Part 2

ASTEROID FAMILIES AND STABILITY

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Asteroid proper elements: recent computational progress

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Abstract. In this work, we review the analytical and semi-analytical tools introduced to deal with resonant proper elements and their applications to the Trojan asteroids, the numerical computation of synthetic proper elements for resonant and non resonant asteroids, and the introduction of proper elements for planet crossing asteroids. We discuss the applications and accuracy of these methods and present some comparisons between them.

Keywords. Minor planets, asteroids, proper elements, perturbation theory

1. Introduction

Proper elements play a major role in the characterization of the long term stability of asteroid orbits, as well as in the identification and definition of asteroidal families. In recent years, the development of new analytical and numerical tools have allowed to extend the computation of proper elements to huge sets of asteroid orbits including main belt, resonant and planet crossing asteroids. This has had a deep impact on our knowledge of the dynamical structure of the asteroid belt, and on the dynamical and collisional processes taking place there.

Among these new tools, the semi-analytical model introduced by Beaugé and Roig (2001) to deal with resonant proper elements represents a major advance in the field. Their method allowed to determine the proper elements of the Trojan asteroids and to confirm the existence of families among these bodies. A purely analytical method to deal with resonant proper elements has been recently introduced by Miloni, Ferraz-Mello and Beaugé (in preparation), who also presented a preliminary application of their method to the Hilda asteroids. On the other hand, Knežević and Milani (2000) elaborated a synthetic theory of the long term asteroidal motion that allowed the numerical computation of highly accurate proper elements for non resonant and resonant asteroids, without the typical limitations of the analytical or semi-analytical models previously used (e.g. Milani and Knežević 1990; Lemaitre and Morbidelli 1994). Another major improvement concerns the computation of proper elements for planet crossing orbits. This method has been introduced by Gronchi and Milani (2001), and has been successfully applied to predict planet collisions of Near-Earth Asteroids (NEAs).

Besides the theoretical development of all these techniques, the access to more powerful computational resources at lower costs has allowed the computation of proper elements (either analytical, semi-analytical or numerically) for very huge sets of orbits and for very different populations, from the main asteroid belt to the trans-Neptunian region. It is also possible to keep large databases periodically updated at the same rhythm of discovery of new asteroids (Knežević and Milani 2003). This has had a major impact in our knowledge of the dynamical structure of the asteroid belt, and especially in the detection of asteroid families.

In this contribution we review these issues. The paper is organized as follows: in Sect. 2 we provide the basic theoretical background about the computation of proper elements. Section 3 is devoted to describe the method of Beaugé and Roig (hereafter B-R). Section 4 summarizes the method of Miloni, Ferraz-Mello and Beaugé The synthetic theory of Knežević and Milani (hereafter K-M) is presented in Sect. 5. In Sect. 6 we present an application of the B-R and K-M methods to the Trojan asteroids. Finally, Sect. 7 describes the method of Gronchi and Milani.

2. Theoretical background

In a strict sense, proper elements should be integrals of motion of the dynamical system representing the motion of an asteroid under the perturbation of the planets. Since this system is not integrable, integrals of motion do not exist at all, but in most cases it is possible to compute quantities that are close to these integrals in the sense that they vary very little over very long time scales. These quasi-integrals of motion are referred to as "the proper elements".

The idea behind the computation of proper elements is to perform a canonical transformation (or a set of canonical transformations) such as to reduce the original Hamiltonian of the system to an integrable approximation. Schematically, suppose that the Hamiltonian can be separated as follows

$$F(\theta, J) = F_0(J) + \varepsilon F_1(\theta, J)$$

where F_0 is an integrable part, F_1 is a perturbation of order $\varepsilon \ll 1$, and θ, J are the angle-action variables of F_0. We search for a canonical transformation

$$(\theta, J) \to (\theta^*, J^*)$$

such that the new Hamiltonian becomes

$$F^*(\theta^*, J^*) = F_0^*(J^*) + \varepsilon^n F_1^*(\theta^*, J^*)$$

with $n > 1$. If the reminder of $\mathcal{O}(\varepsilon^n)$ can be neglected, then the new actions J^* are the proper elements we are looking for.

In practice, this procedure is accomplished by the computation of a time averaging that eliminates the angular dependence of the Hamiltonian. The final result of this averaging method strongly depends on the choice of the averaging "kernel" (the Hori's kernel), that is F_0, which determines how the angles actually vary with time. This choice must be done in such a way that F_0 accounts for the basic dynamical features of the system, or in other words, for the basic topology of the phase space. Thus, the key problem when dealing with the computation of proper elements is how to split the Hamiltonian for a suitable averaging.

In the asteroidal problem, it is possible to separate the angular dependence of the Hamiltonian according to the different time scales of the perturbations, leading to a set of "fast" and "slow" angles. The first ones are related to the mean longitudes of the asteroid (λ) and the planets (λ_i), while the second ones are related to the longitudes of perihelia (ϖ, ϖ_i) and nodes (Ω, Ω_i). Thus, for example, the classical definition of asteroids proper elements, based on Yuasa's theory (Milani and Knežević 1990), involves two averaging: the first to eliminate the fast angles and the second to eliminate the slow

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ones. Schematically, we first write

$$F = F_0(L, L_i) + \varepsilon F_1(\lambda, \varpi, \Omega, \lambda_i, \varpi_i, \Omega_i, L, W, Z, L_i, W_i, Z_i)$$

where F_0 basically represents the two body problem, and $\varepsilon \sim m_i$, which are the masses of the perturbing bodies. Here, L, W, Z are the canonical momenta conjugated to λ, ϖ, Ω, respectively. Then, we introduce a canonical transformation

$$(\varpi, \Omega, \varpi_i, \Omega_i, W, Z, W_i, Z_i) \rightarrow (\bar{\varpi}, \bar{\Omega}, \bar{\varpi}_i, \bar{\Omega}_i, \bar{W}, \bar{Z}, \bar{W}_i, \bar{Z}_i)$$

from "osculating elements" to "mean elements" through a *first* averaging that eliminates λ, λ_i, assuming that these angles vary with time following the solution of F_0. The "averaged" Hamiltonian takes the form

$$\bar{F} = \bar{F}_0 + \varepsilon \bar{F}_1(\bar{\varpi}, \bar{\Omega}, \bar{\varpi}_i, \bar{\Omega}_i, \bar{W}, \bar{Z}, \bar{W}_i, \bar{Z}_i)$$

where $\bar{F}_0$ is a constant that can be disregarded. The averaged perturbation can the be re-written as

$$\bar{F}_1 = \bar{F}_{10}(\bar{\varpi}, \bar{\Omega}, \bar{\varpi}_i, \bar{\Omega}_i, \bar{W}, \bar{Z}, \bar{W}_i, \bar{Z}_i) + \epsilon \bar{F}_{11}(\bar{\varpi}, \bar{\Omega}, \bar{\varpi}_i, \bar{\Omega}_i, \bar{W}, \bar{Z}, \bar{W}_i, \bar{Z}_i)$$

where, again, $\bar{F}_{10}$ is an integrable part basically represented by an harmonic oscillator with a forced term, and ϵ is a small parameter somehow related to the high powers of the eccentricities and inclinations of the bodies. The *second* averaging is performed assuming that the mean angles vary linearly with time with frequencies given by the fundamental frequencies of $\bar{F}_{10}$. After the averaging, we arrive to an integrable Hamiltonian and the proper elements are given by the actions of $\bar{F}_{10}$ plus a correction of order ϵ arising from $\bar{F}_{11}$. The proper frequencies are also given by the fundamental frequencies of $\bar{F}_{10}$ plus a correction of order ϵ. It is then usual to proceed in an iterative way by repeating the average using these corrected frequencies until their values converge.

Other problems in asteroidal dynamics are treated in a similar way. The only differences arise from the form in which the integrable part of the Hamiltonian is separated at the different stages of the procedure, and also the form in which the average is done. Three cases are of particular interest:

• When the asteroid orbit is in a mean motion resonance, some linear combination of the mean longitudes λ, λ_i has a frequency close to zero. This linear combination constitutes the resonant angle, which has to be isolated, so the first average is performed only over the non resonant angles. This procedure leads to an averaged Hamiltonian, where $\bar{F}_0$ has the basic features of a pendulum (actually, an Andoyer Hamiltonian). This must be taken into account when performing the second average, which introduce additional complexity to the problem.

• When the asteroid orbit is in a secular resonance, the corresponding resonant angle has to be isolated from $\bar{F}_1$, and the second average is performed only over the non resonant angles. This procedure leads to an averaged Hamiltonian, where $\bar{F}_{10}$ has the basic features of a pendulum (Morbidelli 1993).

• When the asteroid is in a largely eccentric or largely inclined non resonant orbit, the separation of the averaged Hamiltonian in $\bar{F}_{10}$ and $\bar{F}_{11}$ is no longer valid because ϵ is not small. Other re-arrangements are possible in this case (e.g. Lemaitre and Morbidelli 1994), but the corresponding results are always restricted to limited ranges of the eccentricity and inclination.

• When the asteroid is in a planet crossing orbit, the first average cannot be performed because there is a singularity along the integration path.

In any of these cases, the classical theory to compute proper elements will fail. Therefore, specific techniques have to be developed to treat them, as we will show in the following.

3. Proper elements for resonant orbits

When dealing with resonant orbits in the framework of the restricted three body problem, the Hamiltonian obtained after the elimination of the fast angles has the form

$$\bar{F} = \bar{F}_0(\bar{\sigma}, \bar{L}, \bar{W}, Z, \bar{W}', \bar{Z}') + \varepsilon \bar{F}_1(\bar{\sigma}, \bar{\varpi}, \bar{\Omega}, \bar{\varpi}', \bar{\Omega}', \bar{L}, \bar{W}, \bar{Z}, \bar{W}', \bar{Z}') \tag{3.1}$$

where $\bar{F}_0(\bar{\sigma}, \bar{L}, \bar{W}, \bar{Z}, \bar{W}', \bar{Z}')$ is a pendulum-like Hamiltonian, ε is proportional to the eccentricity and inclination of the perturbing body, $\bar{\sigma}$ is the resonant angle that librates around a certain value $\bar{\sigma}_c$, $\bar{L}$ is the canonical momentum conjugated to $\bar{\sigma}$, and primed variables refer to the perturber. In order to apply a second averaging to eliminate all the angles, we have to take into account that $\bar{\varpi}, \bar{\Omega}, \bar{\varpi}', \bar{\Omega}'$ are linear functions of time but $\bar{\sigma}$ is not. Therefore, the time averaging cannot be directly replaced by an average over $\bar{\sigma}$. A possible solution is to introduce a canonical transformation to find the action-angle variables of $\bar{F}_0$. This is usually accomplished by solving the equations of motion for $\bar{F}_0$ numerically, substituting this solution in $\bar{F}_1$, and computing the time average with a numerical quadrature. However, this has the drawback of being very CPU-time consuming, and does not explicitly yield the proper element associated to the pair $\bar{\sigma}, \bar{L}$.

Another possibility has been introduced by Beaugé and Roig (2001), based on ideas by Jupp (1969) for the Ideal Resonance Problem. This consists into find a canonical transformation from $(\bar{\sigma}, \bar{\varpi}, \bar{\Omega}, \bar{L}, \bar{W}, \bar{Z})$ to new variables $(\theta, \varpi^*, \Omega^* J, W^*, Z^*)$ where all the angles are non resonant. The idea can be summarized as follows: Let us think about the libration region of a resonance as a set of invariant curves around the libration point $\bar{\sigma}_c$. If we only concentrate on this region and disregard the structure of qthe phase space outside the separatrix, we can think of these orbits as distorted circulations around a center which is displaced from the origin of the coordinate system. Now, if we find a canonical transformation $(\bar{L}, \bar{\sigma}) \to (J, \theta)$ that is simply a translation of the origin to the libration center, we will obtain a new angle θ having a frequency different from zero, and the integral of J along any orbit will be the action of that trajectory. In other words, we will have an angle $\bar{\sigma}$ that librates transformed into another angle θ that circulates with frequency $\nu_\theta = \nu_{\bar{\sigma}}$. These new variables will have properties of being "non-resonant" (even though they are a simple translation), and we can use any classical averaging method, such as Hori's method, to determine the corresponding action-angle variables.

A simple way to determine (J, θ) is based on the following series of transformations:

$$\begin{aligned}
(\bar{L}, \bar{\sigma}) &\to (K, H) = \sqrt{2\bar{L}}(\cos\bar{\sigma}, \sin\bar{\sigma}) \\
(K, H) &\to (X, Y) = (K - K_c, H - H_c) \\
(X, Y) = \sqrt{2J}(\cos\theta, \sin\theta) &\to (J, \theta)
\end{aligned} \tag{3.2}$$

where $(K_c, H_c) = \sqrt{2\bar{L}_c}(\cos\bar{\sigma}_c, \sin\bar{\sigma}_c)$ marks the center of libration. This center is nothing but the equilibrium point of $\bar{F}_0$ and can be easily obtained numerically. Beaugé and Roig (2001) introduced a slightly different procedure, in the sense that it can no longer be thought of as a simple translation. The transformation in their case is represented by the relationship:

$$X = \Gamma^{-1/2}\left(\widehat{K} - \left(K_c^2 - \widehat{H}^2\right)^{1/2}\right); \qquad Y = \Gamma^{1/2}\widehat{H} \tag{3.3}$$

where $(\widehat{K}, \widehat{H}) = \sqrt{2\bar{L}}\,(\cos(\bar{\sigma} - \bar{\sigma}_c), \sin(\bar{\sigma} - \bar{\sigma}_c))$ and $\Gamma = \Gamma(K_c)$ is a scaling factor which

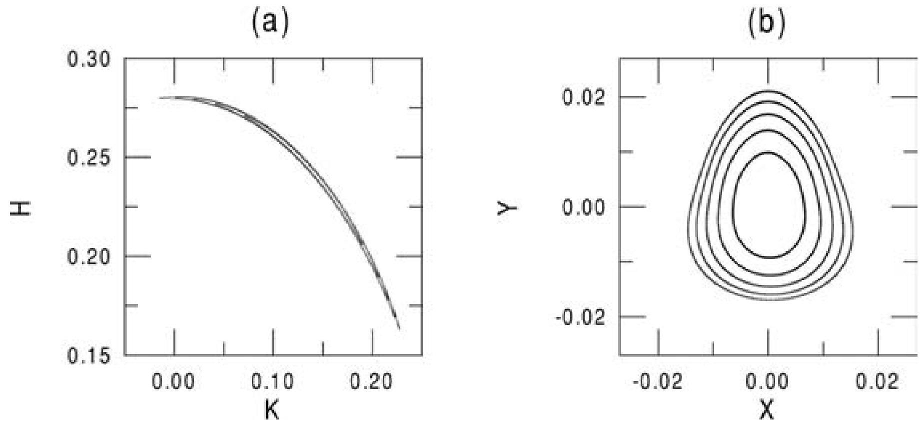

Figure 1. (a) A set of invariant curves representing Trojan-type librations. (b) The corresponding transformation to local variables given by Eq. (3.3).

modifies the shape of the trajectories. This transformation is canonical and is valid as long as $|\bar{\sigma}_{\max} - \bar{\sigma}_c| < \pi/2$ (with both $\bar{\sigma}_{\max}$ and $\bar{\sigma}_c$ defined between $\pm\pi$). Since the transformation is an explicit function of the center of libration, Beaugé and Roig called Eq. (3.3) the *local variables*. An example of this transformation is shown in Fig. 1.

By means of this very simple and purely geometrical "banana-to-pear" transformation, it is possible to bypass the difficulties generated by the libration of σ, and to define variables suitable for the application of Hori's averaging method. The averaged Hamiltonian can then be written in the form:

$$\bar{F} = \bar{F}_0(\theta, J, W^*, Z^*, W^{*\prime}, Z^{*\prime}) + \varepsilon \bar{F}_1(\theta, \varpi^*, \Omega^*, \varpi^{*\prime}, \Omega^{*\prime}, J, W^*, Z^*, W^{*\prime}, Z^{*\prime}) \qquad (3.4)$$

and it would be ready to proceed with the second average.

3.1. *Averaging Methods with Adiabatic Invariance*

In order to treat the second average, Beaugé and Roig (2001) introduced a general procedure to analyze multi-dimensional Hamiltonian systems having a "hierarchical" separation in time of the different degrees of freedom. The procedure can be summarized as follows: Suppose a generic two degrees of freedom system defined by a Hamiltonian

$$F \equiv F(J, \theta) = F_0(J_1, J_2) + F_1(J_1, J_2, \theta_1, \theta_2)$$

where (J, θ) are action-angle variables of F_0. Assuming that neither θ_1 nor θ_2 are resonant angles and that there are no significant commensurabilities between them, this system can be solved using Hori's averaging method. The transformation $(J, \theta) \to (J^*, \theta^*)$ up to first order, is given by the equations:

$$J_k = J_k^* + \frac{\partial B_1}{\partial \theta_k^*}; \qquad \theta_k = \theta_k^* - \frac{\partial B_1}{\partial J_k^*} \qquad (k = 1, 2) \qquad (3.5)$$

where B_1 is the first-order generating function. The idea is to think about Eqs. (3.5) as a system of 4 algebraic equations corresponding to two different sets of variables (the degrees of freedom). Instead of taking all equations simultaneously, the system is broken in two parts and a hypothesis of adiabatic invariance is adopted, assuming that the unperturbed frequencies of each degree of freedom satisfy the condition $\nu_1 \gg \nu_2$. In this way, the two equations corresponding to the first degree of freedom $(k = 1)$ can be solved separately, assuming fixed values for the second degree of freedom and writing the solution in terms of these values.

It is easy to show that this procedure is equivalent to solve the "one degree of freedom" Hamiltonian

$$F \equiv \widetilde{F}(J,\theta) = \widetilde{F}_0(J_1; J_2, \theta_2) + \widetilde{F}_1(J_1, \theta_1; J_2, \theta_2) \tag{3.6}$$

where (J_2, θ_2) are fixed parameters. Note that $\widetilde{F}$ is nothing but the original Hamiltonian F split in a different way. The solution of $\widetilde{F}$ by Hori's method will provide certain values $(\widetilde{J}_1^*, \widetilde{\theta}_1^*)$ of the action and angle, and the Theory of Adiabatic Invariants guarantees that the difference between this solution and the solution (J_1^*, θ_1^*) of the real system in which (J_2^*, θ_2^*) are slowly varying with time is such that

$$\widetilde{J}_1^* - J_1^* \propto \epsilon\, \mathcal{K}(\widetilde{J}_1^*, \widetilde{\theta}_1^*; J_2^*, \theta_2^*); \qquad \widetilde{\theta}_1^* - \theta_1^* \propto \epsilon\, \mathcal{L}(\widetilde{J}_1^*, \widetilde{\theta}_1^*; J_2^*, \theta_2^*)$$

where $\epsilon = \nu_2/\nu_1$, and $\mathcal{K}$ and $\mathcal{L}$ are functions of order unity (Henrard and Roels 1974). Since $\epsilon \ll 1$, both sets of solutions are approximately the same. In other words, using the adiabatic approximation, the new action-angle variables are determined up to order ϵ, and this parameter defines the precision of the method.

The action $\widetilde{J}_1^*$ is an invariant of the "frozen" Hamiltonian $\widetilde{F}$, but not of the full Hamiltonian F. Since (J_2, θ_2) vary slowly with time, so does $\widetilde{J}_1^*$, and according to the Adiabatic Theory, this variation is such that

$$\frac{d\widetilde{J}_1^*}{dt} \sim \epsilon^2.$$

For very small values of ϵ, this second order variation can be neglected and the resulting "constant" value of $\widetilde{J}_1^*$ is called an *adiabatic invariant* of Hamiltonian F. It is worth noting that these second order corrections to the adiabatic invariant are periodic with the same period of (J_2, θ_2). Thus, they could be eliminated by a suitable averaging of $\widetilde{J}_1^*$ over a period of (J_2, θ_2). In most cases, averaging the corrections provides a better approach to the adiabatic invariant than neglecting them.

Once the action-angle variables for the first degree of freedom have been determined (up to order ϵ), it is possible to solve the equations for the second degree of freedom. The idea is to introduce the solution $J_1^* = J_1^*(J_2^*, \theta_2^*)$ and $\theta_1^* = \theta_1^*(J_2^*, \theta_2^*)$ into the generating function B_1 and to solve the sub-system of Eqs. (3.5) corresponding to $k = 2$. This is equivalent to solve a one-degree of freedom non-autonomous Hamiltonian, since θ_1^* is a linear function of time. Actually, this is equivalent to take the original Hamiltonian F, introduce the solution for the first degree of freedom $J_1 = J_1(t, J_2, \theta_2)$, $\theta_1 = \theta_1(t, J_2, \theta_2)$, and average the resulting expression with respect to θ_1. The procedure leads to a new "one degree of freedom" Hamiltonian $\widehat{F}(\langle J_2\rangle_{\theta_1^*}, \langle\theta_2\rangle_{\theta_1^*})$, where $\langle . \rangle_{\theta_1^*}$ represents the average over θ_1^*, whose solution by Hori's method provides the corresponding action-angle variables (J_2^*, θ_2^*). In other words, we can average the original Hamiltonian F over a reference orbit of the first degree of freedom (which is obtained by adiabatic approximation assuming that the second degree of freedom is fixed), and then, we can use this averaged Hamiltonian to solve the second degree of freedom.

The whole procedure can be easily extended to the general case with N degrees of freedom in which the unperturbed frequencies ν_i of each angular variable θ_i are finite and large, and satisfy the condition $\nu_1 \gg \nu_2 \gg .... \gg \nu_N$. Thus, introducing the small parameters $\epsilon_{i,j} = \nu_j/\nu_i$ $(j > i)$ the system can be solved in a hierarchical form, solving one degree of freedom at a time.

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4. Resonant averaging theory

The main limitation of the B-R method is that it can be applied only under the hypothesis of adiabatic invariance. Unfortunately, this situation does not hold in other mean motion resonances, like the 3/2 with Jupiter or the 2/3 with Neptune, both associated to large populations of minor bodies. For these cases, a generalized resonant averaging theory has been formally introduced by Ferraz-Mello (1997, 2002), and has been recently applied by Miloni, Ferraz-Mello and Beaugé (in preparation).

Their method can be summarized as follows: Consider the Hamiltonian of the restricted planar three body problem

$$F = F_0(L, L') + m' F_1(\lambda, \varpi, \lambda', \varpi', L, W, L', W')$$

where F_0 is the Keplerian part, F_1 the disturbing function, and primed variables refer to the perturber. The disturbing function is further expanded using the expansion of Beaugé (1996), which does not have the convergence limitations of the classical expansions, nor the phase space domain limitations of the asymmetric expansions. The Hamiltonian is then given by a harmonic series with constant coefficients and can be manipulated in a fully analytical way.

Introducing the resonant angles

$$\sigma = \frac{p+q}{q}\lambda' - \frac{p}{q}\lambda - \varpi; \qquad \sigma' = \frac{p+q}{q}\lambda' - \frac{p}{q}\lambda - \varpi'$$

(p, q integers) and averaging (up to first order) over the short period angle $\lambda - \lambda'$, the Hamiltonian takes the form

$$\bar{F} = F_0(\bar{L}) + m' \bar{F}_1$$

In order to split this Hamiltonian for further averaging, the authors expand F_0 around the reference value $\bar{L}_0$ corresponding to the exact mean motion resonance, and assume that $\bar{L} - \bar{L}_0 \sim \mathcal{O}(m'^{1/2})$. This assumption is crucial since it allows to re-arrange terms of the same order in m' so as to write:

$$\bar{F} = m' \widetilde{F}_0(\sigma, S, S') + m'^{3/2} \widetilde{F}_1(\sigma, \sigma', S, S')$$

where $m' \widetilde{F}_0$ is a pendulum Hamiltonian, basically constituted by a quadratic term in $S \sim \bar{L} - \bar{L}_0$ plus a term $m' \cos\sigma$. The next step is to find the actions of the pendulum, which is accomplished by expanding the solution by means of elliptic integrals. This allows to explicitly compute the actions analytically in terms of σ, S. The resulting Hamiltonian takes the form:

$$F^* = m' F_0^*(J, J') + m'^{3/2} F_1^*(\theta, \theta', J, J')$$

and is suitable for the application of Hori's method to totally solve it. Note that in this case, the perturbation equations of Hori's method will be grouped in orders of $m'^{3/2}, m'^{5/2}$, and so on. In practice, the averaging is carried out up to the "first" order only. After finding the proper actions of F^*, it is possible to analytically go back with the transformation to compute the proper amplitude of libration S^* and the remaining proper elements.

The method has been successfully applied by the authors for a preliminary computation of proper elements of the Hilda asteroids in the planar case, and at present it is being extended to include the inclinations.

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5. Synthetic proper elements

The idea underneath the computation of synthetic proper elements is to fit the time series of the asteroid orbital elements to some predefined function. This function usually has the form of a harmonic series with a given number of harmonics. The fit consists of determining the frequencies, amplitudes and phases of the different harmonics by linear regression, which is nothing but to decompose the time series through a Fourier transform. After this decomposition, the signal can be easily filtered to remove all the periodic terms, leaving just the constant terms of the series which constitute the proper elements.

At variance with the usual averaging methods, which deals with the *solution of the averaged equations* of motion

$$\left\langle \frac{dq_i}{dt} \right\rangle = \left\langle \frac{\partial F}{\partial p_i} \right\rangle; \qquad \left\langle \frac{dp_i}{dt} \right\rangle = -\left\langle \frac{\partial F}{\partial q_i} \right\rangle,$$

the filtering deals with the *average of the solution*, that is $(\langle q_i \rangle, \langle p_i \rangle)$. The equivalence between both approaches is given by the condition

$$\frac{d\langle q_i \rangle}{dt} = \left\langle \frac{dq_i}{dt} \right\rangle; \qquad \frac{d\langle p_i \rangle}{dt} = \left\langle \frac{dp_i}{dt} \right\rangle.$$

However, this equivalence is strictly valid only if the averages are made over the "perturbed" solution. While the filtering fulfills this condition, the usual averaging methods don't because they are always made over an "unperturbed" or intermediate solution (e.g. the solution of the Hori's kernel). Therefore, the filtering always provides a more accurate approach to the proper elements than the usual averaging theories (both analytical and numerical).

Knežević and Milani (2000) used a synthetic theory to compute asteroid proper elements. They numerically integrated the orbit of the asteroid over intervals of time ranging from 2 to 10 Myr and performed a Fourier analysis of the output. The original output is represented by the time series of the equinoctal elements

$$(k, h) = e(\cos \varpi, \sin \varpi) \qquad (q, p) = \sin \frac{I}{2}(\cos \Omega, \sin \Omega)$$

These are filtered on-line in order to remove the short period variations related to the mean anomalies, which also allows to decimate the output and to reduce the data storage size. The filtered output is then processed in three steps:

(a) The forced secular perturbations with known frequencies $(g_5, g_6, ...)$ and $(s_5, s_6, ...)$ are removed from the filtered series by identifying the corresponding harmonics in the Fourier transform.

(b) The time series of the free angles (ϖ_f, Ω_f) are fitted by straight lines and the proper frequencies are determined from the slope of the fits.

(c) The components with period 2π are extracted from the data series $k(\varpi_f), h(\varpi_f)$ and $q(\Omega_f), p(\Omega_f)$. These constitute the proper modes and their amplitudes define the proper elements e_p and $\sin I_p/2$.

(d) The proper semi-major axis a_p is computed as the average of the filtered semi-major axis.

Simultaneously to the above procedure, the maximum Lyapunov Characteristic Exponent (LCE) is also computed by numerically solving the variational equations of the orbit. This is used as an indicator of chaos and provides an indication of the reliability of the computed proper elements.

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The K-M procedure requires to perform a numerical integration of the orbits over long time scales. This has an advantage in the sense that it automatically provides a stability test for the proper elements: Indeed, with the same output it is possible to compute proper elements over a running box with a shorter time width and to see how these values vary with time. However, the main disadvantage is that updates of the data to incorporate new asteroids are hard to perform, since they are very time consuming. In spite of this, the method provides very precise values of the proper elements.

5.1. *Synthetic resonant proper elements*

The extension of the synthetic theory to the case of resonant orbits is straightforward. In fact, synthetic theories for the computation of proper elements were first developed for resonant orbits, more specifically, for the Trojan case. The origins of the method go back to Bien and Schubart (1984) and Schubart and Bien (1987), and it was fully developed by Milani (1993).

The idea is to consider the time series of $a \exp \iota\sigma$, $e \exp \iota\varpi$ and $\sin I \exp \iota\Omega$. These time series are first filtered on-line to remove the short period variations and then it is applied the same kind of harmonic decomposition as explained above. However, the (filtered) time series of $a \exp \iota\sigma$ cannot be processed as such because σ is librating rather than circulating. Therefore, a transformation $(a, \sigma) \rightarrow (D, \theta)$ to local variables, like Eq. (3.2), is introduced:

$$D(\cos\theta, \sin\theta) = \left(\sigma - \sigma_c,\, \frac{a - a_c}{\gamma}\right) \tag{5.1}$$

where (σ_c, a_c) is the center of libration and γ is a scaling factor that relates D, i.e. the semi-amplitude of libration in σ, with the semi-amplitude of libration in a. The angle θ is no longer librating but circulating with a fundamental frequency equal to the frequency of libration. The synthetic theory is then applied to the time series of $D \exp \iota\theta$, and the proper semi-amplitude of libration D_p is obtained together with the corresponding proper frequency.

6. An application to the Trojan asteroids

The B-R method and the K-M method described above have been applied to compute proper elements for the Trojan asteroids. Many dynamical properties of these asteroids complicate the elaboration of an analytical model for their long-term motion, so the application of the K-M synthetic theory seems to be a better option. The K-M procedure involves the transformation Eq. (5.1) with $\sigma_c = \pm\pi/3$ and $a_c = a_{\mathrm{Jupiter}}$. This is a major limitation of their method because it is well known that, at large eccentricities and inclinations, the true center of libration may be significantly displaced from its "standard" location at $\pm\pi/3$. Thus, the method will produce fake estimates of D_p and, especially, of the proper frequency of libration, whenever D is smaller that the difference between the true center of libration and $\pi/3$. Fortunately, these cases are very rare among the real Trojan asteroids, and the K-M method can be safely applied in most cases.

On the other hand, there is a major dynamical feature that may be exploited by the B-R method: The different degrees of freedom of the system are well separated with respect to their periods. In fact, while the period of libration of the resonant angle σ is typically about 150 yr, the period of oscillation of ϖ is of the order of 3,500 yr, and the period of Ω is even longer: $10^5 - 10^6$ yr. In this way, it is possible to introduce the adiabatic approach to the problem defining the small parameters $\epsilon_{12} = \nu_\varpi/\nu_\sigma, \epsilon_{23} = \nu_\Omega/\nu_\varpi$ and $\epsilon_{13} = \nu_\Omega/\nu_\sigma$.

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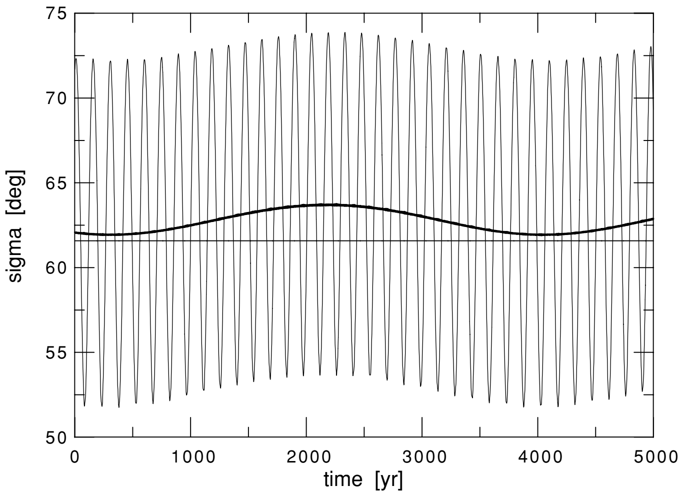

Figure 2. Time evolution of σ from the full Hamiltonian F (thin curve), from the circular planar Hamiltonian F_0 (thin horizontal line), and from the frozen Hamiltonian $\widetilde{F}_0$ (thick curve). Note that the solution of $\widetilde{F}_0$ follows the long term evolution of the center of libration while the solution of F_0 doesn't.

The B-R method starts with the averaged restricted three body problem for the 1/1 mean motion resonance, similar to Eq. (3.1):

$$F = F_0(\sigma, L, W, W') + \varepsilon F_1(\sigma, \varpi, \Omega, \varpi', \Omega', L, W, Z, W', Z')$$

where F_0 corresponds to the circular planar problem. Then, the transformation to local variables (Eq. 3.3) is introduced, which leads to a Hamiltonian like Eq. (3.4)

$$F = F_0(\theta, J, W, W') + \varepsilon F_1(\theta, \varpi, \Omega, \varpi', \Omega', J, W, Z, W', Z').$$

This Hamiltonian is further expanded using an *asymmetric* expansion around the center of libration, that is, a Taylor-Fourier expansion around $J = 0$, $W = 0$, and $Z = 0$†. The secular variation of Jupiter's orbit is introduced through the synthetic planetary theory LONGSTOP 1B (Nobili *et al.* 1989). The direct gravitational effects of Saturn, Uranus and Neptune on the asteroid are also included, assuming that these planets move on fixed circular orbits with zero inclination.

In order to average over the libration period, the Hamiltonian is re-arranged like in Eq. (3.6):

$$F = \widetilde{F}_0(J; \varpi, \Omega, \varpi', \Omega', W, Z, W', Z') + \mu \widetilde{F}_1(\theta, J; \varpi, \Omega, \varpi', \Omega', W, Z, W', Z') \qquad (6.1)$$

where $\varpi, \Omega, \varpi', \Omega', W, Z, W', Z'$ are taken as fixed parameters. Here, $\widetilde{F}_0$ is no longer the Hamiltonian of the circular planar problem and μ is a small parameter somehow related to the amplitude of libration. The key point is that $\widetilde{F}_0$ contains much more information than F_0, since it has embedded the slow variation of the other degrees of freedom. The advantage of this can be appreciated in Fig. 2. The "first order" solution of Hamiltonian Eq. (6.1) provides the proper action and angle (J^*, θ^*).

The averaging over the libration period leads to a new Hamiltonian that corresponds to a two degrees of freedom non autonomous system. This Hamiltonian is split so that

† Note that an expansion around $J = 0$ is indeed asymmetric

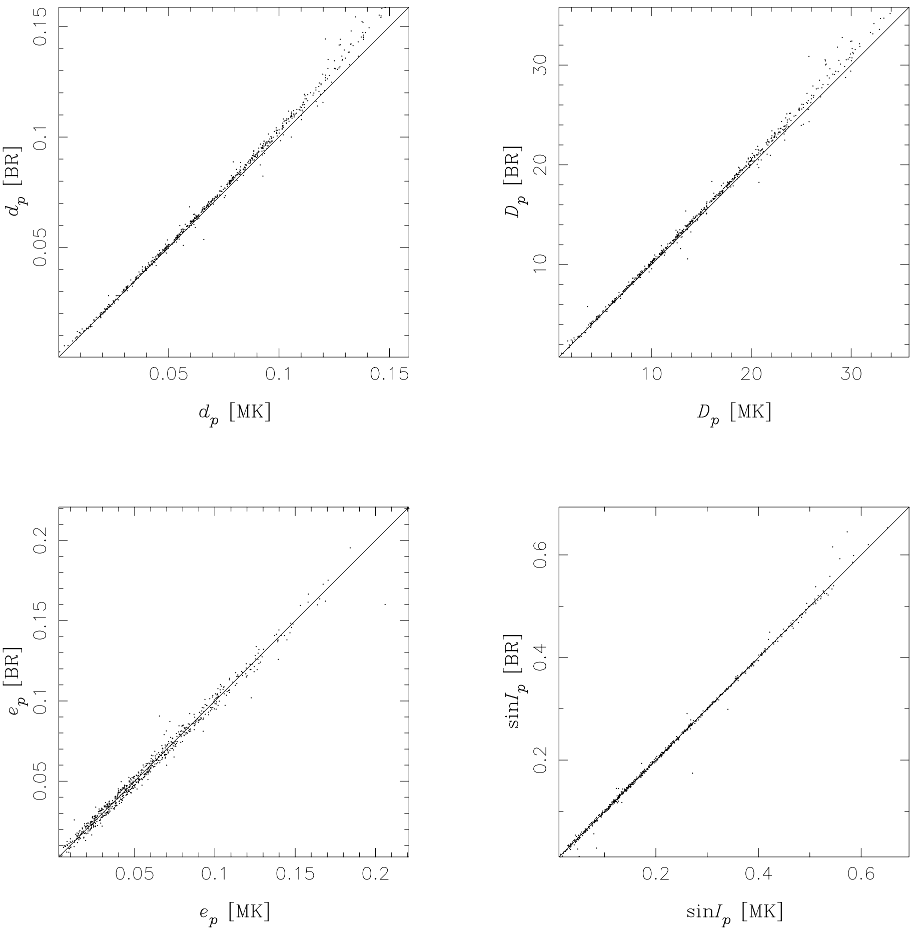

Figure 3. Comparison between the Trojan proper elements obtained with the B-R method (in the ordinates) and the K-M method (in the abscissas). The straight line in each plot represents the identity function. d_p is the proper semi-amplitude of libration in a.

all the terms depending on the angles $(\varpi, \Omega, \varpi', \Omega')$ are grouped in the "perturbation", and the "first order" solution by Hori's method provides the proper actions and angles $(W^*, Z^*, \varpi^*, \Omega^*)$. In the end, the set of proper actions (J^*, W^*, Z^*) depend solely on the initial conditions and on the orbital elements of the perturber, and are transformed back to the set (D_p, e_p, I_p).

A comparison between the results of the B-R method and those of the K-M synthetic theory is shown in Fig. 3. In the case of e_p, I_p, there is practically no differences between both sets. The agreement is also very good for small values of D_p, but there is a systematic bias of the B-R proper elements with respect to the K-M set for large amplitudes of libration. This bias is probably related to the early truncation of the asymmetric expansion of the disturbing function. However, it may also be related to the chaotic character of the large amplitude orbits that precludes the possibility of defining accurate proper elements. This is clearly shown in Fig. 4. In spite of this, we have verified that the systematic character of the bias does not significantly affect the identification of asteroid families.

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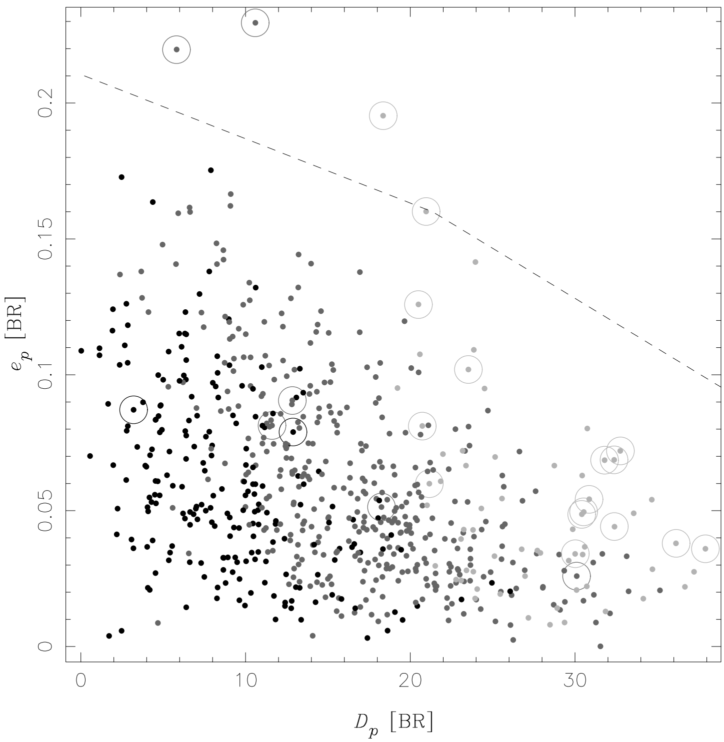

Figure 4. Trojan proper elements computed with the B-R method. The points are coded in a gray-scale according to their LCE (computed from the K-M method): the darker, the more stable. A circle around the dots indicates those cases for which the differences between the B-R and K-M proper elements are larger than 1%. Most of these cases are related to chaotic orbits.

7. Proper elements for planet crossing orbits

As we already said, the application of a perturbative method based on an averaging principle to solve the equations of motion of an asteroid under the perturbation of the planets will fail whenever the perturbation has a singularity along the domain of the solution. To avoid this problem, Gronchi and Milani (1999, 2001) introduced a generalized averaging principle, that can be summarized as follows.

Consider the restricted circular N-body problem, where the planets are assumed to move in circular and coplanar orbits. The Hamiltonian can be written as

$$F = F_0(L, L_i) + \varepsilon F_1(\lambda, \varpi, \lambda_i - \Omega, L, W, Z)$$

where F_1 is the disturbing function expanded in terms of the (modified) Delaunay variables and depends on the combination $\lambda_i - \Omega$. The secular motion of the system, under the absence of any mean motion resonances is given by the equations

$$\left\langle \frac{d\varpi}{dt} \right\rangle = \frac{\partial \langle F \rangle}{\partial W} \qquad \left\langle \frac{dW}{dt} \right\rangle = -\frac{\partial \langle F \rangle}{\partial \varpi}$$
$$\left\langle \frac{d\Omega}{dt} \right\rangle = \frac{\partial \langle F \rangle}{\partial Z} \qquad \left\langle \frac{dZ}{dt} \right\rangle = -\frac{\partial \langle F \rangle}{\partial \Omega} = 0 \tag{7.1}$$

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where $\langle X \rangle = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} X\, d\lambda\, d\lambda_i$. These expressions have embedded the property that

$$\left\langle \frac{\partial F}{\partial x} \right\rangle = \frac{\partial \langle F \rangle}{\partial x} \tag{7.2}$$

for any x provided that F were continuous and differentiable in all the domain of interest. The averaged Hamiltonian is a one degree of freedom Hamiltonian usually known as the Kozai Hamiltonian.

When the orbits of the asteroid and one planet intersect each other, F_1 has a first order pole, arising from the direct perturbation, and the $\partial F_1/\partial x$ have a second order pole. In such case $\langle F_1 \rangle$ is a complete improper integral, but $\langle \partial F_1/\partial x \rangle$ is a divergent integral and the equations for the secular motion make no sense. The strategy of Gronchi and Milani was to realize that this is true only at the point of intersection of the orbits, where the pole exists. At all the other points along the orbit the relation (7.2) is still valid, and so are Eqs. (7.1).

The idea is then to numerically integrate Eqs. (7.1) until arriving very close to the intersection point, then to suitably manipulate the jump across the singularity and to continue the integration. In order to manipulate the singularity, the authors introduce a method of singularity extraction proposed by Kantorovich which consists into decompose the direct part of the disturbing function in two terms whose averages can be computed in terms of convergent integrals. In short, they write

$$\frac{\partial}{\partial x}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{\Delta_i}\,d\lambda\,d\lambda_i = \frac{\partial}{\partial x}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{\delta_i}\,d\lambda\,d\lambda_i + \frac{\partial}{\partial x}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{1}{\Delta_i}-\frac{1}{\delta_i}\right)d\lambda\,d\lambda_i$$

where Δ_i is the actual mutual distance between the asteroid and the i-th planet, and δ_i is the mutual distance computed by assuming that, close to the mutual node, the bodies are moving along straight lines tangent to the orbits at that point (Wetherill 1967). With this method, the principal part of the singularity of $\frac{\partial}{\partial x}\frac{1}{\Delta_i}$ is removed, and the reminder $\frac{\partial}{\partial x}(\frac{1}{\Delta_i}-\frac{1}{\delta_i})$ has now a first order pole, so that

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\partial}{\partial x}\left(\frac{1}{\Delta_i}-\frac{1}{\delta_i}\right)d\lambda\,d\lambda_i = \frac{\partial}{\partial x}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{1}{\Delta_i}-\frac{1}{\delta_i}\right)d\lambda\,d\lambda_i$$

The principal term $\frac{\partial}{\partial x}\frac{1}{\delta_i}$ still has a second order pole, but the computation of

$$\frac{\partial}{\partial x}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{\delta_i}\,d\lambda\,d\lambda_i$$

can be performed analytically, giving rise to analytical expressions for the solution of the equations of motion close to the mutual nodes (Gronchi 2002). These expressions allow to jump over the singularity during the numerical integration of the equations.

The proper elements are then obtained from the solution of the Eqs. (7.1) by computing the average semi-major axis, the maximum and minimum eccentricity and the maximum and minimum inclination over one period of $\omega = \varpi - \Omega$ (either if ω is circulating or librating). The solution also provides the proper frequencies and the encounter circumstances with each planet (radiant, planetocentric velocity and date). This latter information allows to predict the occurrence of node crossings and is particularly interesting in the case of Earth crossing asteroids, since it provides a way to link meteor streams with NEAs and to evaluate potential Earth impactors.

Let's say to close this review that these proper elements do not have the same meaning as the usual proper elements since they are not quasi-integrals of motion in a strict sense.

Their stability is guaranteed only over a short time scale, either of the order of the period of a complete oscillation of ω, or until the next very close approach to a planet.

References

Beaugé, C. 1996, *Cel. Mech. Dyn. Astron.* 64, 313
Beaugé, C. & Roig, F. 2001, *Icarus* 153, 391
Bien, R. & Schubart, J. 1984, *Celest. Mech.* 34, 425
Ferraz-Mello, S. 1997, *Cel. Mech. Dyn. Astron.* 66, 39
Ferraz-Mello, S. 2002, *Cel. Mech. Dyn. Astron.* 83, 275
Gronchi, G. F. 2002, *Cel. Mech. Dyn. Astron.* 83, 97
Gronchi, G. F. & Milani, A. 1999, *Cel. Mech. Dyn. Astron.* 71, 109
Gronchi, G. F. & Milani, A. 2001, *Icarus* 152, 58
Henrard, J. & Roels, J. 1974, *Celest. Mech.* 10, 497
Jupp, A. H. 1969, *Astron. J.* 74, 35
Knežević, Z. & Milani, A. 2000, *Cel. Mech. Dyn. Astron.* 78, 17
Knežević, Z. & Milani, A. 2003, *Astron. Astrophys.* 403, 1165
Lemaitre, A. & Morbidelli, A. 1994, *Cel. Mech. Dyn. Astron.* 60, 29
Milani, A. 1993, *Cel. Mech. Dyn. Astron.* 57, 59
Milani, A. & Knežević, Z. 1990, *Cel. Mech. Dyn. Astron.* 49, 347
Morbidelli, A. 1993, *Icarus* 105, 48
Nobili, A. M., Milani, A. & Carpino, M. 1989, *Astron. Astrophys.* 210, 313
Schubart, J. & Bien, R. 1987, *Astron. Astrophys.* 175, 299
Wetherill, G. W. 1967, *J. Geophys. Res.* 72, 2429