nilpotent. Clearly if A is a nilpotent minimal right ideal then $A^2 = (0)$. Let A_1, A_2 be nilpotent minimal

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right ideals of N. If $A_1 \not\cong A_2$ then $H(A_1) \cap H(A_2) = (0)$ and so $H(A_1)H(A_2) = (0)$ and $A_1A_2 = (0)$. If $A_1 \cong A_2$ let ϕ be the isomorphism and $a_1 \in A_1$, $a_2 \in A_2$, $a_2^* \in A_2$ with $\phi(a_2^*) = a_1$. Then $a_1a_2 = \phi(a_2^*)a_2 = \phi(a_2^*a_2) = \phi(0) = 0$. It follows that $A_1A_2 = (0)$ and $J_2(N)^2 = (0)$.

Corollary 3. *If N is completely reducible then $J_0(N) = D(N) = J_1(N) = J_2(N)$.*

Proof. $J_2(N)$ is a nilpotent ideal so $J_2(N) \subseteq J_0(N)$.

As a second corollary to this we will obtain a proof of Proposition 3 different from that in (7). An element $x \in N$ is *right quasi-regular* (rqr) if and only if the minimal right ideal of N containing all elements of the form $n - xn$ for each $n \in N$ also contains x. If we denote by L_x the right ideal of N generated by $\{n - xn: n \in N\}$ then x is rqr if and only if $x \in L_x$.

Lemma 4. x *is rqr if and only if $L_x = N$.*

Proof. If $L_x = N$ then $x \in L_x$. Conversely if x is rqr then $x \in L_x$ so for $s \in N$, $s = (s - xs) + xs \in L_x$ and $N = L_x$.

A right ideal of N is *quasi-regular* in case each of its elements is rqr. By Ramakotoiah (8; 2.2) $D(N)$ is quasi-regular and contains all the quasi-regular right ideals of N. A right ideal, A, of N is *small* if and only if whenever $B \in L(N)$ with $A + B = N$ then $B = N$.

Lemma 5. *If I is a right ideal of N and N has a left identity e then I is small if and only if $I \subseteq D(N)$.*

Proof. Let $I \subseteq D(N)$ and $B \in L(N)$ with $B + I = N$. Then $e = b + i$. Now $D(N)$ is quasi-regular, so i is rqr; so by Lemma 4, $L_i = N$. But $r \in N$ implies $r - ir = (b + i)r - ir \in B$. Thus $L_i \subseteq B$ and $B = N$ as required. Conversely, if I is a small right ideal let $x \in I$. Then $e - xe \in L_x$ so $e = (e - xe) + xe \in L_x + I$. Hence $L_x = N$ and $I \subseteq D(N)$.

This gives an alternative characterisation of $D(N)$.

Corollary 4. *If N has a left identity then $D(N)$ is a small right ideal of N and is the sum of all the small right ideals of N.*

Corollary 5. *For a near-ring N with left identity the following are equivalent*

(i) *N is completely reducible*;

(ii) *N has no nilpotent N-subgroups and satisfies the minimum con- on N-subgroups.*

Proof. (ii) implies (i) is due to Blackett (3). Suppose (i). Then $D(N) = (0)$ so that $J_2(N) = (0)$ and N has no nilpotent N-subgroups. The minimum condition follows immediately from N having a left identity.

Let us now turn to the case where $J_2(N) = (0)$.

Theorem 3. *If N is completely reducible and $J_2(N) = (0)$ then each homogeneous component is a simple near-ring.*

Proof. Let $N = \bigoplus F_\lambda$ where each F_λ is a homogeneous component. For distinct F_1, F_2 we have $F_1F_2 = (0)$ so if X is an ideal of F_1 then X is a right ideal (in fact an ideal) of N. If $X \neq F_1$ let A be a minimal right ideal of N in F_1 with $X \cap A = (0)$. Since $J_2(N) = (0)$, $A = eN$ for some non-zero idempotent $e \in A$. If $X \neq (0)$ let fN be a minimal right ideal of N in X with $f = f^2 \neq 0$. Then $AX \subseteq A \cap X = (0)$ so $X \subseteq r(A)$. The minimal right ideals of N in F_1 are isomorphic and thus $fN \cong eN$. If ϕ is the isomorphism let $\phi(f) = en$. Then $0 = enf = \phi(f)f = \phi(f)$ which is not true. Thus $X = F_1$ or $X = (0)$ and F_1 is a simple near-ring.

Corollary 6. *If N is completely reducible with $J_2(N) = (0)$ and if A is a two-sided N-subgroup of N there is a two-sided ideal X of N with $A \cap X = (0)$ and $A + X = N$.*

Proof. Let X be an ideal of N maximal subject to $A \cap X = (0)$. Write $N = \bigoplus F_\lambda$ where each F_λ is a homogeneous component and thus an ideal of N and simple as a near-ring. Clearly $(A + X) \cap F_\lambda \neq (0)$ for each λ. If B is a minimal N-subgroup of N contained in F_λ and $B(A + X) = (0)$ then $(A + X) \cap F_\lambda \subseteq r(B) \cap F_\lambda$. Now F_λ is simple and thus has no proper two-sided ideals so $r(B) \cap F_\lambda = (0)$ or F_λ. Since $(A + X) \cap F_\lambda \neq (0)$ we must have $r(B) \cap F_\lambda = F_\lambda$ and so $B \subseteq r(B) \cap F_\lambda$. But then $B^2 = (0)$ contrary to $J_2(N) = (0)$. Since $(A + X) \cap B = (0)$ implies $B(A + X) = (0)$ it follows that $B \subseteq A + X$ for each minimal right ideal of N and thus $A + X = N$ as required.

Theorem 4. *If $J_2(N) = (0)$ and $N = \bigoplus N_\lambda$, where each N_λ is an ideal of N, is simple as a near-ring and contains a minimal right ideal then N is completely reducible.*

Proof. If A is the minimal right ideal of N_λ and B is isomorphic to A as an N-module then $B \subseteq N_\lambda$ since $J_2(N) = (0)$. Apply Zorn's Lemma to the family of all sums of right ideals of N_λ which are isomorphic to A to obtain a maximal such sum T. Then T is an ideal of N_λ so $T = N_\lambda$ and N is completely reducible.

We now obtain the structure of two-sided N-subgroups of a completely reducible near-ring with identity.

Lemma 6. *If N has no nilpotent N-subgroups and A is an N-subgroup of N, B a two-sided N-subgroup of N, then $AB = (0)$ if and only if $A \cap B = (0)$.*

Proposition 10. *If N is completely reducible with identity* 1 *and A is a two-sided N-subgroup of N then $A = eN$ where e is a central idempotent.*

Proof. From Theorem 2 and Corollary 5 we get $J_2(N) = (0)$. From

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Corollary 6 there is an ideal X of N with $A \cap X = (0)$, $A + X = N$. Write $1 = e + x$ ($e \in A$, $x \in X$). Then $e - e^2 = (e+x)e - e^2 \in A \cap X = (0)$. Clearly $A = eN$ and e is central.

So far we have not distinguished between rings and near-rings. We now wish to investigate near-rings which are not rings. These we call *nonrings*. An extremely important result (due to Wielandt and reported by Betsch (**2**; 2.12)) is

Lemma 7. *Let N be a near-ring and Γ a faithful N-module with $\Gamma = \gamma N$ for some $\gamma \in \Gamma$. If B, $C \in L(N)$ satisfy*

$$B + r_N(\gamma) = N = C + r_N(\gamma); \quad B \cap C \subseteq r_N(\gamma).$$

then N is a ring.

Lemma 8. *Let Γ be a type 2 N-module and $\gamma \in \Gamma$ with $\gamma N \neq (0)$. If $I \in SL(N)$ with $r_N(\gamma) \subsetneq I$ then $I = N$.*

Proof. Since $r_N(\gamma) \subsetneq I$ we have $\gamma I = \Gamma$. If $n \in N$, then for some $t \in I$, $\gamma n = \gamma t$ so $n - t \in r_N(\gamma) \subset I$ and thus $n \in I$ and $N = I$.

By a standard argument one can show that if A is a non-nilpotent minimal N-subgroup of a near-ring N then $A = eN$ for some idempotent $e \in A$.

Lemma 9. *If N is a completely reducible nonring, without proper 2-sided ideals, with $J_2(N) = (0)$ and if eN is a minimal right ideal of N and X a right ideal of N with $eN \cap X = (0)$ then $X \subseteq r(e)$.*

Proof. If $x \in X$ with $ex \neq 0$ then $r(e) + X = N = r(e) + eN$, and $eN \cap X = (0) \subseteq r(e)$ contrary to N being a nonring.

Theorem 5. *If N is a completely reducible nonring, without proper 2-sided ideals, with $J_2(N) = (0)$ then the lattice of right ideals of N has unique complements.*

Proof. Let $X \in L(N)$ with A, $B \in L(N)$ such that $X \cap A = (0) = X \cap B$, $X + A = N = X + B$. If eN is a minimal right ideal of N with $eN \cap A = (0)$ then $A \subseteq r(e)$. Since $r(e) \neq N$ we cannot have $X \subseteq r(e)$ and so $X \cap eN \neq (0)$ and $eN \subseteq X$. It follows that A is the sum of all those minimal right ideals of N not in X. Similarly B is also their sum and $A = B$.

Corollary 7. *If N is a completely reducible nonring, without proper 2-sided ideals, with $J_2(N) = (0)$ then the lattice $L(N)$ is distributive.*

The proof of Theorem 5 contains the proofs of the following

Lemma 10. *If N is a completely reducible nonring, without proper 2-sided ideals, with $J_2(N) = (0)$ and $A \in L(N)$ then A is the sum of the minimal right ideals of N which are contained in it.*

Lemma 11. *If N is a completely reducible nonring without proper 2-sided ideals and $N = \bigoplus A_\lambda$ where each A_λ is a minimal non-nilpotent right ideal of N then each minimal right ideal of N is one of these A_λ.*

A near-ring N is v-primitive ($v = 0, 1, 2$) if it has a faithful type v N-module. A simple nonring without nilpotent N-subgroups and with a minimal N-subgroup will be 2-primitive and hence 1-primitive. For 1-primitive nonrings Ramakotaiah proved a density theorem which we wish to use.

Let N be a 1-primitive nonring and Γ be a faithful type 1 N-module. If $x, y \in N$ we define $x \sim y$ if and only if $r_N(x) = r_N(y)$. Clearly $\sim$ is an equivalence relation and C_0, the equivalence class containing 0, consists precisely of those $x \in \Gamma$ with $xN = (0)$. Ramakotaiah (9; Theorem 4) proved

Lemma 12. *Let N be a 1-primitive nonring and Γ be a faithful type 1 N-module. Let $w_1, w_2, \ldots, w_n \in \Gamma \backslash C_0$ with $w_i \not\sim w_j$ if $i \neq j$. For each set $m_1, m_2, \ldots, m_n \in \Gamma$ there is an element $b \in N$ with $w_i b = m_i$ $(1 \leq i \leq n)$.*

Lemma 13. *Let N be a completely reducible nonring, without proper two-sided ideals, in which $J_2(N) = (0)$. Then N has a system of idempotents $\{e_\lambda\}$ such that $e_\lambda e_\mu = 0$ if $\lambda \neq \mu$.*

Proof. Writing $N = \bigoplus_\lambda e_\lambda N$ where each $e_\lambda N$ is a minimal right ideal of N and $e_\lambda^2 = e_\lambda$ we know that $e_\lambda N \cap r_N(e_\lambda) = (0)$ and $e_\lambda N \bigoplus r_N(e_\lambda) = N$. If $\lambda \neq \mu$ then $e_\mu N \cap e_\lambda N = (0)$ and so, from Lemma 9, $e_\mu N \subseteq r_N(e_\lambda)$ and $e_\lambda e_\mu = 0$ as required.

Now suppose that N is a completely reducible nonring with $J_2(N) = (0)$ in which $xt = yt$ for each $t \in N$ implies $x = y$. Writing $N = \bigoplus N_\lambda$ where each N_λ is a homogeneous component of N we see that each N_λ has these properties and in addition has no two-sided proper ideals. Those N_λ which are simple rings are regular by Blair (4). Thus we need only consider those N_λ which are completely reducible nonrings with $J_2(N_\lambda) = (0)$, which have no two-sided proper ideals and in which $x, y \in N$ with $xt = yt$ for each $t \in N$ implies $x = y$.

Theorem 6. *If N is a completely reducible nonring, without proper two-sided ideals, such that $J_2(N) = (0)$ and whenever $x, y \in N$ with $xt = yt$ for each $t \in N$ then $z = y$ then N is regular in the sense that to each $a \in N$ there corresponds $b \in N$ with $a = aba$.*

Proof. Let $a \in N$. Choose non-nilpotent minimal right ideals $e_1N, \ldots, e_kN$ with $a \in e_1N \bigoplus \ldots \bigoplus e_kN$, and k minimal, where $e_i^2 = e_i$ for each i. Then as N-modules, e_iN is isomorphic to e_jN for $1 \leq i, j \leq k$. Let ϕ_j: $e_jN \to e_1N$ be an isomorphism and write $\gamma_j = \phi_j(e_j)$. Clearly $\gamma_i \sim \gamma_j$ if and only if $i = j$. From Lemma 13 we observe that if $N = \bigoplus e_\lambda N$ where each $e_\lambda N$ is a non-nilpotent minimal right ideal of N then $e_\lambda a = 0$ if

$\lambda \neq 1, 2, \ldots, k$ and, since k is minimal, $e_i a \neq 0$ for $1 \leq i \leq k$. Hence $\gamma_i a \neq 0$ for $1 \leq i \leq k$. Let $\gamma_1 a, \gamma_2 a, \ldots, \gamma_q a$ be those $\gamma_i a$ in different equivalence classes under $\sim$. Clearly $e_1 N x = 0$ implies $x = 0$ so N is a 2-primitive near-ring and thus 1-primitive. Appealing to Lemma 12 we can choose $b \in N$ with $\gamma_i ab = \gamma_i$ for $1 \leq i \leq q$. Now consider $\gamma_j a$ where $q < j \leq k$. For some i, $\gamma_j a \sim \gamma_i a$, so $r(\gamma_j a) = r(\gamma_i a)$. Now $\gamma_i abat = \gamma_i at$ for each $t \in N$; so $bat - t \in r(\gamma_i a)$. It follows that $bat - t \in r(\gamma_j a)$ for each $t \in N$. Hence $1 \leq s \leq k$ and $t \in N$ implies $\gamma_s abat = \gamma_s at$. By assumption we have $\gamma_s aba = \gamma_s a$. Hence $aba - a \in r(\gamma_s) = r(e_s)$; so

$$aba - a \in e_1 N \oplus \cdots \oplus e_k N \cap r(e_1) \cap \cdots \cap r(e_k) = (0),$$

or $aba = a$ as required.

Corollary 8. *If N is a completely reducible nonring with $J_2(N) = (0)$ and if $x, y \in N$ with $xt = yt$ for each $t \in N$ implies $x = y$ then N is regular.*

Proof. A direct sum of regular near-rings each of which is an ideal in the sum is regular so we simply apply Blair's result to those direct summands which are rings and Theorem 6 to the nonrings.

Observe that if R is a ring with $J_2(R) = (0)$ then $xR = (0)$ if and only if $x = 0$. Whether this is true for a general near-ring is unknown. However, when N is distributively generated we have

Lemma 14. *If N is distributively generated and has no nilpotent N-subgroups then $xNx = (0)$ implies $x = 0$.*

Proof. Let N be distributively generated by S (i.e. $a, b \in N$, $s \in S$ implies $(a + b)s = as + bs$ and $a \in N$ implies $a = \sigma_1 + \sigma_2 + \cdots + \sigma_n$ where for $1 \leq i \leq n$ either $\sigma_i \in S$ or $-\sigma_i \in S$). From $xNx = (0)$ we get $(xN)^2 = (0)$ and hence $xN = (0)$. Let B be the N-subgroup of N generated by x. If $b \in B$ then $b = n.x$, n an integer, in the obvious notation, since $xN = 0$. Then $(n.x)(m.x) = m.((n.x).x)$. Now $x = \Sigma_j \sigma_j$, where either $\sigma_j \in S$ or $-\sigma_j \in S$. Then $(n.x)(\Sigma_j \sigma_j) = \Sigma \pm (n(\pm x\sigma_j))$, taking the positive signs when $\sigma_j \in S$ and the negative signs when $-\sigma_j \in S$, but $\sigma_j \notin S$. As $x\sigma_j \in xN = 0$, so $(n.x)x = 0$ and $(n.x)(m.x) = 0$. Thus $B^2 = (0)$ and so $B = (0)$.

Corollary 9. *If N is distributively generated by S and has no nilpotent N-subgroups then $x, y \in N$ with $xt = yt$ for each $t \in N$ implies $x = y$.*

Proof. In particular $xs = ys$ for $s \in S$ and so $(x - y)s = 0$. It follows that $S \subseteq r(x - y)$ and hence $N \subseteq r(x - y)$. Then $(x - y)N = (0)$ so $x - y = 0$ and $x = y$.

Combining this with Corollary 8 we obtain

Theorem 7. *If N is a distributively generated, completely reducible near-ring and $J_2(N) = (0)$ then N is regular.*

I should like to take this opportunity of thanking the referee for his many helpful comments on an earlier version of this paper.

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