TAMARKIN–TSYGAN CALCULUS AND CHIRAL POISSON COHOMOLOGY

EMILE BOUAZIZ

Abstract. We construct and study some vertex theoretic invariants associated with Poisson varieties, specializing in the conformal weight 0 case to the familiar package of Poisson homology and cohomology. In order to do this conceptually, we sketch a version of the *calculus*, in the sense of [12], adapted to the context of vertex algebras. We obtain the standard theorems of Poisson (co)homology in this *chiral* context. This is part of a larger project related to promoting noncommutative geometric structures to chiral versions of such.

§1. Introduction

1.1 Motivation

In the celebrated paper (see [10]), the authors construct a sheaf of differential graded vertex algebras on any smooth **C**-variety, M, referred to as the *chiral de Rham complex*. This promotes the classical de Rham complex to a richer object of vertex theoretic nature, and this process of promotion is now traditionally referred to by the somewhat vague term *chiralization*, with the objects being chiralized then referred to as *classical*. We adopt this language throughout. If M is endowed with some additional structure, one may be able to define variations on the classical package of de Rham or Hodge cohomology. Relevant to this note are the case of Poisson varieties, and the so-called Poisson (co)homology studied in [3], [9]. These provide rather subtle invariants of Poisson varieties which, according to a theorem of Brylinski (cf. [3]), reduce to de Rham cohomology when the Poisson form is nondegenerate. Given the chiralization of the de Rham complex from [10], one might then ask whether one can chiralize Poisson (co)homology. This is the question animating this note. The classical Poisson (co)homology package is formulated very neatly in terms of *Gerstenhaber algebras* and their modules, and our approach to chiralization is based on this. In particular, we introduce the notion of a *Gerstenhaber vertex algebra* in order to do this, which has the advantage of allowing us to obtain the basic theorems of chiral Poisson (co)homology rather easily.

In more concrete terms, we produce in this note certain infinite dimensional invariants of a Poisson variety, naturally graded by *conformal* weight. These invariants reduce to cohomology of the chiral de Rham when the Poisson form vanishes and Poisson (co)homology in conformal weight 0. We also identify the fixed points of the homotopical S^1 action coming from the chiral de Rham differential, d_{dR}^{ch}, and prove an analogue of the theorem of Brylinski (see [3]) showing that these invariants vanish in non-zero conformal weight when π is non-degenerate. We view this note as a part of a larger set of questions regarding when Tamarkin–Tsygan calculi can be chiralized, the case of matrix factorizations was treated in the paper [2], where some general expectations are also sketched. A genuinely

noncommutative case would be very interesting in our opinion, and this note can be seen as a first-order approximation to such in the case of almost commutative algebras.

In the following subsections, we recall some definitions and theorems in the classical case, so that the reader can compare with their chiral versions introduced later in this note.

1.2 Recollections on Gerstenhaber algebras

We begin with the abstract definition of a Gerstenhaber algebra and a module for it, so the reader familiar with the classical calculus of polyvectors, in particular the *Schouten–Nijenhuis bracket* and the action of polyvectors on forms, should bear these in mind.

DEFINITION 1.1. Let A be a graded super-commutative $\mathbf{C}$ algebra, equipped with a super Lie bracket on the grading-shifted vector space $A[1]$, denoted $\{\,,\}$. We call it a *Gerstenhaber algebra* if in addition for all homogeneous $a, b, c \in A$, we have

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|-1)|b|} b\{a, c\},$$

where $|a|$ and $|b|$ denote the weight of a and b, respectively.

REMARK. Let us unpack the above a little. $A[1]$ is the graded vector space with underlying vector space A, and grading shifted, so that an element $a \in A$ has degree $|a| - 1$ in $A[1]$. It follows that the operators $\{a, -\}$ have weight $|a| - 1$. Furthermore, the Jacobi identity takes the following form, writing $\epsilon_{a,b} := (-1)^{(|a|-1)(|b|-1)}$, we have

$$\epsilon_{a,c}\{a, \{b, c\}\} + \epsilon_{b,a}\{b, \{c, a\}\} + \epsilon_{c,b}\{c, \{a, b\}\}.$$

We often refer to the bracket $\{\,,\}$ as the *Gerstenhaber bracket* and Gerstenhaber algebras as G-algebras.

DEFINITION 1.2. A module for a $\mathcal{G}$-algebra A, is a graded vector space, M, endowed with the structure of a module over the underlying algebra of A, and a representation of the super Lie algebra $(A, \{\,,\})$. For $a \in A$, the corresponding operators are denoted ι_a and $\mathcal{L}_a$, respectively. We demand that ι_a is of weight $-|a|$, and $\mathcal{L}_a$ is of weight $1 - |a|$. We further demand the following compatibilities, which are to be understood as identities in the super Lie algebra $End(M)$—for all $a, b \in A$, we have:

- $[\mathcal{L}_a, \iota_b] = \iota_{\{a,b\}}$.
- $\mathcal{L}_{ab} = \iota_a \mathcal{L}_b + (-1)^{|a|} \iota_b \mathcal{L}_a$.

REMARK.
- Note that the above definition implies the following identity of operators in $End(M)$—for $a, b \in A$, we have:

$$[\mathcal{L}_a, \mathcal{L}_b] = \mathcal{L}_{\{a,b\}}.$$

- Note further that setting $\mathcal{L}_a := \{a, -\}$ and $\iota_a := a(-)$, we obtain the *adjoint* representation of A on itself (with the negative grading).

As mentioned above, a motivating example is that of polyvectors on some smooth space. Let M denote a smooth $\mathbf{C}$-variety, and let Θ_M denote the exterior algebra on the tangent sheaf T_M, which we refer to as the sheaf of *polyvectors*. This is a graded-commutative sheaf of algebras with product the exterior product. In degrees less than or equal to 1, the underlying sheaf is $\mathcal{O}_M \oplus T_M$. The sheaf T_M acts by Lie derivatives on $\mathcal{O}_M$, and is endowed with a bracket $[\,,]$ coming from the commutator of vector fields. If v is a vector field and f

a function, let us write $\mathcal{L}_v(f)$ for the Lie derivative. Then we have the following well-known and simple lemma.

LEMMA 1.1. *The bilinear operation $\{,\}$ defined on $\mathcal{O}_M \oplus T_M$ by $\{v,f\} = -\{f,v\} := \mathcal{L}_v(f)$ and $\{v_1, v_2\} := [v_1, v_2]$ extends uniquely to a Gerstenhaber bracket on the graded-commutative algebra Θ_M, referred to as the Schouten–Nijenhuis bracket.*

Proof. One notes that the product is of the appropriate degree and then extends in the only possible way, noting that elements of degree at most 1 generate Θ_M. □

Hinted above is the fact that the sheaf of forms, Ω_M, can be endowed with the structure of a module for the $\mathcal{G}$-algebra Θ_M. We state this now as a lemma.

LEMMA 1.2. *Ω_M has the structure of a module for the $\mathcal{G}$-algebra Θ_M. For v a polyvector, the operator ι_v is the natural contraction arising from the equivalence $T_M^* = \Omega_M^1$. The operators $\mathcal{L}_v$ are defined as $[d_{dR}, \iota_v] := \mathcal{L}_v$.*

Proof. This is a standard fact. □

REMARK. Often, the Lie derivative $\mathcal{L}_v$ is defined without reference to the de Rham differential d_{dR}, and the formula $[d_{dR}, \iota_v] = \mathcal{L}_v$ is then the so-called *Cartan formula*.

Motivated by the above, we now define (following [12]) the structure of a *calculus*.

DEFINITION 1.3. A triple (A, M, ∂) consisting of a $\mathcal{G}$-algebra A, a module M for A, and a degree 1 differential ∂ on M is called a *calculus* if, for all $a \in A$, we have

$$[\partial, \iota_a] = \mathcal{L}_a.$$

We have immediately the following corollary.

COROLLARY 1.1. *The triple $(\Theta_M, \Omega_M, d_{dR})$ is a sheaf of calculi on M.*

REMARK. The purpose of the language of [12] is really to deal with noncommutative versions of the above, namely the so-called *Hochschild calculus*. For more on these constructions the reader can further consult [7]. We say nothing of chiral versions of this in this note; however, the broader (and still imprecise) question of when Hochschild theory might admit chiral enhancements was a major motivation for the writing of this note.

1.3 Poisson homology and cohomology

With the above formalism out of the way, we are now in a position to give a very fast introduction to Poisson (co)homology. Let π be a Poisson form on M. It is standard that this is equivalent to an element of Θ_M of cohomological degree 2 satisfying $\{\pi, \pi\} = 0$. One then obtains a cohomological differential $\{\pi, -\} := \partial_\pi$ on Θ_M (cf. the work of Lichnerowicz in [9]), and a corresponding homological differential $\mathcal{L}_\pi$ on Ω_M (cf. the work of Brylinski in [3]).

REMARK. Note that the differentials are of the appropriate degree because π is a degree 2 element. Furthermore, note that the differentials square to zero because of the relation $\{\pi, \pi\} = 0$; for example, we have

$$2\mathcal{L}_\pi^2 = [\mathcal{L}_\pi, \mathcal{L}_\pi] = \mathcal{L}_{\{\pi,\pi\}} = 0.$$

DEFINITION 1.4. The hypercohomology of these complexes are denoted $H_\pi^*(M)$ and $H_*^\pi(M)$, and referred to as *Poisson cohomology* and *Poisson homology*, respectively.

EXAMPLE. We unpack the definition of $\partial_\pi := \{\pi, -\}$ a little. If f is a function on M, then, writing π locally as $\sum_{i,j} \pi_{i,j} \partial_i \partial_j$, we find

$$\partial_\pi(f) = \sum_{i,j} \pi_{i,j}(\partial_i(f)\partial_j - \partial_i(f)\partial_j).$$

This is the *Hamiltonian* vector field associated with f. We further have $\partial_\pi(\partial_k) = \partial_k(\pi_{i,j})\partial_i\partial_j$, and these together determine the differential entirely. We see then that the zeroth cohomology is given by the space of functions whose associated Hamiltonian vanishes, and the first cohomology is given by the space of vector fields preserving π infinitesimally, modulo the subspace of Hamiltonians of functions.

Note that one has $\left[d_{dR}, \mathcal{L}_\pi\right] = 0$, whence one can form the $\mathbf{Z}/2$-graded totalization $(\Omega_M((u)), u d_{dR} + \mathcal{L}_\pi)$ of the *mixed complex* $(\Omega_M, \mathcal{L}_\pi, d_{dR})$. Here, u is a formal variable of cohomological degree -2.

DEFINITION 1.5. The hypercohomology of this totalization is denoted $H_*^\pi(M)^{S^1}$.

The reader is referred to [11, §1] for an explanation as to this notational choice.

REMARK. It is essentially formal that the construction above implies that the tuple

$$(H_\pi^*(M), H_*^\pi(M), d_{dR})$$

forms a calculus.

Let us see some examples to get a better feel for these constructions;

EXAMPLE.
- Let $M = \mathbf{C}^2$ with coordinates (x^1, x^2) be equipped with the standard symplectic form $dx^1 dx^2$, considered as the Poisson form $\pi = \partial_1 \partial_2$. One easily confirms that the Poisson homology complex is isomorphic to the de Rham complex of $\mathbf{C}^2$ flipped upside down, whence we see that $H^\pi(M) \cong \mathbf{C}[2]$ in this case. Similarly, we compute $H_\pi(M) \cong \mathbf{C}[0]$. Note that in this case the answer is just (a regrading of) the de Rham cohomology of M, this is generalized by Brylinski in [3] to all symplectic manifolds, and the argument of Brylinski is sketched below.
- Let $M = \mathbf{P}^2$. Then a Poisson form π is equivalent to a section of the anticanonical bundle $\mathcal{O}_{\mathbf{P}^2}(3)$, so is given by a cubic. Note that $\{\pi, \pi\} = 0$ is automatic as Θ_M vanishes in degree 3 in this case. Then one confirms that $H^\pi(M) \cong H_{dR}(M)$; indeed, the spectral sequence corresponding to the stupid filtration has E_1 page the Hodge cohomology of $\mathbf{P}^2$, so there is no room for any differentials and it collapses. A similar argument works for any smooth projective variety whose Hodge numbers are concentrated on the diagonal.
- Consider the example of $M = \mathbf{C}^2$ with coordinates x^i and equipped with the form $\pi := x^2 \partial_1 \partial_2$. This is an analytic local model for *local* Poisson surfaces constructed from the data of a curve Σ with a nonvanishing vector field ξ, together with a line bundle, L. Such a datum specifies a $\mathbf{C}^*$-equivariant Poisson structure on the total space of L. In the example of $M = \mathbf{C}^2$, the $\mathbf{C}^*$ action corresponds to giving x^2 weight 1. Let us first compute the Poisson homology of $(\mathbf{C}^2, \pi)$. We see immediately that $H_2^\pi = 0$ and $H_0^\pi \cong \mathbf{C}[x^1]$. There is an evident map $\mathbf{C}[x^1] dx^1 \to H_1^\pi$, which one can prove directly is an isomorphism. A slicker way to do this is as follows: one notes that the Euler vector field, η, for the $\mathbf{C}^*$ action constructed above, is in the image of the map $\pi : \Omega_M^1 \to T_M$; indeed, $\eta = x^2 \partial_2 = \pi(dx^1)$.

A simple computation now confirms that we have

$$\left[\mathcal{L}_\pi, dx^1\right] = \mathcal{L}_\eta.$$

Now, $\mathcal{L}_\eta$ is simply the grading operator on homology, we deduce that all the homology comes from the weight 0 subspace, and thus the above is proved. We caution the reader that it is not a general fact that a connected algebraic group acting on the Poisson variety (M, π) must act trivially on $H^\pi(M)$; indeed, one can take $\pi = 0$ in which case one is dealing with Hodge cohomology, on which the group can certainly act nontrivially for (say) an affine M. By the above argument, this is true if the infinitesimal action is given by Hamiltonians of functions on M. This is in fact a formal consequence of the calculus structure, as we are acting on homology classes by vanishing cohomology classes (see [3, (3.4)] for some related discussions).

In general, Poisson (co)homology is somewhat difficult to compute; it is not even known when it is finite (cf. [4] for results in this direction); nonetheless, one can compute it in the case that π is nondegenerate; and in all cases, one can compute the totalization $H_*^\pi(M)^{S^1}$.

LEMMA 1.3. *(Brylinski [3]). If π is nondegenerate, then we have $H_*^\pi(M) \cong H^{d-*}(M)$, where d is the dimension of M. Similarly, we have $H_\pi^*(M) \cong H^*(M)$.*

Proof. We deal with the case of homology. Let ω be the symplectic form corresponding to π. Brylinski shows that the associated symplectic Hodge-$*$ operator (which obviously induces an isomorphism between the graded sheaves Ω_M^* and Ω_M^{d-*}) intertwines the differentials d_{dR} and $\mathcal{L}_\pi$, whence the lemma is proved. The case of cohomology follows similarly. $\square$

LEMMA 1.4. *There is an isomorphism $H_*^\pi(M)^{S^1} \cong H^*(M)((u))$.*

Proof. Noting that the operator ι_π is nilpotent, this is immediate from the identity $e^{\iota_\pi} d_{dR} e^{-\iota_\pi} = d_{dR} + \mathcal{L}_\pi$. This is a special case of the identity $e^{ad_x} y = e^x y e^{-x}$, itself a special case of the Baker–Campbell–Hausdorff formula. $\square$

We state now the main Poisson theoretic results of this note, and some definitions regarding chiral objects can be found in the next sections. These results are direct analogues of the computation in the case of nondegenerate Poisson forms and the computation of S^1-invariants above.

THEOREM 1.5. *If (M, π) is a smooth Poisson variety, then there is an associated triple, $((\Theta_M^{ch}, \partial_\pi^{ch}), (\Omega_M^{ch}, \mathcal{L}_\pi^{ch}), d_{dR}^{ch})$, satisfying the axioms of a sheaf of vertex calculi. Furthermore, we have the following basic computations:*

- *De Rham invariants. The hypercohomology of the S^1-fixed points are identified with $H^{ch}(M)((u))$, the 2-periodization of the hypercohomology of the Ω_M^{ch} with vanishing differential.*
- *Brylinksi-type theorem. If π is nondegenerate, then the hypercohomology of $(\Theta_M^{ch}, \partial_\pi^{ch})$ and $(\Omega^{ch}, \mathcal{L}_\pi^{ch})$ vanish in conformal weight greater than 0.*

§2. Chiral calculus

2.1 Chiral polyvectors

We assume familiarity with the basic theory of vertex algebras, and the reader may consult [10] for an introduction. The vertex algebras with which we deal are all *super* such,

and we adopt the convention where all commutators are assumed to be super-commutators, vertex algebras are assumed to be vertex super-algebras, and so on. Furthermore, we assume our vertex algebras are $\mathbf{Z}_{\geq 0}$-graded by *conformal weight* and further endowed with a cohomological $\mathbf{Z}$-grading. We assume this as part of the structure, but abusively write *vertex algebra* nonetheless. If v is a vector in a vertex algebra, we denote by $|v|$ its cohomological weight and by $\Delta(v)$ its conformal weight. We also assume that the reader is familiar with the construction of the chiral de Rham complex, Ω_M^{ch}, as given, for example, in [10]. This is a sheaf of vertex algebras graded by conformal weight and cohomological degree. The conformal weight 0 subspace is the sheaf Ω_M of forms. There is a differential d_{dR}^{ch} on Ω_M^{ch}, whose definition is recalled below. If v is a vector in a vertex algebra, we define the i-modes $v_{(i)}$ by

$$Y(v,z) = \sum_i v_{(i)} z^{-1-i}$$

as usual, where Y is the state-field correspondence. d_{dR}^{ch} is a derivation with respect to all the (i)-modes, so that $[d_{dR}^{ch}, v_{(i)}] = (d_{dR}^{ch})_{(i)}$. If M is Calabi–Yau, there is a vector $Q \in \Omega_M^{ch}$ such that $Q_{(0)} = d_{dR}^{ch}$. Throughout, the x^i denote étale local coordinates on the variety M. Ω_M^{ch} is then étale locally generated by vectors $x_0^i, y_1^i, \phi_0^i, \psi_1^i$, where lower subscripts denote conformal weight, x, y are bosonic in cohomological weight 0, and ϕ, ψ are fermionic, respectively, of cohomological weights 1 and -1. The reader can consult the formulae of [10] to see the relevant transformation formulae and commutation relations.

REMARK.   With respect to notation, we remark that unbracketed lower subscripts denote conformal weight, and bracketed ones denote modes of vectors, so that, for example, $(y_1^i)_{(0)}$ is the operator of differentiation with respect to the vector x_0^i.

Ω_M^{ch} is a vertex operator algebra (that is to say, admits a Virasoro vector) for any smooth M and an $\mathcal{N} = 2$ such in the Calabi–Yau case (cf. [10]). In this latter case, Q is locally expressed as $\sum_i y_1^i \phi_0^i$. Regardless of whether Q is globally well defined, its 0-mode $Q_{(0)} := d_{dR}^{ch}$ is. We record this below as a definition, referring to [10] for a proof that this operator is well defined.

DEFINITION 2.1.   The operator d_{dR}^{ch}, expressed in local coordinates as

$$\left(\sum_i y_1^i \phi_0^i\right)_{(0)},$$

is referred to as the *chiral de Rham* differential.

The sheaf of chiral polyvectors, denoted Θ_M^{ch}, is defined by regrading Ω_M^{ch}, so that the generating ϕ-vectors are now of conformal weight 1, and the ψ vectors of conformal weight 0. We further take the negative of the cohomological grading, so that the local vectors ψ^j now have cohomological weight 1. When care is needed, we denote the corresponding vectors $\bar{\psi}$ and $\bar{\phi}$.

LEMMA 2.1.   *The conformal weight* 0 *subspace of* Θ_M^{ch} *is the sheaf* Θ_M.

Proof.   This is easily confirmed.   □

REMARK.   This apparently trivial adaptation hides some slight subtleties, and the formulae of [10] now imply that Θ_M^{ch} is *no longer* endowed with a Virasoro vector compatible

with its conformal grading, unless M is Calabi–Yau. Happily, in the CY case, Θ_M^{ch} is still endowed with an $\mathcal{N} = 2$ structure (cf. [5]). We note here that, as remarked in [10], all objects exist canonically as *gerbes* .

DEFINITION 2.2. On a formal D-disc Δ^D, we define the vector $\bar{Q} \in \Theta_{\Delta^D}^{ch}$ of conformal weight 2 by

$$\bar{Q} = y_1^i \bar{\phi}_1^i.$$

LEMMA 2.2. *The 0-mode of this vector is invariant under the action of $Aut(\Delta^D)$, and so $\bar{Q}_{(0)}$ is defined on Θ_M^{ch} for any smooth M. It is a derivation with respect to all the $_{(i)}$ products.*

Proof. This follows from the formulae of [10]. □

REMARK. In the absence of the global vector $\bar{Q}$, we nonetheless write $\bar{Q}_{(0)}$ for the corresponding 0-mode, which always exists according to the above lemma.

2.2 Brackets and chiral Poisson cohomology

We want to introduce some additional structure to Θ_M^{ch}. Now, in the case of Ω_M^{ch}, we know that the weight 0 subspace is endowed with a canonical differential, which extends to the whole space Ω_M^{ch}. The weight 0 subspace of Θ_M^{ch} is endowed with the structure of a $\mathcal{G}$-algebra, and we would like some analogue of this on the whole space Θ_M^{ch}. Assuming for simplicity that M is Calabi–Yau, recall that the differential d_{dR}^{ch} comes from the vector Q. What is crucial is that Q is of conformal weight 1, cohomological degree 1, and satisfies $Q_{(0)}^2 = 0$. We have the corresponding vector $\bar{Q} \in \Theta_M^{ch}$, which is now of conformal weight 2 and cohomological weight -1. Some thought tells us that the operator $(\bar{Q}_{(0)}v)_{(0)}$ is now of conformal weight $\Delta(v)$ and cohomological weight $|v| + 1$, so that it has some chance of being the desired bracket on the weight 0 subspace. We thus define some operators as follows.

DEFINITION 2.3. We define the Gerstenhaber i-products on Θ_M^{ch}, $v \otimes w \mapsto v_{\{i\}}w$, by

$$v_{\{i\}}w := (\bar{Q}_{(0)}v)_{(i)}w.$$

LEMMA 2.3. *The product $v \otimes w \mapsto v_{\{0\}}w$ restricts on the conformal weight 0 subspace to the usual Gerstenhaber bracket of polyvector fields.*

Proof. $v_{\{0\}} = (\bar{Q}_{(0)}v)_{(0)}$ acts a derivation with respect to all (j)-products, in particular with respect to the (-1)-product. We must then verify $(\bar{\psi}_0^i)_{\{-1\}}x_0^j = (-x_0^j)_{\{-1\}}\bar{\psi}_0^i = \delta_{ij}$. This follows immediately from

$$(\bar{Q}_{(0)})_0(\bar{\psi}_0^i) = (y_1^i)_0 = \partial_{x_0^i},$$

$$(\bar{Q}_{(0)}x_0^i)_0 = (-\bar{\phi}_1^i)_0 = -\partial_{\bar{\psi}_0^i}.$$

Finally, note that the products $\bar{\psi}_0^i{}_{\{0\}}\bar{\psi}_0^j = 0$, for all i and j, as follows from

$$(y_1^i)_{(0)}\bar{\psi}_0^j = \partial_{x_0^i}\bar{\psi}_0^j = 0.$$ □

REMARK. The construction of the *Gerstenhaber operations*,

$$v \otimes w \mapsto v_{\{i\}}w,$$

is in the spirit of the work of Lian and Zuckerberg [8], and the product {0} is a special case of the construction of [8]. Note, however, that [8] deals with the construction of a $\mathcal{G}$-algebra on the structure of the BRST complex associated with an $\mathcal{N}=2$ vertex algebra, which cohomology vanishes in conformal weights greater than 0, and it is really the higher conformal weight Poisson (co)homology which interests us in this note.

We are now in a position to define the chiral Poisson cohomology complex. We state its construction as a lemma.

LEMMA 2.4. *If π is a Poisson form, then the operator $\pi_{\{0\}} := \partial_\pi^{ch}$ defines on Θ_M^{ch} a cohomological differential, which is moreover a derivation with respect to all (j)-products, restricting to the Poisson cohomology differential on the subspace of conformal weight 0.*

Proof. It is a derivation with respect to all $_{(j)}$ products because it is a 0-mode. We compute

$$2(\partial_\pi^{ch})^2 = 2\pi_{\{0\}}^2 = \left[\pi_{\{0\}}, \pi_{\{0\}}\right] := \left[(\bar{Q}_{(0)}\pi)_{(0)}, (\bar{Q}_{(0)}\pi)_{(0)}\right],$$

$$\left[(\bar{Q}_{(0)}\pi)_{(0)}, (\bar{Q}_{(0)}\pi)_{(0)}\right] = ((\bar{Q}_{(0)}\pi)_{(0)}\bar{Q}_{(0)}\pi)_{(0)}.$$

Now, we note

$$\left[\bar{Q}_{(0)}, (\bar{Q}_{(0)}\pi)_{(0)}\right] = (\bar{Q}_0^2\pi)_0 = 0,$$

whence we deduce that

$$((\bar{Q}_{(0)}\pi)_{(0)}\bar{Q}_{(0)}\pi)_{(0)} = (\bar{Q}_{(0)}(\bar{Q}_{(0)}\pi)_{(0)}\pi)_{(0)} := (\pi_{\{0\}}\pi)_{\{0\}} = 0.$$

We have seen above that {0} restricts to the Gerstenhaber bracket on polyvector fields, whence the induced map on the conformal weight 0 subspace is as claimed. □

REMARK. The above differential graded vertex algebra contains a large commutative subalgebra, generated by the x and ψ variables. This is simply the algebra of functions on the space of arcs into the super variety $T^*[-1]M$. According to the results of [1], such has the structure of a (shifted) *Poisson vertex algebra*, the Gerstenhaber products $v \otimes w \mapsto v_{\{i\}}w$, for $i \geq 0$, recover this structure. For more on these constructions the reader is referred to [6].

REMARK. The identity $[\pi_{\{0\}}, \pi_{\{0\}}] = (\pi_{\{0\}}\pi)_{\{0\}}$ is the special case of a general identity which is valid in any $\mathcal{G}$- vertex algebra (to be defined below),

$$\left[v_{\{i\}}, w_{\{j\}}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left(v_{\{k\}}w\right)_{\{i+j-k\}},$$

which is a form of the Borcherds identity for the super-commutator of modes in a vertex algebra.

2.3 Gerstenhaber vertex algebras

We now explain how the above identity $(\partial_\pi^{ch})^2 = 0$ fits into the general framework of what we call *Gerstenhaber vertex algebras*, or $\mathcal{G}$-vertex algebras.

DEFINITION 2.4. A *vertex $\mathcal{G}$- algebra* is defined to be a vertex algebra V endowed with bilinear operations of cohomological degree (-1), $v \otimes w \to v_{\{i\}}w$, of conformal weight $-i$, satisfying the following quadratic relations (with the vertex products).

- Borcherds commutation identity,

$$\left[v_{\{i\}}, w_{\{j\}}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left(v_{\{k\}}w\right)_{\{i+j-k\}},$$

- Generalized derivation property,

$$\left[v_{\{i\}}, w_{(j)}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left(v_{\{k\}}w\right)_{(i+j-k)},$$

- Bracket of a product,

$$(v_{(i)}w)_{\{j\}} = \sum_{k=0}^{\infty} (-1)^k \binom{i}{k} \left(v_{\{i-k\}}v_{j+k} + u_{(i-k)}v_{\{j+k\}} - (-1)^k (v_{\{i+j-k\}}u_{(k)} + v_{(i+j-k)}u_{\{k\}})\right).$$

REMARK. The above three axioms are, respectively, the analogues of the formulas for a Gerstenhaber module:

- $[\mathcal{L}_a, \mathcal{L}_b] = \mathcal{L}_{\{a,b\}}$.
- $[\mathcal{L}_a, \iota_b] = \iota_{\{a,b\}}$.
- $\mathcal{L}_{ab} = \iota_a \mathcal{L}_b + (-1)^{|a|} \iota_b \mathcal{L}_a$.

The above discussion can now be summarized in the following lemma.

LEMMA 2.5. *The products* $v \otimes w \mapsto v_{\{i\}}w := (\bar{Q}_{(0)}v)_{(i)}w$ *endow the sheaf of chiral polyvector fields,* Θ_M^{ch}, *with the structure of a* $\mathcal{G}$-*vertex algebra.*

Proof. All of the above identities follow from Borcherds–Jacobi identities by substitution and commutation with $\bar{Q}_{(0)}$. The only properties of $\bar{Q}_0$ which we required are that it be of appropriate conformal weight (2) and cohomological weight (-1), and that $\bar{Q}_{(0)}^2 = 0$. We make explicit the proof in the case of the Borcherds commutation identity

$$\left[v_{\{i\}}, w_{\{j\}}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left(v_{\{k\}}w\right)_{\{i+j-k\}}.$$

First, take the classical Borcherds identity for $\bar{Q}_{(0)}v$ and $\bar{Q}_{(0)}w$. We obtain by definition of the $_{\{i\}}$ brackets that

$$\left[v_{\{i\}}, w_{\{j\}}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left((\bar{Q}_{(0)}v)_{(k)}(\bar{Q}_{(0)}w)\right)_{(i+j-k)}.$$

Now, by definition, we have

$$(v_{\{k\}}w)_{\{i+j-k\}} = \left(\bar{Q}_{(0)}((\bar{Q}_0 v)_{(k)}w))\right)_{(i+j-k)},$$

so that it suffices to prove that we have

$$(\bar{Q}_{(0)}v)_{(k)}(\bar{Q}_{(0)}w) = \bar{Q}_{(0)}((\bar{Q}_0 v)_{(k)}w)),$$

which follows immediately from

$$\left[\bar{Q}_{(0)}, (\bar{Q}_{(0)}v)_{(k)}\right] = (\bar{Q}_{(0)}^2 v)_{(k)} = 0,$$

as $\bar{Q}_{(0)}$ is a derivation with respect to all $_{(i)}$ products and satisfies $\bar{Q}_{(0)}^2 = 0$. $\quad\square$

2.4 Action on chiral forms

We are still lacking a definition of chiral Poisson *homology*. Recalling how the classical Poisson homology can be defined neatly in terms of the structure on Ω_M of a module for the $\mathcal{G}$-algebra Θ_M, it is reasonable to expect that this can be done cleanly by constructing on Ω_M^{ch} the structure of a module (to be defined) for the $\mathcal{G}$-vertex algebra Θ_M^{ch}.

DEFINITION 2.5. Let V be a $\mathcal{G}$-vertex module, and let M be a module for the underlying vertex algebra of V, compatibly graded by conformal and cohomological weight. We say that M is endowed with the structure of a $\mathcal{G}$-vertex module for M if, for each $v \in V$, we are given morphisms

$$v_{\{i\}} : M \longrightarrow M,$$

of cohomological weight $1 - |v| - i$ and conformal weight $1 + \Delta(v) - i$. Writing $v_{(i)}$ for the endomorphisms of M coming from the structure of a vertex module, we demand further the following compatibilities:

$$\left[v_{\{i\}}, w_{\{j\}}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left(v_{\{k\}} w\right)_{\{i+j-k\}},$$

$$\left[v_{\{i\}}, w_{(j)}\right] = \sum_{k=0}^{\infty} \binom{i}{k} \left(v_{\{k\}} w\right)_{(i+j-k)},$$

$$(v_{(i)} w)_{\{j\}} = \sum_{k=0}^{\infty} (-1)^k \binom{i}{k} \left(v_{\{i-k\}} v_{j+k} + u_{(i-k)} v_{\{j+k\}} - (-1)^k (v_{\{i+j-k\}} u_{(k)} + v_{(i+j-k)} u_{\{k\}})\right).$$

REMARK. In accordance with the classical case, we write $v_{\{i\}} := \mathcal{L}_i^v$ and $v_{(i)} := \iota_i^v$. The above compatibilities are the chiral analogues of the classical compatibilities between the operators $\mathcal{L}_a$ and ι_b in the definition of a vertex algebra module.

Recall that we had in the classical case the notion of a *calculus*, which involved, in particular, a cohomological differential on the module M. Note also that in the example of interest to us, where the module is Ω_M^{ch}, we already have a cohomological differential. The following definition is now readily suggested.

DEFINITION 2.6. A triple consisting of a $\mathcal{G}$-vertex algebra V, a module M for it, and a cohomological differential, ∂_M, on M, are said to form a *vertex calculus* if, for all $v \in V, j \in \mathbf{Z}$, the following *chiral Cartan formulae* hold:

$$\mathcal{L}_j^v = \left[\partial, \iota_{j-1}^v\right].$$

We wish now to show that Ω_M^{ch} forms a module over the $\mathcal{G}$-algebra Θ_M^{ch}, and that d_{dR}^{ch} enhances this to a vertex calculus. We make use of the following easy lemma, where we write $Lie(V)$ for the Lie algebra of modes obtained from a vertex algebra V.

LEMMA 2.6. *There is an isomorphism of sheaves of Lie algebras, $Lie(\Theta_M^{ch}) \to Lie(\Omega_M^{ch})$.*

Proof. Dispensing of superscripts to unburden notation, and working formally locally, we send the modes $x_{(i)}, y_{(i)} \in Lie(\Theta_M^{ch})$ to the identically denoted elements of $Lie(\Omega_M^{ch})$. We

stipulate further $\bar{\phi}_{(j)} \mapsto \phi_{(j-1)}$ and $\bar{\psi}_{(i)} \mapsto \psi_{(i+1)}$. This can easily be checked to extend to an isomorphism of Lie algebras. $\qquad \square$

COROLLARY 2.1. *Ω_M^{ch} is a module for the sheaf of vertex algebras Θ_M^{ch}.*

Now, this tells us how to make sense of the contraction operators, and since we are in any case expecting a vertex calculus, we may as well enforce the Cartan formulae in order to define the Lie derivatives.

DEFINITION 2.7. For $v \in \Theta_M^{ch}$ a local section, the contraction operators ι_j^v are defined to be the j-modes of the action of Θ_M^{ch} on Ω_M^{ch} constructed above. The Lie derivative $\mathcal{L}_j^v$ is defined to be $\left[d_{dR}^{ch}, \iota_{j-1}^v \right]$.

REMARK. In [10], an action of changes of coordinates on a local model for Ω^{ch} is constructed. The infinitesimal form of this action is recovered in the case where the local section $v \in \Theta_M^{ch}$ is of conformal weight 0 by taking the associated operator $\mathcal{L}_0$.

We summarize the above in the following theorem.

THEOREM 2.7. *The chiral de Rham differential d_{dR}^{ch} endows the module Θ_M^{ch}-module Ω_M^{ch} with the structure of a sheaf of vertex calculi.*

Proof. Once it is shown that the operators satisfy the axioms of a representation of a $\mathcal{G}$-vertex algebra, it follows by construction that d_{dR}^{ch} endows this with the structure of a calculus. That the ι and $\mathcal{L}$ operators satisfy the requisite axioms can be checked explicitly. Alternatively, we note that they can be checked inside the Lie algebra $Lie(\Omega_M^{ch})$, as the operators $v_{\{\}}$ are themselves the modes of vectors, by their construction. Recall from above that $Lie(\Omega_M^{ch})$ is isomorphic to $Lie(\Theta_M^{ch})$, and so the identities follow from those in $Lie(\Theta_M^{ch})$, which simply state that $Lie(\Theta_M^{ch})$ forms a $\mathcal{G}$-vertex algebra. $\qquad \square$

COROLLARY 2.2. *The operator $\left[d_{dR}^{ch}, \iota_{-1}^\pi \right] = \mathcal{L}_0^\pi$, denoted henceforth $\mathcal{L}_\pi^{ch}$ and referred to as the chiral Poisson differential, is of square zero and cohomological degree -1. The conformal weight 0 subspace reproduces the usual Poisson homology complex of [3]. There is an additional differential, given by d_{dR}^{ch}, which commutes with $\mathcal{L}_\pi^{ch}$.*

The following definition-lemma summarizes the above discussion.

DEFINITION 2.8. The hypercohomology

$$H_\pi^{ch}(M) := H^*(M, (\Theta_M^{ch}, \partial_\pi^{ch}))$$

is referred to as *chiral Poisson cohomology*. The hypercohomology

$$H_{ch}^\pi(M) := H^*(M, (\Omega_M^{ch}, \mathcal{L}_\pi^{ch}))$$

is referred to as *chiral Poisson homology* .

LEMMA 2.8. *The formulae defined above endow the triple $(H_\pi^{ch}(M), H_{ch}^\pi(M), d_{dR}^{ch})$ with the structure of a vertex calculus.*

Proof. Above, we have constructed everything on the sheaf level, and it formally descends to the level of hypercohomology. $\qquad \square$

§3. Chiral Poisson (co)homology

We now turn to the task of proving the expected basic theorems concerning chiral Poisson (co)homology. As was mentioned in the introduction, Poisson (co)homology is a somewhat subtle invariant of a Poisson variety (M, π), and as such, we of course cannot expect to compute its chiral analogue too easily, as this chiral analogue is *at least* as intractable as the classical version.

A benefit to the somewhat lengthy discussion above is that the basic expected properties of chiral Poisson (co)homology can now be verified quite easily. We begin with the identification of the fixed points of the S^1-action.

LEMMA 3.1. *There is an isomorphism*

$$H_{ch}^{\pi, S^1}(M) := H^*\big(M, \big(\Omega_M^{ch}((u)), u d_{dR}^{ch} + \mathcal{L}_\pi^{ch}\big)\big) \cong H_{dR}^*(M)((u)),$$

in particular, there are no nonzero classes of strictly positive conformal weight.

Proof. Recall that $\mathcal{L}_\pi^{ch}$ satisfies $\big[d_{dR}^{ch}, \iota_{-1}^\pi\big]$. Now, note that ι_{-1}^π is of even cohomological degree -2 and further is locally nilpotent, as the cohomological degree is bounded below on each fixed conformal weight piece of Ω_M^{ch}. It follows that

$$\exp\big(\frac{\iota_{-1}^\pi}{u}\big)$$

is a well-defined operator on this 2-periodic complex. This conjugates the differential,

$$d_{dR}^{ch} + u^{-1}\mathcal{L}_\pi^{ch},$$

to the usual chiral de Rham differential. One then applies the results of [10] to note that the resulting hypercohomology is simply $H_{dR}(M)((u))$ in conformal weight 0.   □

We now prove the analogue of the theorem of Brylinksi [3] showing that these invariants are really invariants of the singularities of the form π, which is to say that one obtains nothing of interest when π is nondegenerate.

THEOREM 3.2. *If π is nondegenerate, then there is an isomorphism $H_{ch}^\pi(M) \cong H_{dR}^{d-*}(M)$, placed in conformal weight 0, where $d = 2n$ is the dimension of M. A similar result holds for Poisson cohomology. In particular, there are no classes of nonzero conformal weight.*

Proof. Let ω be the symplectic form dual to π. If x is a point of M, then there are formal coordinates around x where π is in standard Darboux form. (We caution the reader that this is not true in the étale topology; indeed, it fails for the form $d\log(x^1)d\log(x^2)$ on $\mathbf{C}^* \times \mathbf{C}^*$.)

Now, the machinery of Gelfand–Kazhdan formal geometry (see [10] for an introduction) implies that there is an associated torsor $\widehat{M} \to M$ for the (pro-)group scheme $Symp(\Delta^{2n}, \omega_{std})$ of formal symplectomorphisms of the $2n$-disc with its standard symplectic form. The group $Symp(\Delta^{2n}, \omega_{std})$ acts on $(\Omega_{\Delta^{2n}}^{ch}, \mathcal{L}_\pi^{ch})$, and $(\Omega_M^{ch}, \mathcal{L}_\pi^{ch})$ is obtained by reduction along this torsor. Recalling that Brylinksi's result computes the conformal weight 0 subspace, we are thus reduced to showing that the inclusion of the weight 0 subspace

$$(\Omega_{\Delta^{2n}}, \mathcal{L}_{\omega_{std}^{-1}}) \longrightarrow (\Omega_{\Delta^{2n}}^{ch}, \mathcal{L}_{\omega_{std}^{-1}}^{ch})$$

is a quasi-isomorphism.

It suffices to handle the case of $n = 1$. Now, let x^1, x^2 be coordinates on Δ^2. Consider the vector

$$H := x_1^1 \phi_0^2 - x_1^2 \phi_0^1 \in \Omega_{\Delta^2}^{ch}.$$

This has conformal weight 1 and cohomological degree 1. Observe now the following simple identity:

$$H_{(0)}(y_1^1 \psi_1^2 - y_1^2 \psi_1^1) = x_1^i y_1^i + \phi_1^i \psi_1^i,$$

where of course we recognize $x_1^i y_1^i + \phi_1^i \psi_1^i$ as the Virasoro vector L. Let us note further that we have, by construction of $\mathcal{L}_{\omega^{-1}}^{ch}$, that

$$(y_1^1 \psi_1^2 - y_1^2 \psi_1^1)_{(1)} = \mathcal{L}_{\omega^{-1}}^{ch}.$$

Now, $H_{(0)}$ acts as a derivation with respect to all (j)-products, whence we compute

$$\left[H_{(0)}, \mathcal{L}_{\omega^{-1}}^{ch} \right] = L_{(1)},$$

so that $L_{(1)}$ acts trivially on cohomology. Now, this operator is simply the (diagonalizable) grading operator for the conformal grading, whence the theorem is established. □

REMARK. When M is Calabi–Yau such that the associated volume form vol_M is *compatible* with π in the sense that $\mathcal{L}_\pi(vol_M) = 0$, then the $\mathcal{N} = 2$ algebra acts on Ω_M^{ch} compatibly with the differential $\mathcal{L}_\pi^{ch}$. In particular, this is the case for symplectic varieties, in which case there is, in fact, an $\mathcal{N} = 4$ action, and the operator H above is induced from this structure.

3.1 An example
We regret that we are not able to produce any genuinely nontrivial computations of chiral Poisson homology as yet—nonetheless, we can still see that it is not a trivial enhancement of its classical counterpart. Indeed, we have seen that there are no new invariants of (M, π) produced in chiral Poisson homology when π is nondegenerate, and further that, regardless of π, the S^1-invariants produce nothing new either, so we had better check then that there is at least some additional richness to $H_\pi^{ch}(M)$.

In order to do so, we return to the example of $M = \mathbf{C}^2$ equipped with the form $\pi := x^2 \partial_1 \partial_2$. Recall that above we used the $\mathbf{C}^*$ action on this Poisson variety to argue that all the cohomology was the weight 0 subspace with respect to this action (because the associated infinitesimal action is given by a Hamiltonian of a function with respect to π). Now, one might hope that this argument could be applied to easily compute the chiral Poisson homology of $(\mathbf{C}^2, x^2 \partial_1 \partial_2)$. The calculus developed in the previous section readily implies that we have

$$\left[\mathcal{L}_\pi^{ch}, (dx^1)_{(-1)} \right] = \mathcal{L}_0^\eta,$$

and, of course, $\mathcal{L}_0^\eta$ is still the grading operator on homology. However, the weight 0 subspace is now huge, as the annihilation vectors y^2 and ψ^2 are now of negative weight.

This at least cuts down the size of the space we must compute with somewhat, and, for example, we can now compute the conformal weight 1 component. Let us enumerate the $\mathbf{C}^*$ weight 0 variables, and they are generated over $\mathbf{C}[x_0^1, \phi_0^1]$ by the vectors:

- $x_0^2 \psi_1^2$ in cohomological degree -1,
- $x_0^2 y_1^2, \phi_0^2 \psi_0^2, x_1^1, y_1^1$ in cohomological degree 0,

- $y_1^2\phi_0^2$, ϕ_1^1 in cohomological degree 1,
- and vanishing in all other cohomological degrees.

We can compute by hand the following differentials:

- $y_1^2\phi_0^2 \mapsto y_1^1$, $\phi_1^1 \mapsto x_0^2 y_1^2 + \phi_0^2\psi_1^2$,
- $x_0^2 y_1^2 \mapsto -\psi_1^1$, $\phi_0^2\psi_1^2 \mapsto \psi_1^1$, $x_1^1 \mapsto x_0^2\psi_1^2$, $y_1^1 \mapsto 0$,
- and vectors of cohomological degree -1 map to 0 for trivial reasons.

Staring at the above one deduces that there is no cohomology in conformal weight 1 and it is perhaps thus tempting to conjecture that the same is true in all nonzero conformal weights. This is in fact *false*, and there are nonzero classes already in conformal weight 2. One such is given by the vector

$$v := x_1^1\psi_1^1 - x_1^2\psi_1^2,$$

as the reader can confirm. Taking products of the above example, one thus sees that there can be an arbitrarily long string of vanishing cohomology groups in conformal weights $1, 2, \ldots, N$ before some nonzero classes show up.

REMARK. When M is endowed with a π-compatible volume form, then there is a Virasoro element L in (global sections of) Θ_M^{ch} which represents a class in $H_\pi^{ch}(M)$, the chiral Poisson cohomology. Now, the $\{0\}$ product of this class acting on cohomology (resp. the operator $\mathcal{L}_0$ acting on homology) give the gradings on cohomology and homology, respectively, whence we see that this class is precisely the obstruction to nonzero classes of nonzero conformal weight in the case of Calabi–Yau Poisson varieties.

REMARK. It would be interesting to compute chiral Poisson homology in some (perhaps) manageable cases of interest, for example, a simple Lie algebra **g** with its Kostant–Kirillov form.

REFERENCES

[1] A. Beilinson and V. Drinfeld, *Chiral Algebras*, Amer. Math. Soc. Colloq. Publ. **51**, Amer. Math. Soc., Providence, 2004, 375 pp.

[2] E. Bouaziz, *Semi-infinite non-commutative Hodge theory and LG-models*, preprint, arXiv:1806.00854

[3] J. L. Brylinski, *A differential complex for Poisson manifolds*, J. Differential Geom. **28** (1988), 93–114.

[4] P. Etingof and T. Schedler, *Poisson traces, D-modules and symplectic resolutions*, Lett. Math. Phys. **108** (2018), 633–678.

[5] V. Gorbounov and F. Malikov. Moscow Mathematical Journal. 4, 3 p. 729–779.

[6] R. Heluani, *Florence lecture notes*, available at https://w3.impa.br/∼heluani/

[7] M. Kontsevich, *XI Solomon Lefschetz Memorial Series: Hodge Structures in Non-Commutative Geometry*, Contemp. Math. **462**, Amer. Math. Soc., Providence, 2008.

[8] B. H. Lian and G. J. Zuckerman, *New perspectives on the BRST algebraic structure of string theory*, Commun. Math. Phys. **154** (1993), 613–646.

[9] A. Lichnerowicz, *Les Variétés de Poisson et Leurs algébres de Lie associée*, J. Differential Geom. **12** (1977), 253–300.

[10] F. Malikov, V. Schechtman, and A. Vaintrob, *Chiral de Rham complex*, Commun. Math. Phys. **204** (1999), 439–473.

[11] T. Pantev, B. Toen, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328.

[12] D. Tamarkin and B. Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjecture*, Methods Funct. Anal. Topology **1** (2001).

Emile Bouaziz
MPIM Bonn and Cambridge University
emile.g.bouaziz@gmail.com