Some Extensions of Pincherle's Polynomials.

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The polynomials which satisfy linear differential equations of the second order and of the hypergeometric type have been the object of extensive work, and very few properties of them remain now hidden; the student who seeks in that direction a subject for research is compelled to look, not after these functions themselves but after generalisations of them. Among these may be set in first place the polynomials connected with a differential equation of the third order and of the extended hypergeometric type, of which a general theory has been given by Goursat. The number of such polynomials of which properties have been studied in particular is rather small; in fact, Appell's polynomials

$$P_{2n} = \frac{d^n}{dx^n}\left[x^n(x^2-1)^n\right]$$

and Pincherle's polynomials, arising from the expansions

$$(1-3tx+t^3)^{-\frac{1}{3}} = \sum t^n P_n(x)$$

are, so far as I know, the only well-known ones. To show what can be done in these ways, I shall briefly give the definition and principal properties of some polynomials analogous to Pincherle's and of some allied functions.

I. Starting from the expression

$$(1-3tx+t^3)^{-\nu}$$

where ν is an arbitrary quantity I expand it in ascending powers of t, and call $P_n^\nu(x)$ the coefficient of the nth power of t; so

$$(1)\qquad (1-3tx+t^3)^{-\nu} = \sum_n t^n P_n^\nu(x).$$

P_n^ν is obviously a polynomial of degree n in respect of x; and, bearing in mind the definition of Pincherle's polynomial P_n given above, it may be seen that P_n^ν plays with P_n the same part as Gegenbauer's polynomial C_n^ν plays with the ordinary Legendre's polynomial X_n.

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From the fundamental relation (1) may be easily obtained, through the ordinary method in such cases, the following recurrence formula

$$(1) \qquad (n+1)\,P^{\nu}_{n+1} - 3x\,(n+\nu)\,P^{\nu}_{n} + (n+3\,\nu-2)\,P^{\nu}_{n-2} = 0.$$

This formula affects only polynomials with the same superior index; but another one can be readily written, connecting polynomials with different indices, both superior and inferior

$$(n+1)\,P^{\nu}_{n+1} + 3\,\nu\,P^{\nu+1}_{n-2} = 3\,\nu\,x\,P^{\nu+1}_{n}.$$

Other recurrence formulae introduce the differential coefficient of the functions:

$$n\,P^{\nu}_{n} + \frac{dP^{\nu}_{n-2}}{dx} - x\,\frac{dP^{\nu}_{n}}{dx} = 0$$

$$(n+3\nu)\,P^{\nu}_{n} = \frac{dP^{\nu}_{n+1}}{dx} - 2x\,\frac{dP^{\nu}_{n}}{dx}$$

and

$$(2) \qquad \frac{dP^{\nu}_{n}}{dx} = 3\,\nu\,P^{\nu+1}_{n-1}.$$

This last formula is of special interest. If we write it

$$(3) \qquad \frac{dP^{\nu-1}_{n+1}}{dx} = 3\,(\nu-1)\,P^{\nu}_{n}$$

and compare (3) with (2) we obtain

$$\frac{d^2\,P^{\nu-1}_{n+1}}{dx^2} = 9\nu\,(\nu-1)\,P^{\nu+1}_{n-1}$$

and so on. Any polynomial P^{ν}_{n} may then be expressed in terms of a differential coefficient of a polynomial whose superior index lies between 0 and 1. In particular, if ν is of the form $\alpha+\frac{1}{2}$, α being an integer, we have

$$P^{\nu}_{n} = \left(\frac{2}{3}\right)^{\alpha} \frac{1}{1\,.\,3\,.\,5\,\ldots\,(2\alpha-1)}\,\frac{d^{\alpha}}{dx^{\alpha}}\,P_{n+\alpha},$$

$P_{n+\alpha}$ being the ordinary Pincherle's polynomial of order $n+\alpha$; and if ν is itself an integer, we have

$$P^{\nu}_{n} = \frac{1}{3^{\nu-1}}\,\frac{1}{(\nu-1)\,!}\,\frac{d^{\nu-1}}{dx^{\nu-1}}\,P^{1}_{n+\nu-1},$$

or

$$P_n^\nu = \frac{1}{3^\nu \,.\, \nu\,!} \frac{d^\nu}{dx^\nu} P^0_{n+\nu}$$

P_n^0 being the coefficient of t^n in the expansion, in ascending powers of t, of

$$-\log(1 - 3tx + t^3).$$

From the recurrence formulae we shall obtain a differential equation of the third order and of the extended hypergeometric type, satisfied by P_n^ν, namely

$$(4x^3 - 1)\,y''' + 6x^2(2\nu + 3)\,y'' - xy'\left[3n^2 + 3n(2\nu + 1) - (3\nu + 2)(3\nu + 5)\right]$$
$$- n(n + 3\nu)(n + 3\nu + 3)\,y = 0.$$

II. A certain number of functions are connected with these polynomials P_n^ν; for instance, if we consider the finite difference-equation

$$(n+1)f_{n+1} - 3x(n - \nu + 1)f_n + (n - 3\nu + 1)f_{n-2} = 0$$

which is satisfied by $P_n^{1-\nu}$, we can see that another solution of it is the function

$$\lambda_u^\nu(x) = \int_0^\omega \frac{(\wp' u)^{2\nu - 1}}{(\wp\, u)^{n+1}}\, du,$$

the Weierstrassian elliptic function $\wp\, u$ being defined by the radical $\sqrt{t^3 - 3tx + 1}$, and ω being its real period. This function λ_n^ν is also the coefficient of x^n in the expansion in ascending powers of z of the integral

$$\int_{e_3}^{\infty} \frac{dt}{(z - t)\,(t^3 - 3tx + 1)^{1-\nu}}$$

e_3 being the greatest root of $t^3 - 3tx + 1 = 0$.

Another function, $\sigma_n^\nu(x)$, may be introduced by the formula

$$\sigma_n^\nu = \int_0^{e_1} t^n\,(t^3 - 3tx + 1)^{\nu - 1}\, dt$$

e_1 being the smallest root of the polynomial of the third order; this function satisfies the recurrence formula

$$(n + 3\nu - 1)\,\sigma_{n-1}^\nu - 3x(n + \nu - 1)\,\sigma_{n-1}^\nu + (n - 1)\,\sigma_{n-2}^\nu = 0$$

which is the inverse of the difference-equation satisfied by P_n^ν. The presence of a polynomial of the third order in the definition of these functions leads to the introduction of elliptic functions in all these theories.

III. Leaving now the field of differential equations of the third order, we can introduce new and more general functions; considering, for instance, the formula

$$\frac{dP_n^\nu}{dx} = 3\,\nu\, P_{n-1}^{\nu+1}$$

and comparing it with the relation

$$\frac{dC_n^\nu}{dx} = 2\,\nu\, C_{n-1}^{\nu+1}$$

for Gegenbauer's polynomials, we are led to consider the polynomials $\Pi_{n,m}^\nu$, where m is an integer, defined by the expansion

$$(1 - mtx + t^m)^{-\nu} = \sum_n t^n\, \Pi_{n,m}^\nu(x)\,.$$

These satisfy the recurrence formulae

$$(n+1)\Pi_{n+1,m}^\nu - mx(n+\nu)\,\Pi_{n,m}^\nu - (n + m\nu - m + 1)\,\Pi_{n\,m+1,m}^\nu = 0$$

$$\frac{d}{dx}\Pi_{n,m}^\nu = m\,\nu\,\Pi_{n-1,m}^{\nu+1}$$

which generalise the formulae (1)′ and (2), and also a differential equation of order m.

IV. In conclusion, we may remark that it would be easy to extend these properties to certain polynomials with two variables; for instance, just as Hermite generalised Legendre's polynomials, we can start from the expansion

$$(1 - 3ax - 3by + a^3 + b^3)^{-1} = \sum_m \sum_n a^m\, b^n\, V_{m,n}(x, y)$$

and obtain formulae such as

$$\frac{\partial V_{m,\,n+1}}{\partial x} = \frac{\partial V_{m+1,\,n}}{\partial y}$$

$$\frac{\partial V_{m-1,\,n}}{\partial x} - x\frac{\partial V_{m,\,n}}{\partial x} + m V_{m,\,n} = 0,$$

and so forth.

Note.—The properties of Pincherle's polynomials are given by S. Pincherle in *Memorie della R. Accademia di Bologna*, seria quinta, t. I., 1890, p. 337.

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