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WEAKLY PURELY FINITELY ADDITIVE MEASURES

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ABSTRACT. Let L be an orthomodular poset. A positive measure ξ on L is said to be *weakly purely finitely additive* if the zero measure is the only completely additive measure majorized by ξ. It was shown in [15] that, in an arbitrary orthomodular poset L, every positive measures μ is the sum $\nu + \xi$ of a positive completely additive measure ν and a weakly purely finitely additive measure ξ. We give sufficient conditions for this Yosida-Hewitt-type decomposition to be unique.

A positive measure λ on L is said to be *filtering* if every non-zero element p in L majorizes a non-zero element q on which λ vanishes. A filtering measure is weakly purely finitely additive. Filtering measures play a mediator role throughout these investigations since some of the aforementioned conditions are given in terms of these.

The results obtained here are then viewed in the context of Boolean lattices and applied to lattices of idempotents of non-associative JBW-algebras.

1. **Introduction.** Let L be an orthomodular lattice or an orthocomplete orthomodular poset. Notice that a Boolean lattice together with its uniquely determined orthocomplementation is an orthomodular lattice. The complete lattice of self-adjoint idempotents, resp. idempotents, of a W^*-algebra, resp. JBW-algebra, admits a natural orthocomplementation which makes it into an orthocomplete orthomodular poset.

A positive measure ξ on L is said to be *weakly purely finitely additive* if the zero measure is the only completely additive measure majorized by ξ. Let $J^+(L)$, $J_c^+(L)$ and $J_{\text{wpfa}}^+(L)$ be the sets of positive measures, positive completely additive measures and weakly purely finitely additive measures on L, respectively. Using functional analytic methods it was shown by the author in [15] that, for an arbitrary orthomodular poset L,

$$J^+(L) = J_c^+(L) + J_{\text{wpfa}}^+(L)$$

holds true. Techniques developed in [9] lead to examples of orthomodular lattices for which this Yosida-Hewitt-type decomposition lacks the uniqueness feature. It is one of the goals of this paper to present sufficient conditions for this decomposition to be unique.

A positive measure λ on L is said to be *filtering* if for every non-zero element p in L there exists a non-zero element q in L such that $q \leq p$ and $\lambda(q)$ is equal to zero. Filtering measures may be viewed as anti-completely additive measures. In fact, a filtering measure

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is weakly purely finitely additive. The collection $J_f^+(L)$ of filtering measures on L forms a face of the cone $J^+(L)$.

The orthomodular poset L is said to be *c-positive* if for every non-zero element p in L there exists a positive completely additive measure ν such that $\nu(p) > 0$. We call L wpfa-*hereditary* provided that the property of a measure to be weakly purely finitely additive is preserved under restriction to segments of L. Complete Boolean lattices are wpfa-hereditary and, as shown at the end of this paper, the complete orthomodular lattice of idempotents of the most interesting examples of JBW-algebras are both c-positive and wpfa-hereditary.

Provided that L is a c-positive orthomodular lattice or a c-positive orthocomplete orthomodular poset, L is wpfa-hereditary if and only if the two sets $J^+_{\mathrm{wpfa}}(L)$ and $J_f^+(L)$ coincide. For an orthomodular lattice or an orthocomplete orthomodular poset the latter condition is equivalent to the condition that

$$J^+(L) = J_c^+(L) \oplus J_f^+(L)$$

from which the uniqueness of the aforementioned decomposition follows.

These results are then applied to the orthomodular lattice of idempotents of a JBW-algebra. As a by-product we obtain an extension to the context of JBW-algebras of the measure-theoretic version of a result by M. Takesaki [16], Theorem 3.8, on the characterization of singular states on W^*-algebras.

2. **Preliminaries.** Let $(L, \leq, ')$ be an orthocomplemented partially ordered set with 0 as the least and 1 as the greatest element. A pair (p, q) of elements in L is said to be *orthogonal*, denoted by $p \perp q$, provided that $p \leq q'$. By an *orthogonal subset* M of L we mean a subset having the property that each pair (p, q) of elements in M with $p \neq q$ is orthogonal. An element p in L is called an *atom* if, for non-zero elements q in L, $q \leq p$ implies that q equals p. The orthocomplemented poset $(L, \leq, ')$ is said to be *atomic*, if every non-zero element in L majorizes an atom.

An *orthomodular poset* [3], [6], [8] is an orthocomplemented partially ordered set $(L, \leq, ')$ satisfying the following conditions:

(i) if $p \perp q$ then $p \vee q$ exists and

(ii) if $p \leq q$ then $q = p \vee (q \wedge p')$.

If an orthomodular poset is indeed a lattice, *i.e.* suprema and infima of finite subsets exist, then it is referred to as an *orthomodular lattice*. An orthomodular poset is said to be *orthocomplete* provided that the supremum of every orthogonal subset exists. A pair (p, q) of elements in an orthomodular poset $(L, \leq, ')$ is called *compatible*, denoted by pCq, if

$$p = (p \wedge q) \vee (p \wedge q').$$

An orthomodular poset $(L, \leq, ')$ is said to be *Boolean* if every pair of elements in L is compatible. In this case $(L, \leq)$ is a Boolean lattice. Let $(B, \leq)$ be a Boolean lattice and let $': B \longrightarrow B$ be its uniquely determined orthocomplementation. Then $(B, \leq, ')$ is a Boolean orthomodular poset.

A non-empty subset M of a poset $(L, \leq)$ with 0 and 1 is called an *order filter* if, for elements p in M and elements q in L, $p \leq q$ implies that q belongs to M. An *order ideal* is defined dually. Let p be an element in L. Then the order ideal generated by the set $\{p\}$ coincides with the set $\{q \in L : q \leq p\}$, denoted by L_p. Let $(L, \leq, ')$ be an orthomodular poset and let p be an element in L. Then the mapping ${}^{p'}: L_p \longrightarrow L_p$ defined, for elements q in L_p, by

$$q^{p'} = q' \wedge p$$

is an orthocomplementation on the poset $(L_p, \leq|_{L_p})$ which makes it into an orthomodular poset. Quite frequently we write L, resp. L_p, to mean the orthomodular poset $(L, \leq, ')$, resp. $(L_p, \leq|_{L_p}, {}^{p'})$. If the orthomodular poset L is a lattice or orthocomplete, so is L_p, for every element p in L.

Let L be an orthomodular poset. An element μ in the product vector space $\mathbb{R}^L$ is said to be a *measure on* L provided that

$$\mu(p \vee q) = \mu(p) + \mu(q),$$

for elements p, q in L with $p \perp q$. The collection of measures on L is a linear subspace of $\mathbb{R}^L$. A measure μ is said to be *positive* if $\mu(p) \geq 0$ for all elements p in L. By orthomodularity, a positive measure is an isotone map on the poset $(L, \leq)$. Notice that the kernel $\ker \mu$ of a positive measure μ is an order ideal in L which is closed under the formation of finite orthogonal suprema. A measure is said to be *Jordan* if it can be represented as a difference of two positive measures. With $J(L)$, resp. $J^+(L)$, we denote the linear subspace of Jordan measures, resp. the cone of positive measures, on L. The partial ordering induced in $J(L)$ by the cone $J^+(L)$ is called the *natural ordering* in $J(L)$. A positive measure μ such that $\mu(1)$ equals 1 is said to be a *probability measure*; $\Omega(L)$ denotes the convex subset of $\mathbb{R}^L$ of probability measures on L. The cone $J^+(L)$ is closed and the set $\Omega(L)$ is compact in the product topology τ of $\mathbb{R}^L$, a locally convex Hausdorff topology. Moreover, $\Omega(L)$ is a cone base of the cone $J^+(L)$. If $\Omega(L) \neq \emptyset$ then the mapping $\|\cdot\|: J(L) \longrightarrow J(L)$ defined, for elements μ in $J(L)$, by

$$\|\mu\| = \inf\{s + t : \mu = s\eta - t\kappa, s, t \in \mathbb{R}_+, \eta, \kappa \in \Omega(L)\}$$

is a complete norm on $J(L)$, referred to as the *base norm on* $J(L)$. Notice that the base norm induces a topology on $J(L)$ which is finer than the topology $\tau|_{J(L)}$ and is additive on the cone $J^+(L)$. Since $\Omega(L)$ is convex and τ-compact it follows that the unit ball of $J(L)$ coincides with the convex hull of the set $\Omega(L) \cup -\Omega(L)$.

For details of these and other properties of measures on orthomodular posets the reader is referred to [10], [11], [12], [13], [14] and [15].

3. **Filtering measures.** Let $(L, \leq, ')$ be an orthomodular poset. A positive measure λ on L is said to be a *filtering measure* if, for elements p in L,

$$\ker \lambda \cap L_p = \{0\} \Longrightarrow p = 0.$$

In the case that $|L| > 2$, a positive measure λ is filtering if and only if the order filter generated by the collection of non-zero elements in the kernel $\ker \lambda$ of λ coincides with the set $L \setminus \{0\}$. The set of filtering measures on L is denoted by $J_f^+(L)$. Clearly, the zero measure is filtering. Notice that a filtering measure on L vanishes on the set $\mathcal{A}(L)$ of atoms in L. Therefore, if L contains a finite maximal chain then the zero measure is the only filtering measure on L. It also follows that a positive measure μ on an atomic orthomodular poset L is filtering if and only if μ vanishes on the set $\mathcal{A}(L)$.

PROPOSITION 3.1. *Let L be an orthomodular poset. The set $J_f^+(L)$ of filtering measures on L is a non-empty face of the cone $J^+(L)$ of positive measures on L.*

PROOF. Let κ, λ be elements in $J_f^+(L)$ and let p be a non-zero element in L. By hypothesis, there exists a non-zero element q in L such that $q \leq p$ and $\kappa(q)$ is equal to zero. Repeating the argument, there exists a non-zero element r in L such that $r \leq q$ and $\lambda(r)$ equals zero. By positivity of κ and λ, it follows that

$$r \in \ker \kappa \cap \ker \lambda \cap L_p \subseteq \ker(\kappa + \lambda) \cap L_p.$$

Therefore $\kappa + \lambda$ is an element in $J_f^+(L)$.

Suppose now that λ is an element in $J_f^+(L)$ and μ is an element in the cone $J^+(L)$ such that $\mu \leq \lambda$. Then, for every non-zero element p in L,

$$\{0\} \neq \ker \lambda \cap L_p \subseteq \ker \mu \cap L_p$$

and therefore μ is a filtering measure on L.

We shall need the following technical lemmata.

LEMMA 3.2. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Let λ be a filtering measure on L and let p be an element in L. Then p is the supremum of every maximal orthogonal subset of the set* $\ker \lambda \cap L_p$.

PROOF. Let L be an orthomodular poset. Let M be a maximal orthogonal subset of the set $\ker \lambda \cap L_p$. Clearly, p is an upper bound of M. Suppose now that p is not a least upper bound of M in L.

If L is an orthomodular lattice then there exists an element q in L such that $M \leq q$ and $p \not\leq q$. Then $M \leq p \wedge q < p$ and it follows, by orthomodularity, that $p \wedge (p \wedge q)'$ is different from zero. Since λ is a filtering measure there exists a non-zero element r in L such that

$$r \leq p \wedge (p \wedge q)' \qquad \text{and} \qquad \lambda(r) = 0.$$

Then r is a non-zero element in $\ker \lambda \cap L_p$ which is orthogonal to M; this contradicts maximality of M.

If L is orthocomplete then $\bigvee M < p$ and it follows, again by orthomodularity, that $p \wedge (\bigvee M)'$ is different from zero. A similar argument as before completes the proof.

LEMMA 3.3. *Let L be an orthomodular poset and let μ be a positive measure on L. If for every element p in L there exists an orthogonal subset M of L such that p is the supremum of M and μ vanishes on M then μ is a filtering measure on L.*

PROOF. Let p be a non-zero element in L and let M be a subset of L having the required properties. Then necessarily $M \neq \{0\}$ and therefore there exists a non-zero element q in L such that $q \leq p$ and $\mu(q)$ equals zero.

In the case of an orthocomplete orthomodular poset or an orthomodular lattice L the following result characterizes the filtering measures as precisely those positive measures the kernel of which is ortho-join dense in L.

COROLLARY 3.4. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Let μ be a positive measure on L. Then μ is filtering if and only if for every element p in L there exists an orthogonal subset M of the kernel* $\ker \mu$ *of μ such that p is the supremum of M.*

PROOF. This follows, by Zorn's Lemma, Lemma 3.2 and Lemma 3.3.

Notice that, in an orthomodular poset L, the property of a measure λ to be filtering is hereditary, *i.e.*, for every element p in L, the restriction to the orthomodular poset L_p of the filtering measure λ is a filtering measure on L_p.

4. **Weakly purely finitely additive measures.** Let $(L, \leq, ')$ be an orthomodular poset. A measure μ on L is said to be *completely additive* if, for every orthogonal subset M of L for which the supremum $\bigvee M$ exists, the real net

$$\Big(\mu\big(\bigvee N\big)\Big)_{N \in M^f}$$

converges to $\mu(\bigvee M)$, where $(M^f, \subseteq)$ is the collection of finite subsets of M directed by set-inclusion. Let $J_c(L)$ be the vector space of completely additive Jordan measures and let $J_c^+(L)$ be the cone of positive completely additive measures on L. By [11], Theorem 2.2, and [15], Lemma 2.4, $J_c^+(L)$ is a non-empty base norm semi-exposed face of the base norm closed cone $J^+(L)$.

LEMMA 4.1. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice and let p be an element in L. Then the restriction to L_p of a completely additive measure ν on L is a completely additive measure on the orthomodular poset L_p.*

PROOF. Let M be an orthogonal subset of L_p and suppose that q is a least upper bound of M in L_p. We show that q is a least upper bound in L. The assertion then follows immediately since an orthogonal subset of L_p is an orthogonal subset of L.

Let r be an upper bound of M in L. If L is a lattice then $q \wedge r$ is an element in L_p with $M \leq q \wedge r$. Then $q \leq q \wedge r$, hence $q \leq r$. If L is an orthocomplete orthomodular poset then the supremum $\bigvee M$ of M exists in L and, clearly, $\bigvee M \leq p$. Therefore $q \leq \bigvee M$ and since q is an upper bound if M it follows that q is equal to $\bigvee M$.

A positive measure ξ on L is said to be a *weakly purely finitely additive measure* if

$$\nu \leq \xi, \quad \nu \in J_c^+(L) \Longrightarrow \nu = 0.$$

Let $J_{\mathrm{wpfa}}^+(L)$ denote the collection of weakly purely finitely additive measures on L. It follows, by [15], Lemma 2.1, that $J_{\mathrm{wpfa}}^+(L)$ coincides with the union of all faces F of the cone $J^+(L)$ with the property that the intersection $F \cap J_c^+(L)$ is equal to $\{0\}$.

It was shown by the author in [15], Corollary 3.3, that, for every orthomodular poset L, the following Yosida-Hewitt-type decomposition [17] of the cone of positive measures holds true:

$$J^+(L) = J_c^+(L) + J_{\text{wpfa}}^+(L). \tag{1}$$

Since $J_c^+(L)$ is a face of the cone $J^+(L)$ we conclude, by [15], Lemma 2.3, that

$$J_c^+(L) = \{\mu \in J^+(L) : \xi \leq \mu, \xi \in J_{\text{wpfa}}^+(L) \Rightarrow \xi = 0\}. \tag{2}$$

The relationship between filtering measures and weakly purely finitely additive measures is given in the following lemma.

LEMMA 4.2. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Then every filtering measure on L is a weakly purely finitely additive measure on L.*

PROOF. Let λ be an element in $J_f^+(L)$ and suppose that ν is an element in $J_c^+(L)$ such that $\nu \leq \lambda$. By Corollary 3.4, there exists an orthogonal subset M of $\ker \lambda$ such that 1 is the supremum of M. Then ν vanishes on M and therefore, by complete additivity of ν, $\nu(1)$ is equal to zero. Since ν is positive it follows that ν is the zero measure.

5. Wpfa-**Heredity.** An orthomodular poset $(L, \leq, ')$ is said to be wpfa-*hereditary* if, for every element p in L and every element ξ in $J_{\text{wpfa}}^+(L)$, the restriction to L_p of ξ, denoted by $\xi|_{L_p}$, is an element in $J_{\text{wpfa}}^+(L_p)$. The orthomodular poset L is said to be *c-positive* if for every non-zero element p in L there exists an element ν in $J_c^+(L)$ such that $\nu(p)$ is different from zero.

PROPOSITION 5.1. *A complete Boolean orthomodular lattice B is* wpfa-*hereditary.*

PROOF. Let ξ be an element in $J_{\text{wpfa}}^+(B)$ and let p be an element in B. Let ν be an element in $J_c^+(B_p)$ and suppose that, for all elements r in B_p, $\nu(r) \leq \xi(r)$.

Define an element $\tilde{\nu}$ in $\mathbb{R}^B$, for elements q in B, as follows

$$\tilde{\nu}(q) = \nu(q \wedge p).$$

Using full compatibility in B we conclude, by standard arguments, that $\tilde{\nu}$ belongs to $J_c^+(B)$. Then, for all elements q in B,

$$\tilde{\nu}(q) = \nu(q \wedge p) \leq \xi(q \wedge p) \leq \xi(q \wedge p) + \xi(q \wedge p') = \xi\big((q \wedge p) \vee (q \wedge p')\big) \leq \xi(q).$$

Therefore $\tilde{\nu}$ is the zero measure on B and, hence, ν is the zero measure on B_p.

PROPOSITION 5.2. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. If the sets $J_{\text{wpfa}}^+(L)$ and $J_f^+(L)$ coincide then L is* wpfa-*hereditary.*

PROOF. Let ξ be an element in $J^+_{\mathrm{wpfa}}(L)$. Then $\xi|_{L_p}$ belongs to $J^+_f(L_p)$. Since $(L_p, \leq|_{L_p}, {}'^p)$ is an orthocomplete orthomodular poset or an orthomodular lattice it follows, by Lemma 4.2, that $\xi|_{L_p}$ belongs to $J^+_{\mathrm{wpfa}}(L_p)$.

The following result is crucial.

LEMMA 5.3. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Let μ be an element in $J^+(L)$ and let p be an element in L.*

Then for every element ν in $J^+_c(L)$ with $\nu(p) > 0$ there exists a non-zero element q in L_p and a real number $t > 0$ such that

$$\mu(r) < t\nu(r),$$

for all non-zero elements r in L_q.

PROOF. Let L be an orthomodular poset. Let μ be an element in $J^+(L)$ and let p be an element in L. Let ν be an element in $J^+_c(L)$ and suppose that $\nu(p) > 0$. Then there exists a real number $t > 0$ such that $\mu(p) < t\nu(p)$. Let P be the subset of L_p defined by

$$P = \{u \in L_p : t\nu(u) \leq \mu(u)\}.$$

The zero element of L is contained in P and p is an upper bound of the set P.

Let M be a maximal orthogonal subset of P. Then p cannot be a least upper bound of M in L. Suppose, to the contrary, that the supremum of M exists in L and is equal to p. Now, for all elements u in M,

$$t\nu(u) \leq \mu(u) \leq \mu(p).$$

It follows that, for all elements N in M^f,

$$t\nu\Big(\bigvee N\Big) \leq \mu\Big(\bigvee N\Big) \leq \mu(p)$$

and therefore

$$t\nu(p) = \lim_{N\in M^f} t\nu\Big(\bigvee N\Big) \leq \mu(p)$$

which is a contradiction.

If L is an orthomodular lattice or an orthocomplete orthomodular poset then, by orthomodularity, there exists a non-zero element q in L_p such that $q \perp M$. Consequently, by maximality of M,

$$P \cap L_q = \{0\}$$

and therefore $\mu(r) < t\nu(r)$, for all non-zero elements r in L_q.

THEOREM 5.4. *Let L be a c-positive orthocomplete orthomodular poset or a c-positive orthomodular lattice. Let $J^+_{\mathrm{wpfa}}(L)$ be the collection of weakly purely finitely additive measures on L and let $J^+_f(L)$ be the collection of filtering measures on L.*

Then L is wpfa*-hereditary if and only if the sets $J^+_{\mathrm{wpfa}}(L)$ and $J^+_f(L)$ coincide.*

PROOF. Suppose that L is wpfa-hereditary and let ξ be an element in $J^+_{\mathrm{wpfa}}(L)$. Let p be a non-zero element in L. By c-positivity of L, there exists an element ν in $J^+_c(L)$ with $\nu(p) \neq 0$. It follows, by Lemma 5.3, that there exists a non-zero element q with $q \leq p$ and a real number $t > 0$ such that $\xi \leq t\nu$ on L_q. By hypothesis, $\xi|_{L_q}$ is an element in $J^+_{\mathrm{wpfa}}(L_q)$ and, by Lemma 4.1, $t\nu|_{L_q}$ belongs to $J^+_c(L_q)$. By (2), we conclude that $\xi|_{L_q}$ is the zero measure on L_q. Therefore ξ is an element in $J^+_f(L)$.

It follows that $J^+_{\mathrm{wpfa}}(L)$ is a subset of $J^+_f(L)$. Lemma 4.2 completes the proof in one direction. The converse is a consequence of Lemma 5.2.

6. Decomposition theorems.

THEOREM 6.1. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice and let $J^+(L)$ be the cone of positive measures on L. Furthermore, let $J^+_c(L)$, $J^+_{\mathrm{wpfa}}(L)$ and $J^+_f(L)$ be the collection of positive completely additive measures, the collection of weakly purely finitely additive measures and the collection of filtering measures on L, respectively.*

Then TFAE:

(i) $J^+_{\mathrm{wpfa}}(L) = J^+_f(L)$;
(ii) $J^+(L) = J^+_c(L) + J^+_f(L)$;
(iii) $J^+(L) = J^+_c(L) \oplus J^+_f(L)$.

PROOF. (i) $\Rightarrow$ (ii): This follows from (1).

(ii) $\Rightarrow$ (iii): Let μ be an element in $J^+(L)$ and suppose that, for elements ν_1, ν_2 in $J^+_c(L)$ and elements λ_1, λ_2 in $J^+_f(L)$,

$$\mu = \nu_1 + \lambda_1 = \nu_2 + \lambda_2$$

holds true. Let p be an element in L and let M be a maximal orthogonal subset of the set $\ker(\lambda_1 + \lambda_2) \cap L_p$. By Proposition 3.1, $\lambda_1 + \lambda_2$ is a filtering measure on L and therefore, by Lemma 3.2, p is a supremum of M.

Since ν_1 and ν_2 agree on $\ker(\lambda_1 + \lambda_2)$ it follows, by complete additivity of ν_1 and ν_2, that $\nu_1(p)$ equals $\nu_2(p)$. Therefore

$$\nu_1 = \nu_2 \quad \text{and} \quad \lambda_1 = \lambda_2.$$

(iii) $\Rightarrow$ (i): By Lemma 4.2, $J^+_f(L)$ is a subset of $J^+_{\mathrm{wpfa}}(L)$. Let ξ be an element in $J^+_{\mathrm{wpfa}}(L)$. Then there exists an element ν in $J^+_c(L)$ and an element λ in $J^+_f(L)$ such that

$$\xi = \nu + \lambda.$$

Then $\nu \leq \xi$ and therefore ν is the zero measure which shows that ξ belongs to $J^+_f(L)$.

Combining this result with Theorem 5.4 we obtain the following corollary.

COROLLARY 6.2. *Let L be a c-positive orthocomplete orthomodular poset or a c-positive orthomodular lattice. Let $J^+(L)$, $J^+_c(L)$, $J^+_{\mathrm{wpfa}}(L)$ and $J^+_f(L)$ be as in Theorem 6.1. Then TFAE:*

(i) *L is* wpfa-*hereditary;*
(ii) $J^+_{\mathrm{wpfa}}(L) = J^+_f(L)$;
(iii) $J^+(L) = J^+_c(L) + J^+_f(L)$;
(iv) $J^+(L) = J^+_c(L) \oplus J^+_f(L)$.

PROOF. This follows, by Theorem 6.1 and Theorem 5.4.

LEMMA 6.3. *Let L be an orthomodular poset with $\Omega(L) \neq \emptyset$. Let $(J(L), \|\cdot\|)$ be the vector space of Jordan measures on L equipped with the base norm on $J(L)$.*

For every element μ in $J(L)$ there exist positive measures η, κ such that

$$\mu = \eta - \kappa \quad \textit{and} \quad \|\mu\| = \|\eta\| + \|\kappa\|.$$

PROOF. Let μ be a non-zero element in $J(L)$. The unit ball of $J(L)$ coincides with the set $\mathrm{con}\big(\Omega(L) \cup -\Omega(L)\big)$. Therefore there exist elements η, κ in $\Omega(L)$ and a real number t in the unit interval $[0, 1]$ such that

$$\frac{\mu}{\|\mu\|} = t\eta - (1-t)\kappa.$$

It follows that

$$\|t\|\mu\|\eta\| + \|(1-t)\|\mu\|\kappa\| = t\|\mu\| + (1-t)\|\mu\| = \|\mu\|.$$

Moreover, $t\|\mu\|\eta$ and $(1-t)\|\mu\|\kappa$ are elements in $J^+(L)$.

THEOREM 6.4. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Suppose that the vector space of Jordan measures $J(L)$ is different from $\{0\}$ and let $\|\cdot\|$ be the base norm on $J(L)$. Let $J^+(L)$ be the closed cone of positive measures on L and let $J^+_c(L)$ be the base norm semi-exposed face of $J^+(L)$ of positive completely additive measures on L.*

If the set $J^+_{\mathrm{wpfa}}(L)$ of weakly purely finitely additive measures coincides with the face $J^+_f(L)$ of $J^+(L)$ of filtering measures on L then there exists a unique linear projection $\mathcal{P}$ on $J(L)$ such that

$$\mathcal{P}J^+(L) = J^+_c(L) \quad \textit{and} \quad (\mathrm{id}_{J(L)} - \mathcal{P})J^+(L) = J^+_{\mathrm{wpfa}}(L). \tag{3}$$

Moreover,
(i) $\|\mu\| = \|\mathcal{P}\mu\| + \|(\mathrm{id}_{J(L)} - \mathcal{P})\mu\|$, $\forall \mu \in J(L)$;
(ii) $\mathrm{im}\,\mathcal{P} = J^+_c(L) - J^+_c(L)$;
(iii) $\ker \mathcal{P} = J^+_{\mathrm{wpfa}}(L) - J^+_{\mathrm{wpfa}}(L)$;
(iv) $J^+_{\mathrm{wpfa}}(L)$ *is a closed face of* $J^+(L)$;
(v) *The sets $J^+_c(L) \cap \Omega(L)$ and $J^+_{\mathrm{wpfa}}(L) \cap \Omega(L)$ constitute a pair of complementary split faces [1] of the τ-compact convex set $\Omega(L)$ of probability measures on L.*

PROOF. Let μ be an element in $J^+(L)$. By Theorem 6.1, there exists a unique element $Q\mu$ in $J_c^+(L)$ with the property that there is an element ξ in $J_{\mathrm{wpfa}}^+(L)$ such that μ is equal to $Q\mu + \xi$. Then the mappings $Q: J^+(L) \longrightarrow J^+(L)$ and $(\mathrm{id}_{J^+(L)} - Q): J^+(L) \longrightarrow J^+(L)$ are clearly idempotent and preserve positive linear combinations since $J_c^+(L)$ and $J_f^+(L)$ are faces of $J^+(L)$. The base norm is additive on the cone $J^+(L)$ and therefore, for every element μ in $J^+(L)$,

(4)
$$\|\mu\| = \|Q\mu\| + \|(\mathrm{id}_{J^+(L)} - Q)\mu\|.$$

Let κ_1, κ_2, η_1 η_2 be elements in $J^+(L)$ and suppose that

$$\kappa_1 - \eta_1 = \kappa_2 - \eta_2.$$

Then $\kappa_1 + \eta_2$ equals $\kappa_2 + \eta_1$ and therefore $Q\kappa_1 + Q\eta_2$ is equal to $Q\kappa_2 + Q\eta_1$, hence

$$Q\kappa_1 - Q\eta_1 = Q\kappa_2 - Q\eta_2.$$

We now define a mapping $\mathcal{P}: J(L) \longrightarrow J(L)$, for elements μ in $J(L)$, by

$$\mathcal{P}\mu = Q\kappa - Q\eta,$$

where κ, η are elements in $J^+(L)$ such that

$$\mu = \kappa - \eta.$$

It follows that $\mathcal{P}$ and $\mathrm{id}_{J(L)} - \mathcal{P}$ are linear projections on $J(L)$ extending Q and $\mathrm{id}_{J^+(L)} - Q$, respectively. Condition (3) is now easily verified.

(i): Let μ be an element in $J(L)$. Select elements η and κ in $J^+(L)$ such that the conditions of Lemma 6.3 are met. Then, by (4),

$$\begin{aligned}
\|\mu\| &= \|\mathcal{P}\mu + (\mathrm{id}_{J(L)} - \mathcal{P})\mu\| \leq \|\mathcal{P}\mu\| + \|(\mathrm{id}_{J(L)} - \mathcal{P})\mu\| \\
&= \|\mathcal{P}(\eta - \kappa)\| + \|(\mathrm{id}_{J(L)} - \mathcal{P})(\eta - \kappa)\| \\
&\leq \|Q\eta\| + \|(\mathrm{id}_{J^+(L)} - Q)\eta\| + \|Q\kappa\| + \|(\mathrm{id}_{J^+(L)} - Q)\kappa\| \\
&= \|\eta\| + \|\kappa\| = \|\mu\|.
\end{aligned}$$

(ii) and (iii): These follow immediately from (3).

(iv): By (i), the projection $\mathcal{P}$ is continuous. The assertion now follows from Proposition 3.1 and since the set $J_{\mathrm{wpfa}}^+(L)$ coincides with $\ker \mathcal{P} \cap J^+(L)$.

(v): This is now immediate since $\Omega(L)$ is a cone base of the cone $J^+(L)$.

THEOREM 6.5. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Let $J_c(L)$ be the vector space of completely additive Jordan measures on L and let $J_c^+(L)$ be the cone of positive completely additive measures on L.*

If the set $J_{\mathrm{wpfa}}^+(L)$ of purely finitely additive measures coincides with the set $J_f^+(L)$ of filtering measures on L then

$$J_c(L) = J_c^+(L) - J_c^+(L).$$

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PROOF. Let μ be an element in the vector subspace $J_c(L)$. By Theorem 6.4 (ii), the element $\mu - \mathcal{P}\mu$ belongs to $J_c(L)$. Then there exist, by Theorem 6.4 (iii), elements ξ_1 and ξ_2 in $J_f^+(L)$ such that

$$(\mathrm{id}_{J(L)} - \mathcal{P})\mu = \xi_1 - \xi_2.$$

Let p be an element in L. Let A and B be maximal orthogonal subsets in $\ker \xi_1 \cap L_p$ and $\ker \xi_2 \cap L_p$, respectively. Then, by Lemma 3.2,

$$\begin{aligned}(\mathrm{id}_{J(L)} - \mathcal{P})\mu(p) &= \lim_{M \in A^f} \Big(\xi_1\big(\bigvee M\big) - \xi_2\big(\bigvee M\big)\Big) \\ &= -\lim_{M \in A^f} \xi_2\big(\bigvee M\big) \le 0\end{aligned}$$

and, similarly,

$$\begin{aligned}(\mathrm{id}_{J(L)} - \mathcal{P})\mu(p) &= \lim_{N \in B^f} \Big(\xi_1\big(\bigvee N\big) - \xi_2\big(\bigvee N\big)\Big) \\ &= \lim_{N \in B^f} \xi_1\big(\bigvee N\big) \ge 0.\end{aligned}$$

Therefore $(\mathrm{id}_{J(L)} - \mathcal{P})\mu(p)$ is equal to zero, for all elements p in L, and it follows that μ coincides with $\mathcal{P}\mu$. The assertion is now a consequence of Theorem 6.4 (ii).

COROLLARY 6.6. *Let L be an orthocomplete orthomodular poset or an orthomodular lattice. Let $J^+(L)$, $J_c(L)$, $J_c^+(L)$ and $J_{\mathrm{wpfa}}^+(L)$ be the cone of positive measures, the subspace of completely additive Jordan measures, the cone of positive completely additive measures and the set of weakly purely additive measures on L, respectively.*

If L is c-positive and wpfa-*hereditary then*

$$J^+(L) = J_c^+(L) \oplus J_{\mathrm{wpfa}}^+(L) \quad \textit{and} \quad J_c(L) = J_c^+(L) - J_c^+(L).$$

PROOF. This follows, by Corollary 6.2 and Theorem 6.5.

7. **Applications to** JBW**-algebras.** A real algebra A, not necessarily associative, for which

$$a \circ b = b \circ a, \quad a \circ (b \circ a^2) = (a \circ b) \circ a^2,$$

holds true and which is also a Banach space with respect to a norm $a \longrightarrow \|a\|$ satisfying

$$\|a \circ b\| \le \|a\| \cdot \|b\|, \|a^2\| = \|a\|^2 \quad \text{and} \quad \|a^2\| \le \|a^2 + b^2\|$$

is said to be a JB-*algebra*.

An element a in A is called *central* if, for all elements b, c in A,

$$a \circ (b \circ c) = (a \circ b) \circ c.$$

An element a in A is called *positive* if there exists an element b such that

$$a = b \circ b.$$

The set A_+ consisting of positive elements in A forms a generating cone in A. An *idempotent* is an element p in A satisfying

$$p \circ p = p;$$

$U(A)$ denotes the collection of idempotents in A. Trivially, the zero-element 0 and the unit 1 of the algebra A are idempotents.

A JB-algebra A which possesses a, necessarily unique, Banach space pre-dual A_* is called a JBW-*algebra*. A JBW-algebra has a unit.

Let A be a JBW-algebra with Banach space pre-dual A_*. For each element a in A, the positive weak* continuous linear mapping $U_a\colon A \longrightarrow A$ is defined, for elements b in A, by

$$U_a b = \{a\, b\, a\},$$

where for elements a, b and c in A, the Jordan triple product is given by

$$\{a\, b\, c\} = a \circ (b \circ c) - b \circ (c \circ a) + c \circ (a \circ b).$$

Let $\leq$ denote the linear order relation on A induced by the cone A_+. The zero, resp. the unit, in the algebra A is the least, resp. the greatest, element in the poset $\big(U(A), \leq\big)$. Denote with a' the element $1 - a$. Then the mapping $'\colon U(A) \longrightarrow U(A)$ is an orthocomplementation which makes $\big(U(A), \leq, '\big)$ into a complete orthomodular lattice. Notice that

$$p \leq q \Leftrightarrow U_p U_q = U_p, \quad p \perp q \Leftrightarrow U_p U_q = 0, \quad pCq \Leftrightarrow U_p U_q = U_q U_p.$$

Moreover, if pCq then

$$p \wedge q = U_p q = p \circ q$$

and if $p \perp q$ then

$$p \vee q = p + q.$$

An idempotent z is central if and only if zCq for all idempotents q in A. The orthomodular poset $U(A)$ is Boolean if and only if A is an associative algebra. Notice that $U_p A$ is a sub-JBW-algebra, for every idempotent p in A, and that

$$U(U_p A) = U(A)_p.$$

Let the Banach space A be canonically embedded into its Banach space bi-dual A^{**}. Then the product on A, a bilinear map from A into A, admits a unique extension to a separately weak* continuous product on A^{**}. With respect to this product A^{**} is a JBW-algebra.

Let A^* be the Banach space dual of A. Let A^*_+ be the generating cone in A^* of all elements x such that $x(a) \geq 0$, for all elements a in A_+. Let the Banach space pre-dual A_* of A be canonically embedded into A^*. For every idempotent p in A there exists a norm one element x in $A^*_+ \cap A_*$ such that $x(p)$ is equal to one.

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There exists a unique central idempotent z in the JBW-algebra A^{**} such that the image of the pre-dual projection $U_{z,*}$ on A^* of the weak* continuous projection U_z on A^{**} coincides with A_*. It follows that

$$A^*_+ = U_{z,*}A^*_+ + U_{z',*}A^*_+ \quad \text{and} \quad U_{z,*}A^*_+ = A^*_+ \cap A_*. \tag{5}$$

For details and proofs of these the reader is referred to [7]; also compare [2], [5].

Let x be an element in A^*_+. Then the restriction to $U(A)$ of x, denoted by $\mathcal{R}x$, is a positive measure on the orthomodular lattice $U(A)$. If the JBW-algebra A does not contain a Type I_2-summand then, by [4], Theorem 2.1, for every element μ in $J^+\big(U(A)\big)$ there exists a unique element x in A^*_+ such that μ equals $\mathcal{R}x$. Using the weak* continuity of the linear functionals in $A^*_+ \cap A_*$ we conclude that

$$\mathcal{R}(A^*_+ \cap A_*) \subseteq J^+_c\big(U(A)\big). \tag{6}$$

Therefore $U(A)$ is a c-positive complete orthomodular lattice.

THEOREM 7.1. *Let A be a* JBW*-*algebra not containing a Type I_2-summand. Let $U(A)$ be the complete orthomodular lattice of idempotents in A. Let $J^+_{\mathrm{wpfa}}\big(U(A)\big)$ be the set of weakly purely finitely additive measures and let $J^+_f\big(U(A)\big)$ be the set of filtering measures on $U(A)$.*

Then, in the notation used above,

$$J^+_{\mathrm{wpfa}}\big(U(A)\big) = J^+_f\big(U(A)\big) = \mathcal{R}(U_{z',*}A^*_+).$$

PROOF. Let x be an element in $U_{z',*}A^*_+$ and let p be a non-zero idempotent in A. By a remark made above, Lemma 5.3 and (6), there exists an element y in $A^*_+ \cap A_*$ and a non-zero element $q \leq p$ such that $x(r) \leq y(r)$, for all elements r in $U(A)_q$. It follows, by spectral theory, that $x(a) \leq y(a)$, for all elements a in $(U_qA)_+$. Therefore, for all elements a in A_+,

$$U^*_q x(a) = x(U_q a) \leq y(U_q a) = U^*_q y(a).$$

Since the positive operator $U_{z',*}$ commutes with U^*_q we conclude that

$$0 \leq U^*_q x = U^*_q U_{z',*}x = U_{z',*}U^*_q x \leq U_{z',*}U^*_q y = U^*_q U_{z',*}y = 0,$$

where $\leq$ denotes the linear order relation on A^* induced by the cone A^*_+. Then, for all elements r in $U(A)_q$,

$$x(r) = U^*_q x(r) = 0.$$

Therefore $\mathcal{R}x$ belongs to $J^+_f\big(U(A)\big)$.

Let x be an element in A^*_+ such that $\mathcal{R}x$ belongs to $J^+_{\mathrm{wpfa}}\big(U(A)\big)$. By (5), there exists an element y in $A^*_+ \cap A_*$ and an element z in $U_{z',*}A^*_+$ such that x is equal to $y + z$. Then $\mathcal{R}y \leq \mathcal{R}x$ and since $\mathcal{R}y$ belongs to $J^+_c\big(U(A)\big)$ it follows that $\mathcal{R}y$ is the zero-measure on $U(A)$. Therefore $\mathcal{R}x$ is an element in $\mathcal{R}(U_{z',*}A^*_+)$.

Lemma 4.2 completes the proof.

COROLLARY 7.2. *Let A be a* JBW**-algebra not containing a Type I_2-summand. Let $U(A)$ be the c-positive complete orthomodular lattice of idempotents in A. Let $J^+\big(U(A)\big)$, $J^+_c\big(U(A)\big)$ and $J^+_{\mathrm{wpfa}}\big(U(A)\big)$ be the sets of positive measures, positive completely additive measures and weakly purely finitely additive measures on $U(A)$, respectively.*

Then, in the notation used above,

(i) $U(A)$ *is* wpfa*-hereditary;*

(ii) $J^+\big(U(A)\big) = J^+_c\big(U(A)\big) \oplus J^+_{\mathrm{wpfa}}\big(U(A)\big)$;

(iii) $J^+_c(L) = \mathcal{R}(A^*_+ \cap A)$.

PROOF. (i) and (ii): These follow, by c-positivity of $U(A)$, Theorem 7.1 and Corollary 6.2.

(iii): This follows by (ii), (5), (6) and Theorem 7.1.

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