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FIXED POINTS AS EQUATIONS AND SOLUTIONS

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Introduction. In the literature about the definition of data types there exist many approaches using some concept of fixed point. Wand [**13**] and Lehmann, Smyth [**9**] e.g. constructed data types as least fixed points of functors $F:K \to K$. Arbib and Manes [**3**] showed that some data types turn out to be the greatest fixed points of such endofunctors. In this paper we regard least and greatest fixed points that have a given property.

We study the concepts of an algebra and an equation of type F, where $F:K \to K$ is a functor. An algebra consists of an object A of K and a morphism $\alpha:FA \to A$; whereas, an equation consists of an object X and a morphism $f:X \to FX$. For example, if $K = \text{Set}$ and F is the cartesian-square functor ($FX = X \times X$; $Ff = f \times f$), then an algebra is a groupoid, $x:A \times A$. An equation $f:X \to X \times X$ can be replaced by the following collection of usual (universal algebraic) equations:

$$x = y + z \text{ if and only if } f(x) = (y, z)$$

(for all $x \in X$). A solution of this collection of equations is a groupoid $(A, *)$ and a map $a:X \to A$ such that

$$x = y + z \text{ implies } a(x) = a(y) * a(z).$$

We study the duality between *universal solutions* of an equation (X, f) (i.e., solutions through which each solution factors uniquely) and *universal equations* of an algebra (A, α) (i.e., equations, solved by (A, α) and such that each equation with this property factors uniquely through it).

It turns out that each universal solution as well as each universal equation is a fixed point of F (i.e., an object X, isomorphic to FX). Thus, a new role of fixed points of functors comes to the light. On the other hand, we present a construction of universal solutions (or universal equations) which yields a construction of fixed points of F. For example, if F is a constructive varietor (i.e., free algebras exist constructively, see [**12**]) then the universal-solution construction always converges. This is connected to a recent investigation of M. A. Arbib and E. G. Manes, announced by the latter in [**3**]. Using the free algebra construction, we answer affirmatively the problem of E. G. Manes whether each Peano algebra is free. We are grateful to the referee for his valuable comments, particularly those concerning Section 2.2.

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1. Free algebras.

1.1. Let $F:K \to K$ be a functor. An F-algebra is a pair (A, α), where A is an object and $\alpha:FA \to A$ is a morphism.

Given F-algebras (A, α) and (B, β), a *homomorphism* is a K-morphism $h:A \to B$ for which the following commutes:

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Fh"'] & A \arrow[d, "h"] \\ FB \arrow[r, "\beta"] & B \end{tikzcd} \tag{1.1}$$

The category of F-algebras and homomorphisms is denoted by $\mathrm{Alg}(F)$. See **[4]**, **[2]**.

Examples. (i) Let K be the category of sets. For each type Σ (i.e., a set of operation symbols with prescribed arities $\mathrm{ar}\,\sigma \in \mathrm{Card}$ for $\sigma \in \Sigma$) we denote by

$$H_\Sigma:\mathrm{Set} \to \mathrm{Set}$$

the following functor – a coproduct of powers. On objects X put

$$HX = \coprod_{\sigma \in \Sigma} X^{\mathrm{ar}\sigma};$$

We denote the element $(x_i)_{i<k}$ in the σ-summand $X^{\mathrm{ar}\sigma}$ (with $k = \mathrm{ar}\,\sigma$) by

$$\sigma(x_i)_{i<k} \in H_\Sigma X$$

(to distinguish various summands). On morphisms $f:X \to Y$ put

$$H_\Sigma f:H_\Sigma X \to H_\Sigma Y;\ \sigma(x_i)_{i<\mathrm{ar}\sigma} \mapsto \sigma(f(x_i))_{i<\mathrm{ar}\sigma}.$$

An H_Σ-algebra consists of a set A and a map

$$\alpha: \coprod_{\alpha \in \Sigma} A^{\mathrm{ar}\sigma} \to A;$$

denoting by α_σ the restriction of α to $A^{\mathrm{ar}\sigma}$ ($\sigma \in \Sigma$), we see that α_σ is an operation on A of arity $\mathrm{ar}\,\sigma$. Hence, H_Σ-algebras are precisely the universal algebras of type Σ. Also homomorphisms in the above sense coincide with those of universal algebra. Thus, $\mathrm{Alg}(H_\Sigma)$ is the usual category of universal algebras of type Σ.

(ii) We denote by

$$P:\mathrm{Set} \to \mathrm{Set}$$

the power-set functor, defined on objects X by

$$PX = \exp X = \{A;\ A \subseteq X\}$$

and on morphisms $f:X \to Y$ by

$$Pf\colon \exp X \to \exp Y;\ Pf(A) = \{f(a);\ a \in A\}.$$

A P-algebra is a set A with an operation $\alpha:\exp A \to A$. This can be viewed as a "complete semilattice without the associativity and idempotency axioms". More precisely, the category of complete semilattices and complete homomorphisms is a full subcategory of Alg(P).

1.2. A *free F-algebra* generated by an object I of K is an F-algebra $(I^{\#}, \phi)$ together with a morphism $\eta:I \to I^{\#}$ universal in the usual sense: for each F-algebra (A, α) and each morphism $f:I \to A$ there exists a unique homomorphism

$$f^{\#}:(I^{\#}, \phi) \to (A, \alpha)$$

with $f = f^{\#} \cdot \eta$.

(1.2)
$$\begin{tikzcd}
FI^{\#} \arrow[r, "\phi"] \arrow[d, "Ff\#"'] & I^{\#} \arrow[d, "f^{\#}"] & I \arrow[l, "\eta"'] \arrow[dl, "f"] \\
FA \arrow[r, "\alpha"] & A &
\end{tikzcd}$$

A functor F is called a *varietor* if each object I generates a free F-algebra.

Example. H_Σ is a varietor. The free algebra generated by $\emptyset$ (i.e., the initial algebra) is the Σ-algebra of all finite-path Σ-labelled trees. Σ-labelled means that each node with k successors is labelled by $\tau \in \Sigma$ with ar $\tau = k$. Each set I generates the free Σ-algebra which is the initial Σ' algebra, where Σ' is the type obtained from Σ by adjoining elements of I as nullary symbols.

If the free F-algebra exists then $I^{\#} = I + FI^{\#}$ is a coproduct with injections $\phi:FI^{\#} \to I^{\#}$ and $\eta:I \to I^{\#}$ (see **[4]**).

If I is the initial object of K then the free F-algebra over I is the *initial F-algebra*, i.e., an F-algebra (A_0, α_0) such that for each F-algebra (A, α) there exists a unique homomorphism

$$h:(A_0, \alpha_0) \to (A, \alpha).$$

The initial algebra (A_0, α_0), if it exists, has the property that $\alpha_0:FA_0 \to A_0$ is an isomorphism. Thus, A_0 is a *fixed point* of F (i.e., an object, isomorphic to its F-image). This shows, for example, that the initial P-algebra does not exist: since card $X <$ card PX for each set X, P has no fixed point.

The free F-algebra over I is precisely the initial $(F + C_I)$-algebra, where

$$(F + C_I)X = FX + I \text{ and } (F + C_I)f = Ff + 1_I.$$

This follows from the fact that an $(F + C_I)$-algebra (Q, δ) consists of an F-algebra $FQ \to Q$ and a morphism $I \to Q$.

1.3. *The free-algebra construction.* Let K have colimits of chains (including the initial object $\perp = \operatorname{colim} \emptyset$). For each functor $F:K \to K$ we denote by $F^k \perp$ ($k \in \mathrm{Ord}$) the following chain (with connecting maps $w_{k,n}:F^k \perp \to F^n \perp$):

I. $F^0\perp = \perp$ and $F^1 \perp = F \perp$; $w_{0,1}$ is the unique map.

II. $F^{k+1}\perp = F(F^k \perp)$ and $w_{k+1,n+1} = Fw_{k,n}$;

III. Given $F^k \perp$ for all $k < n$, where n is a limit ordinal, then

$$F^n \perp = \operatorname{colim} F^k \perp$$

(with connecting maps $w_{k,n}$).

If this chain stops after k steps, i.e., if $w_{k,k+1}$ is an isomorphism, then

$$(F^k \perp, w_{k,k+1}^{-1})$$

is the initial F-algebra.

Consequently, if F is such that for each object I the chain $(F + C_I)^k \perp$ stops, then F is a varietor. This has been proved in [**1**]. Such F are called constructive varietors.

1.4. Given a class M of monos in K, a *subalgebra* of an F-algebra (Q, δ) is an M-homomorphism

$$m_0: (Q_0, \delta_0) \to (A, \delta).$$

An M-subobject $m:I \to Q$ *generates* the F-algebra (Q, δ) if each subalgebra m_0 with $m \leqq m_0$ is an isomorphism. If M contains all coproduct injections $(A \to A + B$ for $A, B \in K)$, then each free algebra (Q, δ) over $\eta:I \to Q$ has the following *Peano properties*:

(i) $Q = I + FQ$ with coproduct injections η and δ;

(ii) (Q, δ) is generated by $\eta:I \to Q$.

E. G. Manes [**10**] stated the problem whether for $K = \mathrm{Set}$, Peano properties imply freeness. We answer this in the affirmative by using the proof techniques of [**12**].

A class M of monos is *constructive* provided that

C_1: M is closed under colimits of chains in K;

C_2: M contains all coproduct injections;

C_3: K is M-well-powered

THEOREM. *Let M be a constructive class, and let F be a varietor preserving M-monos. Then*

(i) *a fixed point (Q, δ) of F is an initial F-algebra if and only if (Q, δ) has no proper subalgebra;*

(ii) *each F-algebra with Peano properties is free;*

(iii) *F is a constructive varietor.*

Proof. We prove (i) by showing that given a fixed point (Q_0, δ_0) with no proper subalgebra, then the chain $F^k \perp$ stops and the resulting initial algebra is isomorphic to (Q_0, δ_0). By applying this to the functor $F + C_I$ ($I \in K$), we conclude that (ii) and (iii) hold too.

Define M-monos

$$m_k:F^k \perp \to Q_0 \ (k \in \mathrm{Ord})$$

compatible with the maps $w_{k,n}$ of 1.3 by transfinite induction:

$$m_0:\perp \to Q_0 \text{ is the unique map;}$$

$$m_{k+1} = \delta_0 \cdot F_{m_k}:F^{k+1} \perp \to Q_0;$$

$$m_n = \operatorname*{colim}_{k<n} m_k:\mathrm{colim}\ F^k \perp \to Q_0 \text{ (limit ordinal } n).$$

The compatibility, i.e., $m_k = m_{k+1} \cdot w_{k,k+1}$ is easily seen by induction: for $k = 0$ there is nothing to prove, the limit step is also clear and if

$$m_{k-1} = m_k \cdot w_{k-1,k}$$

then

$$m_k = \delta_0 \cdot Fm_{k-1} = \delta_0 \cdot Fm_k \cdot Fw_{k-1,k} = m_{k+1} \cdot w_{k,k+1}.$$

Also $m_k \in M$ is clear by induction since M is constructive: for $k = 0$, apply C_2 to $A = \perp$; for the limit step use C_1; the isolated step is obvious since F preserves M-monos.

Since K is M-well-powered, there exists k with m_{k+1} equivalent to m_k $(= m_{k+1} \cdot w_{k,k+1})$; it follows that $w_{k,k+1}$ is an isomorphism. Consequently, $(F^k \perp, \overset{-1}{w}_{k,k+1})$ is the initial F-algebra. Since (Q_0, δ_0) has no proper subalgebras and

$$m_k:(F^k \perp, \overset{-1}{w}_{k,k+1}) \to (Q_0, \delta_0)$$

is subalgebra, m_k is an isomorphism and hence, (Q_0, δ_0) is also the initial F-algebra.

COROLLARY. *For each set functor F, free algebras are precisely those with the Peano properties.*

In fact, the class M of all monos is constructive, and each functor preserves monos (except, possibly, the empty monos, but this makes no difference, as proved in **[12]**).

2. Universal solutions.

2.1. We recall the following notions introduced in [**11**]. Let $F:K \to K$ be a functor. An *F-equation* is a pair (X, f), where X is an object of K and $f:X \to FX$ is a morphism.

Let (X, f) and (Y, g) be F-equations. A morphism $p:X \to Y$ is called a morphism of F-equations if (2.1) commutes:

$$\text{(2.1)}\qquad \begin{array}{ccc} X & \xrightarrow{f} & XF \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pF} \\ Y & \xrightarrow[g]{} & YF \end{array}$$

Let Equ(F) denote the category of F-equations.

Remark. Every functor $F:K \to K$ defines a functor on the dual category K^{op}. Define $F^{\mathrm{op}}:K^{\mathrm{op}} \to K^{\mathrm{op}}$ on objects by

$$F^{\mathrm{op}}X = FX$$

and on morphisms $f:X \to Y$ (in K^{op}; i.e., morphisms $f:Y \to X$ in K) by

$$F^{\mathrm{op}}f = Ff.$$

Then F-equations are just F^{op}-algebras and Equ(F) is just the category Alg(F^{op}).

We purposely avoid the term "F-coalgebra", replacing that of F-equation, since the basic example H_Σ would not work. A coalgebra of type Σ consists of a set X and co-operations $\sigma:X \to X_n$ $(= X + X + \ldots$ n-fold coproduct) for each $\sigma \in \Sigma$ of arity n. This can be viewed as a single morphism

$$\alpha:X \to \coprod_{\sigma\in\Sigma} X_{\mathrm{ar}\sigma}.$$

Hence, a Σ-coalgebra is something other than an H_Σ-equation.

A relation between Σ-coalgebras and Σ-equations is discussed in [**11**] in more detail.

A *solution* of an F-equation (X, f) is an F-algebra (A, α) together with a morphism $a:X \to A$ such that $a = \alpha \cdot Fa \cdot f$. We write $a:X \to (A, \alpha)$ for short.

$$(2.2)\qquad \begin{array}{ccc} X & \xrightarrow{f} & FX \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle Fa} \\ A & \xleftarrow{\alpha} & FA \end{array}$$

Examples. (i) For $F = H_\Sigma:\text{Set} \to \text{Set}$ we get the most intuitive example. We can view $H_\Sigma X$ as the set of simple Σ-expressions over X (trees of height 1). An H_Σ-equation $f:X \to H_\Sigma X$ assigns Σ-expressions over X to variables $x \in X$. An H_Σ-algebra $\alpha:H_\Sigma A \to A$ evaluates Σ-expressions over A (with values in A). Now a solution $a:X \to A$ is an assignment of values to variables such that the expression assigned to x by f evaluates to $a(x)$:

$$f(x) = \sigma(x_i)_{i<n} \quad \text{implies}$$

$$a(x) = \sigma(a(x_i))_{i<n}.$$

This is the usual meaning of a solution of an equation (of a system of equations).

(ii) A P-equation is a set X and a map $f:X \to PX$. This can be viewed as a directed graph (binary relation) $\rho \subseteq X \times X$ where

$$f(x) = \{y \in X;\, y\rho x\}, \quad \text{or,}$$

$$\rho = \{(x, y);\, y \in X,\, x \in f(y)\}.$$

A solution of a graph (X, ρ) is a P-algebra (A, α) and an interpretation $a:X \to A$ of variables such that, for each $x \in X$,

$$a(x) = \alpha\{a(y);\, y \in X,\, y\rho x\}.$$

Remark. In **[11]** it is shown that finite automata may also be regarded as F-equations where F is a special case of H_Σ. If I is the input alphabet of an automaton then the set $P(I^*)$ of languages over I is given the structure of a Σ-algebra such that we achieve the usual Arden-equations of the automaton.

2.2. For each equation $f:X \to FX$, all solutions form a full subcategory S_f of the category of $(F + C_X)$-algebras, see 1.2. The initial object of S_f is called the *universal solution* of f. This is a solution of f through which each solution uniquely factors.

PROPOSITION. *Let K be a complete and cowell-powered category. Each epi-preserving varietor F has universal solution for every equation, and this solution is a fixed point of F.*

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Proof. Each morphism in K factors as an epi followed by an extremal mono, see [**5**]. Given an equation $f:X \to FX$, for each solution $a:X \to (A, \alpha)$ we factor the free homomorphism as

$$a^{\#} = m \cdot e:(X^{\#}, \phi) \to (A, \alpha).$$

We use the diagonal fill-in to obtain a subalgebra $m:(A_0, \alpha_0) \to (A, \alpha)$:

$$(2.3)\qquad \begin{tikzcd} FX^{\#} \arrow[r, "Fe"] \arrow[d, "\phi"'] & FA_0 \arrow[d, "Fm"] \arrow[ddl, "\alpha_0"'] \\ X^{\#} \arrow[d, "e"'] & FA \arrow[d, "\alpha"] \\ A_0 \arrow[r, "m"'] & A \end{tikzcd}$$

Then $a_0 = e \cdot \eta:X \to (A_0, \alpha_0)$ is obviously a solution through which the given solution factors. Since A_0 is a quotient of $X^{\#}$ and K is cowell-powered, all these sub-solutions clearly have a small choice-set (with respect to isomorphism in S_f). By the (dual of) Adjoint Functor Theorem, S_f has an initial object.

Let $a:X \to (A, \alpha)$ be the universal solution of f. We prove that α is an isomorphism.

$$(2.4)\qquad \begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "a"] \arrow[ddl, bend right, "Fa \cdot f"'] & FX \arrow[d, "Fa"] \arrow[ddr, bend left, "F(Fa \cdot f)"] & \\ & A \arrow[dl, dashed, "p"'] & FA \arrow[l, "\alpha"'] \arrow[dr, dashed, "Fp"] & \\ FA & & & F^2A \arrow[lll, "F\alpha"'] \end{tikzcd}$$

Consider the F-algebra $(FA, F\alpha)$. Then $Fa \cdot f:X \to (FA, F\alpha)$ is a solution because

$$\begin{aligned} F\alpha \cdot F(Fa \cdot f)f &= F(\alpha \cdot Fa \cdot f)f \\ &= Fa \cdot f. \end{aligned}$$

Thus, there is exactly one homomorphism $p:(A, \alpha) \to (FA, F\alpha)$ with $p \cdot a = Fa \cdot f$.

Now we have

$$(\alpha p)\alpha = \alpha(p\alpha) = \alpha(F\alpha)(Fp) = \alpha F(\alpha p)$$

so αp is an endomorphism $(A, \alpha) \to (A, \alpha)$ in $\mathrm{Alg}(F)$. Further

$$(\alpha p)a = \alpha(p\alpha) = \alpha \cdot Fa \cdot f = a$$

and hence αp is an endomorphism in S_f. Thus, $\alpha p = 1_A$ (identity on A). Next we have

$$p\alpha = F(\alpha p) = F(1_A) = 1_{FA}$$

and thus $p = \alpha^{-1}$.

Remark. The iteration of the equation $\perp \to F\perp$ carried through in 1.3 was generalized in **[11]** to an arbitrary equation $f:X \to FX$. Define $F^k X$ (for $k \in \mathrm{Ord}$) and $w_{k,n}:F^k X \to F^n X$ by the following induction:

I. $w_{0,1} = f:X \to FX$;
II. $F^{k+1}X = F(F^k X)$ and $w_{k+1,n+1} = Fw_{k,n}$;
III. given $F^k X$ for all $k < n$, where n is a limit ordinal, then

$$F^n X = \operatorname*{colim}_{k<n} F^k X$$

(with connecting maps $w_{k,n}$).

If this chain stops after k steps, i.e., $w_{k,k+1}$ $(= F^k f)$ is an isomorphism, then

$$w_{0,k}:X \to (F^k X, (F^k f)^{-1})$$

is the universal solution.

F is said to have *constructive universal solutions* if the chain stops for each equation. The following is a constructive counterpart of the preceding proposition.

2.3. PROPOSITION. *Let K be a complete and cowell-powered category. Each epi-preserving constructive varietor has constructive universal solutions.*

Proof. Let $f:X \to FX$ be an equation. The free-algebra construction over X can be described as the following chain $W:\mathrm{Ord} \to K$:

$$\bar{W}_0 = X;\ \bar{W}_{k+1} = F\bar{W}_k + X \text{ and } \bar{W}_n = \operatorname*{colim}_{k<n} \bar{W}_k$$

(n limit ordinal) and analogously on morphisms: $\bar{w}_{01}$ is the coproduct injection;

$$\bar{w}_{k+1,n+1} = Fw_{k,n} + 1_X$$

and for n limit ordinal, $\bar{w}_{k,n}$ are the colimit injections. We know that $\bar{W}$ stops. Since K is cowell-powered, each quotient chain of $\bar{W}$ also stops (possibly later than $\bar{W}$). Thus, it is sufficient to find an epitransformation

$$e_k:\bar{W}_k \to F^k X = W_R.$$

Put

$$e_0 = 1_X : X \to X$$

and given e_k, let

$$e_{k+1} : F\bar{W}_k + X \to F(F^k X)$$

have components Fe_k and $w_{0,k+1} : X \to F^{k+1}X$. (If e_k is epi, then also Fe_k is epi and hence, so is e_{k+1}.)

For limit ordinals, the morphism

$$e_n : \operatorname*{colim}_{k<n} \bar{W}_k \to \operatorname*{colim}_{k<n} W_k$$

is uniquely determined by the naturality condition:

$$\text{(2.5)} \qquad \begin{tikzcd} \bar{W}_i \arrow[r, "\bar{w}_{ij}"] \arrow[d, "e_i"'] & \bar{W}_j \arrow[d, "e_j"] \\ W_i \arrow[r, "w_{ij}"'] & W_j \end{tikzcd} \qquad (i \leqq j)$$

Moreover if each $e_k (k \leqq n)$ is epi then e_n is obviously epi. It remains to prove the naturality condition. We can restrict ourselves to $j = i + 1$, and we use induction in i. The case $i = 0$ is clear since the second component of e_1 is $w_{0,1} = f$:

$$\text{(2.6)} \qquad \begin{tikzcd} X \arrow[r, "\text{injection}"] \arrow[d, "1_X"'] & FX + X \arrow[d, "e_1"] \\ X \arrow[r, "f"'] & FX \end{tikzcd}$$

For each isolated ordinal i, the morphism

$$w_{i,i+1} \cdot e_i = Fw_{i-1,i} \cdot e_i : F\bar{W}_{i-1} + X \to F$$

has components

$$Fw_{i-1,i} \cdot Fe_{i-1} = F(e_i \cdot \bar{w}_{i-1,i})$$

and

$$Fw_{i-1,i} \cdot w_{0,i} = w_{0,i+1}.$$

These are also the components of

$$_{i+1} \cdot \overline{w}_{i,i+1} = e_{i+1} \cdot (F\overline{w}_{i-1,i} + 1_X).$$

For limit ordinals i the situation is clear.

This concludes the proof.

2.4. *Example* (see **[11]**). Let

$$f:X \rightarrow H_\Sigma X$$

be an equation in the category of sets. Denote by Σ' the type, obtained from Σ by adding a new nullary symbol τ_x for each $x \in X$. The universal solution of f is the initial algebra of the variety of all Σ'-algebras, satisfying the equation

$$\tau_x = \sigma(\tau_{x_i})_{i<n}$$

for each $x \in X$ with $f(x) = \sigma(x_i)_{i<n}$.

More generally, let K be a concrete category (a category of structured sets) which has both products and coproducts constructed on the level of sets. For instance, K = posets or K = topological spaces. For each type Σ we obtain a functor $H_\Sigma:K \rightarrow K$, defined as in sets. Then H_Σ-algebras are structured (universal) Σ-algebras. The universal solution of an equation will be again an initial algebra of some variety of structured algebras.

2.5. *Example*. No P-equation has a universal solution since P has no fixed point. Consider its subfunctor P_0, defined by

$$P_0X = PX - \{\phi\}.$$

A P_0-equation is a graph (X, ρ) such that for each $x \in X$ the set

$$f(x) = \{y \in X; x\rho y\}$$

is non-void. Let us call an equivalence $\sim$ on the set X *simple* if it has at least two classes and for arbitrary nonequivalent points x_1, x_2 we have

$$\{[y] \in X/\sim; x_1\rho y\} \neq \{[y] \in X/\sim; x_2\rho y\}.$$

A P_0-equation (X, ρ) has a universal solution if and only if it has no simple equivalence.

Proof. Let (X, ρ) have a universal solution

$$a_0:X \rightarrow (A_0, \alpha_0).$$

Since A_0 is a fixed point of P_0, it has at most one point (indeed, card $A_0 \geqq 2$ implies card $P_0A_0 = 2^{\text{card}A_0} - 1 >$ card A_0). Hence, a_0 is a constant map; consequently, for each solution $a:X \rightarrow (A, \alpha)$, the map a is constant. Assume that, nevertheless, there exists a simple equivalence $\sim$ on (X, ρ).

Put $A = X/\sim$ and choose an arbitrary map

$$\alpha:P_0A \to A$$

such that for each $x \in X$

$$\alpha\{[y]; x\rho y\} = [x].$$

The simplicity of $\sim$ guarantees that such α exists. Clearly, the canonical map $a:X \to X/\sim = A$ defines a solution $a:X \to (A, \alpha)$ of (X, ρ). Yet, a is non-constant, a contradiction.

Conversely, let (X, ρ) have no simple equivalence. Then for each solution $a:X \to (A, \alpha)$ the map a is constant. Else, the kernel equivalence of a (defined by $x_1 \sim x_2$ if and only if $a(x_1) = a(x_2)$) is clearly simple: if $x_1 \nsim x_2$ then

$$\alpha\{a(y); x_1\rho y\} = a(x_1) \neq a(x_2) = \alpha\{a(y); x_2\rho y\},$$

which implies

$$\{a(y); x_1\rho y\} \neq \{a(y); x_2\rho y\},$$

hence

$$\{[y]; x_1\rho y\} \neq \{[y]; x_2\rho y\}.$$

Since each solution is a constant map, the universal solution is obviously the singleton P_0-algebra.

2.6. *Remark*. If F preserves α-colimits for some infinite cardinal α (i.e., F preserves colimits of all diagrams $D:\alpha \to K$) then the solution construction stops after α steps. Indeed, since α is a limit ordinal, we have

$$\begin{aligned} F^\alpha X &= \operatorname*{colim}_{k<\alpha} F^{k+1}X = \operatorname*{colim}_{k<\alpha} F(F^kX) \\ &= F \operatorname*{colim}_{k<\alpha} F^kX = F(F^\alpha X) = F^{\alpha+1}X. \end{aligned}$$

Example. The functor H_Σ: Set $\to$ Set with Σ a finitary type preserves ω-colimits. Thus, the universal solution A_0 of an equation

$$f:X \to H_\Sigma X$$

is obtained after a countable iteration:

$$X \xrightarrow{f} H_\Sigma X \xrightarrow{H_\Sigma f} H(H_\Sigma X) \xrightarrow{H_\Sigma^2 f} \ldots \operatorname{colim} = A_0.$$

For each infinitary type Σ there exist equations such that countable iterations are not sufficient. Yet, H_Σ preserves α-colimits for any regular cardinal α, larger than all arities. Hence

$$A_0 = H_\Sigma^\alpha x.$$

2.7. *Counter-examples.* (i) Universal solutions exist but the construction does not stop.

Let K be the ordered class of all ordinals with a largest element ∞ added. Let $F:K \to K$ be the functor defined by

$$Fi = i + 1 \text{ for each ordinal } i; \ F\infty = \infty.$$

This functor has only one F-algebra;

$$1_\infty:F\infty \to \infty.$$

Therefore, this algebra is the universal solution to every equation. Yet, neither the solution construction nor the free-algebra construction stop for $i \neq \infty$.

(ii) Universal solutions exist (constructively) but free algebras do not.

Let K be the category of graphs. Objects are pairs (X, ρ), where X is a set and $\rho \subseteq X \times X$ is a binary relation. Morphisms

$$f:(X, \rho) \to (Y, \sigma)$$

are maps such that

$$x_1 \rho x_2 \text{ implies } f(x_1)\sigma f(x_2).$$

Note that if the graph (Y, σ) is discrete (i.e., $\sigma = \emptyset$) then so is (X, ρ). Define a functor $F:K \to K$, using the power-set functor P (see 1.1 example (ii)); on objects (X, ρ) put

$$F(X, \rho) = (PX, \emptyset) \text{ if } \rho \neq \emptyset; \ F(X, \emptyset) = (\emptyset, \emptyset);$$

on morphisms

$$f:(X, \rho) \to (Y, \rho)$$

put

$$Ff = Pf \text{ if } \rho \neq \emptyset$$

(hence, $\sigma \neq \emptyset$);

$$Ff = \text{the void map if } \rho = \emptyset.$$

This functor has a unique equation:

$$1_\emptyset:(\emptyset, \emptyset) \to (\emptyset, \emptyset) = F(\emptyset, \emptyset).$$

Indeed, for any graph (X, ρ) which is not discrete, the existence of a morphism $F(X, \rho) \to (X, \rho)$ implies that also $F(X, \rho)$ is not discrete, yet, $F(X, \rho) = (PX, \emptyset)$; a contradiction for any discrete graph $(X, \emptyset)$ with $X \neq \emptyset$; there again is no morphism

$$f:(X, \emptyset) \to F(X, \emptyset) = (\emptyset, \emptyset).$$

The unique equation has a universal solution: the algebra (A, α) where

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$$A = (\emptyset, \emptyset) \text{ and } \alpha = 1_\emptyset:(\emptyset, \emptyset) \to (\emptyset, \emptyset).$$

The functor F has no free algebra over a non-discrete graph $I = (X, R)$. Indeed, if $I^{\#}$ existed, it could not be a discrete graph, since $I^{\#} \cong I + FI^{\#}$ (see 1.2); but then the cardinality of $FI^{\#}$ is larger than that of $I^{\#}$, hence, $I^{\#}$ cannot be the coproduct (= disjoint union) of I and $FI^{\#}$.

Situations, as described in these counter-examples, cannot occur at least in the category of sets:

2.8. THEOREM. *The following conditions are equivalent for each functor* f:Set $\to$ Set:

(i) *F has universal solutions*;

(ii) *F has constructive universal solutions*;

(iii) *F is a varietor*;

(iv) *F has arbitrarily large fixed points* (*i.e., for each cardinal α there exists a fixed point of F of power* $\geqq \alpha$) *or F is constant.*

Proof. The equivalence of (iii) and (iv) is proved in [7]; moreover, each varietor in Set is constructive (see 1.4). Hence (iii) implies (ii) by 2.3: in Set, each epi splits, and thus, all functors preserve epis. Since ii $\Rightarrow$ i is trivial, it remains to prove i $\Rightarrow$ iv.

Thus, let F be a non-constant functor with universal solutions. For each cardinal α we are to exhibit a fixed point of F of power $\geqq \alpha$. It is proved in [**6**] that, since F is non-constant, there exists a cardinal β such that for any set X

$$\text{card } X \geqq \beta \text{ implies card } FX \geqq \text{card } X.$$

Choose any set X of power $> \max(\alpha, \beta)$. Then card $X \leqq$ card FX, hence, there exists a one-to-one equation

$$f:X \to FX.$$

Let

$$X \xrightarrow{a_0} (A_0, \alpha_0)$$

be its universal solution. Then A_0 is a fixed point of F (2.2) and it suffices to prove that a_0 is one-to-one: then

$$\text{card } A_0 \geqq \text{card } X > \alpha.$$

Since $X \neq \emptyset$ (indeed, card $X > \max(\alpha, \beta) \geqq 0$), the map f splits: we can choose any $x_0 \in X$ and define $g:FX \to X$ by $g(f(x)) = x$ for each $x \in X$; $g(y) = x_0$ for each $y \in FX - f(x)$ then

$$g \cdot f = 1_X.$$

Then $1_X:X \to (X, g)$ is a solution of the equation f:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1_X"'] & FX \arrow[d, "1_{FX}"] \\ X \arrow[r, "g"'] & FX \end{tikzcd} \tag{2.7}$$

Hence, there exists $f^{\#}:A_0 \to X$ with $f = f^{\#} \cdot a$. This proves that a_0 is one-to-one: if $a_0(x) = a_0(y)$ then $f(x) = f(y)$, hence, $x = y$. This concludes the proof.

Remark. Precisely the same theorem holds for the category K-Vect of vector spaces over an (arbitrary) field K: a functor F: K-Vect $\to$ K-Vect has (constructive) universal solutions if and only if it has arbitrarily large fixed points or it is constant. The equivalence of (iii) and (iv) is proved in [**12**].

3. Universal equations.

3.2. Recall that Equ(F) denotes the category of F-equations and morphisms of F-equations. Equ(F) may also be regarded as Alg(F^{op}).

The following dualizes the notion of a universal solution:

3.2. A *universal equation* for an F-algebra (A, α) is an equation (X_0, f_0) and a morphism $a_0:X_0 \to A$ such that

(i) $X \xrightarrow{a} (A, \alpha)$ is a solution of the equation (X_0, f_0);

(ii) for each equation (X, f) and each solution of the form $X \xrightarrow{a} (A, \alpha)$ there exists a unique morphism

$$a_{\#}:(X, f) \to (X_0, f_0) \quad \text{with } a = a_0 \cdot a_{\#}.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_{\#}"] \arrow[dd, bend right=60, "a"'] & FX \arrow[d, "Fa_{\#}"'] \arrow[dd, bend left=60, "Fa"] \\ X_0 \arrow[r, "f_0"] \arrow[d, "a_0"] & FX_0 \arrow[d, "Fa_0"'] \\ A & FA \arrow[l, "\alpha"'] \end{tikzcd} \tag{3.1}$$

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A universal equation is always a fixed point of F (this is the dual of 2.2). If K has limits of well-ordered spectra, then universal equations can be obtained by the construction dual to 2.5:

$$A \xleftarrow{\alpha} FA \xleftarrow{F\alpha} F^2 A \xleftarrow{F^2\alpha}$$
$$\dots F^\omega A \dots = \lim_{n<\omega} F^n A \leftarrow F^{\omega+1} A = F(F^\omega A) \leftarrow \dots$$

In particular, dualizing the Remark in 2.6, we get the following lemma for the ordinal ω.

3.3. LEMMA. *Let F be a functor preserving ω-limits. Then each F-algebra (A, α) has a universal equation, which is the limit*

$$F^\omega A = \lim_{n<\omega} F^n A$$

of the following ω-chain:

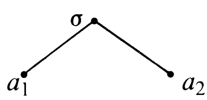
$$A \xleftarrow{\alpha} FA \xleftarrow{F\alpha} F^2 A \xleftarrow{F^2\alpha} \dots .$$

Remark. More generally, the corresponding lemma for functors preserving α-limits can be formulated.

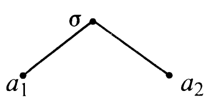
2.4. *Example.* Consider the case of groupoids, i.e., the functor H_Σ where Σ contains just one binary operation σ. Then H_Σ preserves all limits, thus, the universal equation for a groupoid $(A, *)$ is obtained as the following ω-limit

$$A \xleftarrow{*} H_\Sigma A \xleftarrow{H_\Sigma *} H_\Sigma^2 A \leftarrow \dots .$$

The elements of $H_\Sigma A$ can be represented as the trees

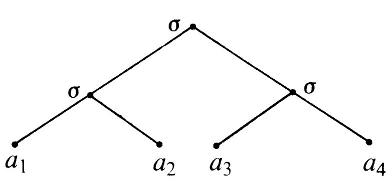


where $a_1, a_2 \in A$; then the first map of this spectrum is the "computation" of trees:

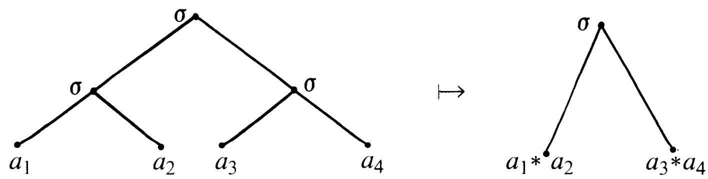


$\mapsto \quad a_1 * a_2.$

Analogously, $H_\Sigma^2 A$ are the following trees:



and the second map is the computation of these trees:



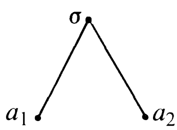
In general, $H^n_\Sigma A$ are all binary trees of depth n, where each leaf has distance n from the root and carries a label from A. The operation $*$ defines a natural map

$$\alpha_n : H^n_\Sigma A \rightarrow H^{n-1}_\Sigma A.$$

A limit of (any) ω-spectrum $\{\alpha_n\}_{n<\omega}$ in Set is the set of all sequences $\{t_n\}_{n<\omega}$ which fulfill

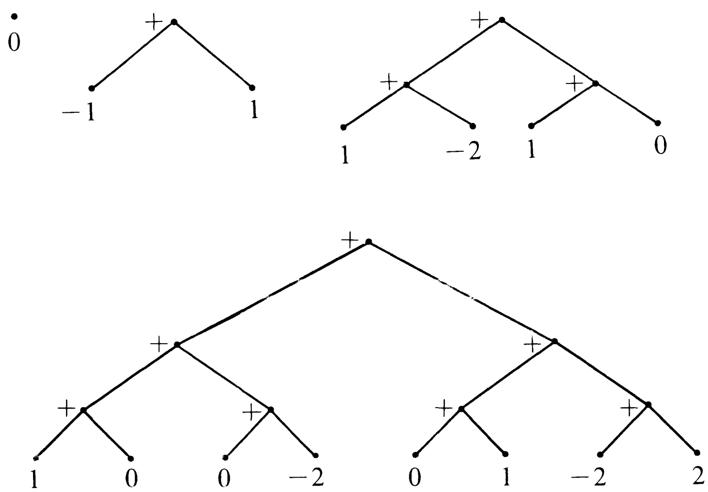
$$\alpha_n(t_n) = t_{n-1} \quad \text{for all } n > 0.$$

Thus, the universal equation for a groupoid $(A, *)$ is the set of all sequences of labelled binary trees $t_0, t_1, t_2, \ldots$ where $t_0 \in A$ and t_{n+1} is obtained from t_n by substituting any leaf, labelled by $a \in A$, by some subtree



where $a_1, a_2 \in A$ fulfil $a_1 * a_2 = a$.

For instance, consider the additive groupoid $(Z, +)$ of integers. The following sequence is a typical element of the universal equation.



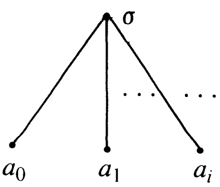
3.5. *Example.* More generally, for each type Σ (even infinitary), the functor H_Σ preserves ω-limits. This follows from the fact that H_Σ is a coproduct of the functors H_σ, $\sigma \in \Sigma$; each H_σ (defined by $X \mapsto X^n$ where $n = \operatorname{ar} \sigma$) preserves all limits, in particular ω-limits. And coproducts in Set commute with ω-limits, hence, H_Σ preserves ω-limits. Consequently, the universal equation of a Σ-algebra (A, α) is obtained as the limit of the following spectrum.

$$A \xleftarrow{\alpha} H_\Sigma A \xleftarrow{H_\Sigma \alpha} H_\Sigma^2 A \xleftarrow{H_\Sigma^2 \alpha} H_\Sigma^3 A \leftarrow \dots .$$

Again, the elements of $H_\Sigma^n A$ can be represented by finite labelled trees. First, represent each $a \in A$ by the tree

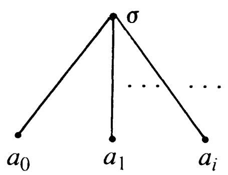
$$\overset{\bullet}{a}$$

and each $\sigma(a_i)_{i<n} \in H_\Sigma A$ by



$(i < n \neq 0)$ or $\overset{\bullet}{\sigma}$ $(n = 0)$.

The map α is the computation of this tree:



$\mapsto \alpha_\sigma(a_i)_{i<n}$ or $\overset{\bullet}{\sigma} \mapsto \alpha_\sigma \in A$.

In general, $H_\Sigma^n A$ is the set of all labelled trees such that

(i) each node with $k > 0$ successors is labelled by some $\sigma \in \Sigma$ with arity k;

(ii) each leaf is labelled by some $\sigma \in \Sigma$ with arity 0 or some $a \in A$;

(iii) each path from the root to a leaf labelled by x has lengths

a. n if $x \in A$

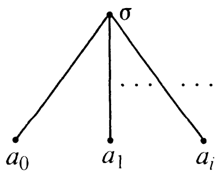
b. smaller than n if $x \in \Sigma$ and $\operatorname{ar} x = 0$.

The mapping

$$H_\Sigma^n \alpha : H_\Sigma^{n+1} A \to H_\Sigma^n A$$

assigns to each tree $t \in H_\Sigma^{n+1} A$ the tree $\bar{t} \in H_\Sigma^n A$ obtained by the computation of all operation symbols with distance n from the root. Thus, if a node is labelled by $\sigma \in \Sigma$, $\operatorname{ar}(\sigma) = k$, and its distance is n then (by (iii) above) each successor of this node is labelled by some element of A, say, a_i

for the i-th successor. We substitute the subtree



$(k > 0)$ or $\overset{\bullet}{\sigma}$ $(k = 0)$

by the subtree

$$\overset{\bullet}{a}$$

where $a = \alpha_\sigma(a_i)_{i<k}$ or $a = \alpha_\sigma$.

The universal equation of (A, α) is the set X_0 of all sequences of trees

$$(t_0, t_1, t_2, \dots)$$

where $t_0 \in A$, $t_{n+1} \in H_\Sigma^{n+1}A$ and $H^n\alpha(t_{n+1}) = t_n$ for each $n < \omega$. Such a sequence can be viewed as an "unfolding" of the element $t_0 \in A$.

We have the following natural bijection

$$f_0: X_0 \to H_\Sigma X_0.$$

Note first that, given an unfolding

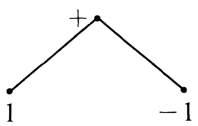
$$(a, t_1, t_2, t_3, \dots) \in X_0$$

where

$$t_1 = \sigma(a_i)_{i<k} \quad (\sigma \in \Sigma, \text{ ar } \sigma = k),$$

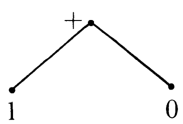
then for each $i \in k$ the i-th maximal branches $\partial_i(t_n)$ of the trees t_n, $n = 1, 2, 3, \dots,$ form an unfolding of the element a_i. For example, in the unfolding of 0 in $(Z, +)$ above, we have

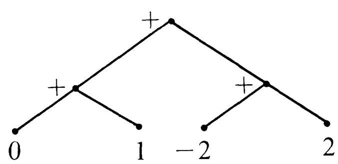
t_1:



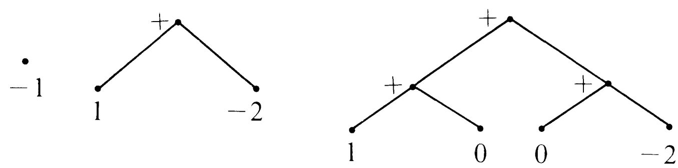
and we obtain an unfolding of 1 as $\partial_0(t_n)$:

$\overset{\bullet}{1}$





as well as an unfolding of -1 as $\partial_1(t_n)$:



The mapping $f_0 : X_0 \to H_\Sigma X_0$ assigns to each unfolding

$$x = (t_0, t_1, t_2, t_3, \dots)$$

with

$$t_1 = \sigma(a_i)_{i<k} \quad (\sigma \in \Sigma,\ \text{ar}\ \sigma = k)$$

the k-tuple of unfoldings

$$f_0(x) = \sigma(x_i)_{i<k}$$

where

$$x_i = (\partial_i(t_1), \partial_i(t_2), \partial_i(t_3), \dots) \in X_0 \quad \text{for each } i < k.$$

This is indeed a bijection, since for each k-tuple of unfoldings

$$x_i = (a^{(i)}, t_1^{(i)}, t_2^{(i)}, t_3^{(i)}, \dots) \in X_0$$

we can define an unfolding

$$x = f_0^{-1}(\sigma(x_i)_{i<k})$$

of the element

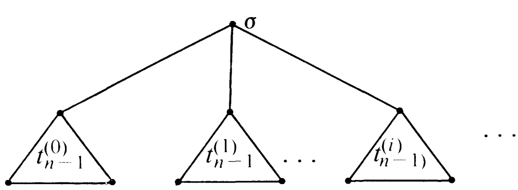
$$a = \alpha_\sigma(a^{(i)})_{i<k}$$

as follows:

$$x = (a, t_1, t_2, t_3, \dots)$$

where for each $n < \omega$ the tree t_n is the following one:

t_n:



Moreover, f_0 is the natural bijection: for each projection

$$\pi_k : X_0 \to H_\Sigma^k A, \quad \pi_k(t_0, t_1, t_2, \dots) = t_k$$

clearly

$$H_\Sigma \pi_k \cdot f_0 = \pi_{k+1} \quad (k < \omega).$$

Thus, the universal equation for (A, α) is

$$f_0: X_0 \to H_\Sigma X_0$$

with respect to the solution map $\pi_0: X_0 \to A$.

4. Fixed points.

4.1. Again let $F: K \to K$ be a functor. A *fixed point of* F is a pair (Y, η) where $\eta: Y \to FY$ is an isomorphism. Thus a fixed point (Y, η) is at the same time an F-equation and an F-algebra (Y, η^{-1}). A morphism of fixed points may equally well be defined as a morphism of F-algebras or of F-equations. Let $\mathrm{Fix}(F)$ denote the category of fixed points of F. We have inclusion functors:

$$\begin{tikzcd} & \mathrm{Alg}(F) \\ \mathrm{Fix}(F) \arrow[ur, "I"] \arrow[dr, "J"'] & \\ & \mathrm{Equ}(F) \end{tikzcd}$$

with $\mathrm{Fix}(F)$ a full subcategory of $\mathrm{Alg}(F)$ and of $\mathrm{Equ}(F)$ $(= \mathrm{Alg}(F^{\mathrm{op}}))$.

The above concepts of universal solutions and universal equations can now be reformulated as follows.

4.2. Proposition. *If F has universal solutions then* $J: \mathrm{Fix}(F) \to \mathrm{Equ}(F)$ *has a left adjoint, i.e.,* $\mathrm{Fix}(F)$ *is a reflective subcategory.*

We regard the local universal problem of J at $(X, f) \in \mathrm{Equ}(F)$, i.e., the comma category $(X, f) \downarrow J$ with objects morphisms

$$p: (X, f) \to (Y, \eta)$$

from (X, f) to fixed points. Let $((A_0, \alpha_0), a_0)$ be the universal solution of (X, f) then (A_0, α_0^{-1}) is a fixed point and

$$a_0: (X, f) \to (A_0, \alpha_0^{-1})$$

an object of $(X, f) \downarrow J$. For any

$$p: (X, f) \to (Y, \eta)$$

$((Y, \eta^{-1}), p)$ is a solution of (X, f) and thus there exists a unique morphism of fixed points

$$p_\#: (A_0, \alpha_0^{-1}) \to (Y, \eta) \quad \text{with } p = p_\# a_0 \quad (4.1).$$

This shows that a_0 is initial in $(X, f) \downarrow J$ and J has a left adjoint S with unit the universal solution.

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$$\begin{tikzcd}
A_0 \arrow[ddd, dashed, "p_\#"'] & & & FA_0 \arrow[lll, "\alpha_0"'] \arrow[ddd, dashed, "Fp_\#"] \\
& X \arrow[ul, "a_0"'] \arrow[r, "f"] \arrow[ddl, "p"] & FX \arrow[ur, "Fa_0"] \arrow[ddr, "Fp"'] & \\
& & & \\
Y \arrow[rrr, "\eta"'] & & & FY
\end{tikzcd}$$

4.3. Dually we conclude

PROPOSITION. *If F has universal equations then*

$$I:\mathrm{Fix}(F) \to \mathrm{Alg}(F)$$

has a right adjoint E, i.e., $\mathrm{Fix}(F)$ *is a coreflective subcategory.*

We now want to ask whether the converse of 4.2 (and 4.3) also holds. The following counter example will show that this is not the case in general.

4.4. *Counter example.* P_0:Set $\to$ Set denotes the restricted powerset-functor $P_0X = PX - \{\emptyset\}$. The only fixed points of P_0 are the singletons and the void set $\emptyset$ and, consequently, constant functions are mapped onto constant functions by P_0. Let (X, f) be a P_0-equation, then the terminal morphism $t:X \to 1$ is clearly a morphism of equations

$$(4.2)\qquad \begin{tikzcd}
1 \arrow[r] & 1 \\
X \arrow[u, "t"] \arrow[r, "f"] & P_0X \arrow[u, "Pt"']
\end{tikzcd}$$

It also solves the local universal problem of J. Thus J has a left adjoint with all unit components terminal and which trivially maps an equation $f:X \to P_0X$ to the fixed point $1 \to P_01$. Now if $\alpha:P_0A \to A$ is a P_0-algebra and $a:X \to A$ a solution then a can be factored through t only if it is constant. But example 2.5 had shown that non-constant solutions exist and thus P_0 does not have universal solutions.

4.5. THEOREM. *Let F have universal equations. Then it has universal solutions if and only if*

$$J:\mathrm{Fix}(F) \to \mathrm{Equ}(F)$$

has a left adjoint.

Proof. Assuming that J has a left adjoint, we shall prove that each equation (X, f) has a universal solution. Let

$$a_0:(X, f) \to (A, \alpha_0^{-1})$$

be the reflection of (X, f) in Fix(F). We are going to verify that the universal solution is

$$X \xrightarrow{a_0} (A_0, \alpha_0).$$

Let

$$X \xrightarrow{a} (A, \alpha)$$

be an arbitrary solution of (X, f) and let

$$(X_0, f_0) \xrightarrow{p_0} A$$

be a universal equation for (A, α) (4.3).

(4.3)
$$\begin{tikzcd}
X_0 \arrow[rrr, "f_0"] \arrow[ddd, "p_0"'] & & & FX \arrow[ddd, "Fp_0"] \\
& X \arrow[ul, "a_\#"'] \arrow[r, "f"] \arrow[d, "a_0"] \arrow[ddl, "a"] & FX \arrow[ur, "Fa_\#"'] \arrow[d, "Fa_0"'] \arrow[ddr, "Fa"] & \\
& A_0 \arrow[uul, "h"] & FA_0 \arrow[l, "\alpha_0"] \arrow[uur] & \\
A & & & FA \arrow[lll, "\alpha"]
\end{tikzcd}$$

Since (X, f) has a solution (A, α), there exists a unique morphism

$$a_\#:(X, f) \to (X_0, f_0)$$

in Equ(F) with

$$a = p_0 \cdot a_\#.$$

Since $a_\#:(X, f) \to J(X_0, f_0^{-1})$ is a morphism, and a_0 is the reflection there exists a unique $h:(A_0, \alpha_0) \to (X_0, f_0^{-1})$ in Fix(F) with

$$a_\# = h \cdot a_0.$$

Then

$$p_0 \cdot h:(A_0, \alpha_0) \rightarrow (A, \alpha)$$

is the unique morphism with

$$(p_0 \cdot h) \cdot a_0 = a.$$

This proves that

$$X \xrightarrow{a_0} (A_0, \alpha_0)$$

is a universal solution.

COROLLARY. *Each pair of the following conditions implies the remaining condition*:

(i) *F has universal solutions*;

(ii) *F has universal equations*;

(iii) *both I and J have adjoints.*

Indeed, by duality we conclude from the preceding theorem that, in the presence of universal solutions, I has a right adjoint if and only if F has universal equations.

Remark. If F has both universal solutions and universal equations then we obtain an adjoint pair:

$$\mathrm{Alg}(F) \underset{I \cdot L}{\overset{J \cdot R}{\rightleftarrows}} \mathrm{Equ}(F).$$

Let an equation (X, f) have a universal solution (A_0, α_0), and an algebra (A, α) have a universal equation (X_0, f_0). Then this adjunction establishes a bijection between all solutions of (X, f) in (A, α) and all fixed-point morphisms from (X_0, f_0) to (A_0, α_0).

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