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THE EXISTENCE OF A CLASS OF KIRKMAN SQUARES OF INDEX 2

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Abstract

A Kirkman square with index λ, latinicity μ, block size k and v points, $KS_k(v; \mu, \lambda)$, is a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) defined on a v-set V such that (1) each point of V is contained in precisely μ cells of each row and column, (2) each cell of the array is either empty or contains a k-subset of V, and (3) the collection of blocks obtained from the nonempty cells of the array is a (v, k, λ)-BIBD. For $\mu = 1$, the existence of a $KS_k(v; \mu, \lambda)$ is equivalent to the existence of a doubly resolvable (v, k, λ)-BIBD. In this case the only complete results are for $k = 2$. The case $k = 3$, $\lambda = 1$ appears to be quite difficult although some existence results are available. For $k = 3$, $\lambda = 2$ the problem seems to be more tractable. In this paper we prove the existence of a $KS_3(v; 1, 2)$ for all $v \equiv 3 \pmod{12}$.

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1. Introduction

A Kirkman square with index λ, latinicity μ, block size k and v points, $KS_k(v; \mu, \lambda)$, is a $t \times t$ ($t = \lambda(v-1)/\mu(k-1)$) array defined on a v-set V such that

(1) each point of V is contained in precisely μ cells of each row and column,
(2) each cell of the array is either empty or contains a k-subset of V, and
(3) the collection of blocks obtained from the nonempty cells of the array is a (v, k, λ)-BIBD.

The existence question for $KS_2(v; \mu, \lambda)$ has been completely settled [5]. For $\mu = 1$, the existence of a $KS_k(v; \mu, \lambda)$ is equivalent to the existence of a doubly resolvable (v, k, λ)-BIBD. A doubly resolvable (v, k, λ)-BIBD is denoted by

$DR(v,k,\lambda)$-BIBD. The existence question for $DR(v,k,\lambda)$-BIBDs with $k \geqslant 3$ is open. Of particular interest to us is the case $k = 3$. A necessary condition for the existence of a $KS_3(v;1,1)$ is $v \equiv 3 \pmod 6$. The best result, thus far, for $KS_3(v;1,1)$s is asymptotic.

THEOREM 1.1 [8]. *There exists a constant v_1 such that for all $v \geqslant v_1$ and $v \equiv 3 \pmod 6$ there exists a $KS_3(v;1,1)$.*

In this paper, we consider the next case $k = 3$ and $\lambda = 2$. $KS_3(v;1,2)$s are equivalent to $DR(v,3,2)$-BIBDs and have been called doubly resolvable twofold triple systems of order v (DRTTS(v)) ([1]). A necessary condition for the existence of a $KS_3(v;1,2)$ is $v \equiv 0 \pmod 3$. A $KS_3(3;1,2)$ defined on $\{\infty,0,1\}$ is

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It is known that there do not exist $KS_3(6;1,2)$ and $KS_3(9;1,2)$ [6]. The next smallest design has recently been constructed. A $KS_3(12;1,2)$ appears in [4]. $KS_3(v;1,2)$s are also known to exist for $v = 15$, 18, 21, 24, 27, 30 and 33. These designs were constructed using starters and adders ([1], for $v = 33$, Lemma 3.6). In the next section, we give some recursive constructions for $KS_3(v;1,2)$s. In the last section, we apply these constructions to prove the existence of $KS_3(v;1,2)$s for $v \equiv 3 \pmod{12}$.

2. Constructions

Let V be a set of v elements. Let $G_1, G_2, \dots, G_m$ be a partition of V into m sets. A $\{G_1, G_2, \dots, G_m\}$-frame F with block size k, index λ and latinicity μ is a square array of side v which satisfies the properties listed below. We index the rows and columns of F by the elements of V.

(1) Each cell is either empty or contains a k-subset of V.

(2) Let F_i be the subsquare of F indexed by the elements of G_i. F_i is empty for $i = 1, 2, \dots, m$.

(3) Let $j \in G_i$. Row j of F contains each element of $V - G_i$ μ times and column j of F contains each element of $V - G_i$ μ times.

(4) The collection of blocks obtained from the nonempty cells of F is a $GDD(v;k;G_1G_2,\dots,G_m;0,\lambda)$ (see [14] for GDD notation).

If $|G_i| = h$ for $i = 1, 2, \dots, m$, we call F a $(\mu,\lambda;k,m,h)$-frame.

We will use frames to provide some product constructions for $KS_3(v;1,2)$s. The first result uses a $(1,2,3;m,1)$-frame.

THEOREM 2.1. *If there exists a* $(1,2;3,m,1)$*-frame, a* $KS_3(n+1;1,2)$ *and three mutually orthogonal Latin squares of side* n*, then there is a* $KS_3(mn+1;1,2)$ *which contains as a subarray a* $KS_3(n+1;1,2)$.

PROOF. Let $V=\{1,2,\ldots,n\}$ and let $V_i = V \times \{i\}$ for $i=1,2,\ldots,m$. Let L_1, L_2 and L_3 be a set of three mutually orthogonal Latin squares of side n defined on V. L will denote the array of triples formed by the superposition of L_1, L_2 and L_3. L_{ijk} is the $n \times n$ array of triples formed by replacing each triple (a,b,c) in L with the triple (a_i, b_j, c_k) where $a_i \in V_i$, $b_j \in V_j$ and $c_k \in V_k$.

Let K_i be a $KS_3(n+1;1,2)$ defined on $V_i \cup \{\infty\}$. Let F be a $(1,2;3,m,1)$-frame defined on $\{1,2,\ldots,m\}$ such that i is missing from cell (i,i) for $i = 1,2,\ldots,m$.

We construct a $KS_3(mn+1;1,2)$ on $(V \times \{1,2,\ldots,m\}) \cup \{\infty\}$ as follows. Replace each triple (i,j,k) in F with the $n \times n$ array L_{ijk}. In each cell (i,i) of F, place the $n \times n$ array K_i for $i=1,2,\ldots,m$. The resulting array A has size $mn \times mn$. Each distinct pair in $(V \times \{1,2,\ldots,m\}) \cup \{\infty\}$ occurs twice in A. Each element in $(V \times \{1,2,\ldots,m\}$ occurs once in each row and each column of A. Thus, A is a $KS_3(mn+1;1,2)$.

The next result will be used for $(1,2;3,m,h)$-frames with $h=1,3$ and 6. This construction also appears in [2].

THEOREM 2.2. *If there exists a* $(1,2;3,m,h)$*-frame, a* $KS_3(hn+w;1,2)$ *which contains as a subarray a* $KS_3(w;1,2)$ $(w \geqslant 3)$ *and three mutually orthogonal Latin squares of side* n*, then there is a* $KS_3(hmn+w;1,2)$ *which contains as a subarray a* $KS_3(w;1,2)$.

PROOF. Let $V = \{x_1^i, x_2^i, \ldots, x_h^i \mid 1 \leqslant i \leqslant m\}$ and let $G_i = \{x_1^i, x_2^i, \ldots, x_h^i\}$ for $i=1,2,\ldots,m$. Let $W = \{\infty_1, \infty_2, \ldots, \infty_w\}$ and let $N=\{1,2,\ldots,n\}$.

Let L_1, L_2 and L_3 be a set of three mutually orthogonal Latin squares of side n defined on N. L will denote the array of triples formed by the superposition of L_1, L_2 and L_3. L_{ijk} is the $n \times n$ array of triples formed by replacing each triple (a,b,c) in L with the triple (a_i,b_j,c_k) where $a_i \in N \times \{i\}$, $b_j \in N \times \{j\}$ and $c_k \in N \times \{k\}$.

Let F be a $(1,2,3;m,h)$-frame defined on V. F is a $\{G_1, G_2, \ldots, G_m\}$-frame. Construct an $hmn \times hmn$ array H from F by replacing each triple (x,y,z) in F with the $n \times n$ array L_{xyz}. H contains a diagonal of m $hn \times hn$ empty arrays.

Let K_i denote a $KS_3(hn + w; 1, 2)$ defined on $(N \times G_i) \cup W$ which contains as a subarray a $KS_3(w; 1, 2)$ defined on W. Let A denote the subarray defined on W. K_i can be partitioned as follows.

$$K_i = \begin{array}{|c|c|}\hline A & B_i \\ \hline C_i & D_i \\ \hline \end{array} \begin{array}{l} \}w - 1 \\ \}hn \end{array}$$

where A and D_i are square arrays of side $w - 1$ and h respectively.

We now construct a new array K from H and the K_i's for $i = 1, 2, \ldots, m$. K is defined on $(N \times V) \cup W$.

$$K = \begin{array}{|ccccc|}\hline A & B_1 & B_2 & \cdots & B_m \\ C_1 & D_1 & & H & \\ C_2 & & D_2 & & \\ | & & & \ddots & \\ | & & & & \\ | & & & & \\ C_n & & & & D_m \\ \hline \end{array}$$

K is a square array of side $hnm + w - 1$. Each element of $(N \times V) \cup W$ occurs precisely once in each row and each column of K. Every distinct pair in $(N \times V) \cup W$ occurs twice in K. Thus, K is a $KS_3(hmn + w; 1, 2)$ which contains as a subarray a $KS_3(w; 1, 2)(A)$.

The last construction in this section is an indirect product for $KS_3(v; 1, 2)$s. Before describing the construction, we recall the definition of an $IA(n, k, s)$. Let V be a finite set of size n. Let K be a subset of size k of V. An incomplete orthogonal array $IA(n, k, s)$ is an $n^2 - k^2 \times s$ array written on the symbol set V such that every ordered pair of symbols in $V \times V - (K \times K)$ occurs in any ordered pair of columns from the array. We may think of an $IA(n, k, s)$ as a set of $s - 2$ mutually orthogonal Latin squares of order n which are missing a subsquare of order k. We need not be able to fill in the $k \times k$ missing subsquares with Latin squares of side k.

THEOREM 2.3. *Let u, v and w be non-negative integers such that $0 \leqslant u < w < v$. Suppose that $v - u \equiv 0 \pmod h$ and $w - u \equiv 0 \pmod h$. If there exists a $(1, 2, 3; m, h)$-frame, and $IA((v - u)/h, (w - u)/h, 5)$, a $KS_3(v + 1; 1, 2)$ which contains as a subarray a $KS_3(w + 1; 1, 2)$, and a $KS_3(m(w - u) + u + 1; 1, 2)$, then there exists a $KS_3(m(v - u) + u + 1; 1, 2)$.*

PROOF. Let $V = \{x_1^i, x_2^i, \ldots, x_h^i \mid 1 \leqslant i \leqslant m\}$, $W = \{1, 2, \ldots, (v - u)/yh\}$, $W_1 = \{1, 2, \ldots, (w - u)/h\}$ and $U = \{\infty_1, \infty_2, \ldots, \infty_{u+1}\}$. Let $G_i = \{x_1^i, x_2^i, \ldots, x_h^i\}$.

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Let F be a $(1,2;3,m,h)$-frame defined on V such that F is a $\{G_1, G_2, \ldots, G_m\}$-frame.

We construct a set of three mutually orthogonal Latin squares of order $(v-u)/h$ defined on W which are missing subsquares of order $(w-u)/h$ defined on W_1 in the upper left hand corners of the arrays from the $IA((v-u)/h, (w-u)/h, 5)$. Let I be the $(v-u)/h \times (v-u)/h$ array of triples formed from the superposition of these three squares. The array I_{ijk} will be the array of triples formed by replacing each triple (a,b,c) in I with the triple (a_i, b_j, c_k) where $a_i \in W \times \{i\}$, $b_j \in W \times \{j\}$ and $c_k \in W \times \{k\}$.

Next we construct an $m(v-u) \times m(v-u)$ array from F by replacing each triple (i,j,k) in F by the $(v-u)/h \times (v-u)/h$ array I_{ijk}. (Empty cells in F are replaced by $(v-u)/h \times (v-u)/h$ empty arrays.) Call the resulting array H'. H' contains a diagonal of m $(v-u) \times (v-u)$ empty arrays. We can partition H' into m^2 $(v-u) \times (v-u)$ arrays. Denote these subarrays by H'_{ij} for $i, j = 1, 2, \ldots, m$. We can permute the rows and columns of H' so that each subarray H'_{ij} contains an empty $(w-u) \times (w-u)$ array in the upper left hand corner. Call this array H. H also contains a diagonal of m $v-u \times v-u$ empty arrays. H is defined on $W \times V$.

Let A_i be a $KS_3(v+1;1,2)$ on $(W \times G_i) \cup U$ such that the subarray $KS_3(w+1;1,2)$ is defined on $(W_1 \times G_i) \cup U$. We can partition A_i as follows.

$$A_i = \begin{array}{c} w\left\{\begin{array}{c} u\{ \\ \\ \end{array}\right. \\ v-w\,\{ \end{array} \underbrace{\begin{array}{|c|c|c|} \hline & & R_i \\ \hline & & S_i \\ \hline C_i & T_i & K_i \\ \hline \end{array}}_{v-u} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} v-u$$

We now construct a square array of side $m(v-u)+u$ using the A_i and H. This array will be called B_1 and has the following form.

E	E	R_1	E	R_2	E	E	R_m
E	E	S_1	E		E	E	
C_1	T_1	K_1					H
E C_2	E		E T_2	S_2 K_2			
					$\ddots$		
E C_m	E					E T_m	S_m K_m

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The arrays labelled E in B_1 are empty. They form an $m(w-u)+u \times m(u-u)+u$ array. Place a $KS_3(m(w-u)+u+1;1,2)$ defined on $(W_1 \times V) \cup U$ in this array. The resulting array B is a $KS_3(m(v-u)+u+1;1,2)$ on $(W \times V) \cup U$. Every pair of distinct elements in $(W \times V) \cup U$ occurs precisely twice in B since F and the Kirkman squares used to construct B had index $\lambda = 2$. It can be verified that each element in $(W \times V) \cup U$ occurs once in each row and each column of B.

3. Applications

In order to apply the constructions from the previous section, we will need the following results on frames from [2].

THEOREM 3.1. [2] *There exist* $(1,2;3,m,3)$*-frames for* $m \geqslant 5$ *except possibly for* $m \in \{6, 10, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 38, 39, 42, 43, 44, 46, 47, 48, 51, 52, 59, 118, 123\}$.

THEOREM 3.2. [2] *There exist* $(1,2;3,m,6)$*-frames for* $m \geqslant 5$ *except possibly for* $m \in \{10, 11, 14, 15, 17, 18, 19, 20, 23, 24, 27, 28, 32, 34, 39\}$.

We note that one more value can be deleted from the list of exceptions to Theorem 3.1.

LEMMA 3.3. *There exists a* $(1,2;3,48,3)$*-frame.*

PROOF. Apply the frame singular direct product [2] using a $(1,2;3,6,6)$-frame, three mutually orthogonal Latin squares of side 4 and a $(1,2;3,8,3)$-frame.

The constructions also require the existence of some $KS_3(v;1,2)$s which contain as subarrays $KS_3(w;1,2)$s where $w \geqslant 3$.

LEMMA 3.4. *There exists a* $KS_3(v;1,2)$ *which contains as a subarray a* $KS_3(3;1,2)$ *for* $v = 15, 21, 27, 39, 51, 63$ *and* 81. *Furthermore, there exists a* $KS_3(63;1,2)$ *which contains as a subarray a* $KS_3(15;1,2)$.

PROOF. A $KS_3(15;1,2)$ is displayed in Figure 1. A starter and adder for a $KS_3(21;1,2)$ are listed in [1]. Since there exist $KS_3(v;1,1)$ for $v = 27, 39, 51, 63$ and 81 ([3], [9], [11], [12]), there exists $KS_3(v;1,2)$ which contain as subarrays $KS_3(3;1,2)$ for $v = 27, 39, 51, 63$ and 81. To construct a $KS_3(63;1,2)$ which contains as a subarray a $KS_3(15;1,2)$, we apply Theorem 2.2 using a $(1,2;3,5,3)$-frame, a $KS_3(15;1,2)$ which contains a $KS_3(3;1,2)$ and 3 mutually orthogonal Latin squares of side 4.

$\infty 0 \bar{0}$	$2 \bar{4} 5$	$4 \bar{2} \bar{1}$		$1 \bar{6} \bar{2}$			$3 5 6$						
	$\infty 1 \bar{1}$	$3 \bar{5} \bar{6}$	$5 \bar{4} \bar{2}$		$2 \bar{0} \bar{2}$			$4 6 0$					
		$\infty 2 \bar{2}$	$4 \bar{6} \bar{0}$	$6 \bar{5} \bar{2}$		$2 \bar{1} \bar{4}$			$5 0 1$				
$4 \bar{2} \bar{5}$			$\infty 2 \bar{2}$	$5 \bar{0} \bar{1}$	$0 \bar{6} \bar{4}$					$6 1 2$			
	$5 \bar{2} \bar{6}$			$\infty 4 \bar{4}$	$6 \bar{1} \bar{2}$	$1 \bar{0} \bar{5}$					$0 2 3$		
$2 \bar{1} \bar{6}$		$6 \bar{4} \bar{0}$			$\infty 5 \bar{5}$	$0 \bar{2} \bar{3}$						$1 3 4$	
$1 \bar{2} \bar{4}$	$3 \bar{2} \bar{0}$		$0 \bar{5} \bar{1}$			$\infty 6 \bar{6}$							$2 4 5$
$3 5 6$							$\infty 0 \bar{0}$	$2 \bar{4} 5$	$4 \bar{3} \bar{1}$		$1 \bar{6} \bar{2}$		
	$4 6 0$							$\infty 1 \bar{1}$	$3 \bar{5} \bar{6}$	$5 \bar{4} \bar{2}$		$2 \bar{0} \bar{3}$	
		$5 0 1$							$\infty 2 \bar{2}$	$4 \bar{6} \bar{0}$	$6 \bar{5} \bar{3}$		$2 \bar{1} \bar{4}$
			$6 1 2$				$4 \bar{2} \bar{5}$			$\infty 3 \bar{2}$	$5 \bar{0} \bar{1}$	$0 \bar{6} \bar{4}$	
				$0 2 3$				$5 \bar{3} \bar{6}$			$\infty 4 \bar{4}$	$6 \bar{1} \bar{2}$	$1 \bar{0} \bar{5}$
					$1 32 4$		$2 \bar{1} \bar{6}$		$6 \bar{4} \bar{0}$			$\infty 5 \bar{5}$	$0 \bar{2} \bar{3}$
						$2 4 5$	$1 \bar{3} \bar{4}$	$3 \bar{2} \bar{0}$		$0 \bar{5} \bar{1}$			$\infty 6 \bar{6}$

FIGURE 1.
A $KS_3(15;1,2)$ which contains a $KS_3(3;1,2)$.

Finally, we require three designs which we constructed directly using starters and adders and the following result. For definitions and results on 1-rotational $(v, 3, 1)$-BIBDs, see [7].

LEMMA 3.5. *Let $k = (v-3)/6$. Let $(B_0, B_1, \ldots, B_k)$ be a starter for a 1-rotational $(v, 3, 1)$-BIBD defined on $Z_{v-1} \cup \{\infty\}$. Let $A = (a_0, a_1, \ldots, a_k)$ be an adder for S. Suppose S and A have the following properties.*

(1) $B_0 = \{\infty, 0, (v-1)/2\}$ *and* $a_0 = 0$.
(2) *If* $b \in B_i$ *for some* i, $1 \leqslant i \leqslant k$, *then* $-b \notin B_j$ *for* $j = 0, 1, \ldots, k$.
(3) *For* $i = 1, 2, \ldots, k$, $a_i \neq 0$ *or* $(v-1)/2$.
(4) $a_i + a_j \not\equiv 0 \pmod{v-1}$ *for* $1 \leqslant i, j \leqslant k$.
Then there exists a $KS_3(v; 1, 2)$.

PROOF. If $B_i = \{x, y, z\}$, define $-B_i = \{-x, -y, -z\} = (v-1-x, v-1-y, v-1-z\}$. A starter for a $KS_3(v; 1, 2)$ is $S \cup \{-B_1, -B_2, \ldots, -B_k\}$ and a corresponding adder is $A \cup \{-a_1, -a_2, \ldots, -a_k\}$.

It is known that 1-rotational $(v, 3, 1)$-BIBDs exist if and only if $v \equiv 3$ or $9 \pmod{24}$, [7].

LEMMA 3.6. *There exist* $KS_3(v; 1, 2)$ *for* $v = 33$, 57 *and* 75.

PROOF. In Table 3, we list the starters and adders required to apply Lemma 3.5.

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We are now in a position to prove our main result.

THEOREM 3.7. *There exist a* $KS_3(v; 1, 2)$ *which contains a subarray* $KS_3(3; 1, 2)$ *for* $v \equiv 3 \pmod{12}$.

TABLE 3

Starters and adders for $KS_3(v; 1, 2)$ for $v = 33, 57$ and 75

$v = 33$						
Starter	∞ 0 16	1 2 8	7 9 21	3 6 14	15 19 28	5 10 20
Adder	0	4	22	1	27	14
$v = 57$						
Starter	∞ 0 28	1 2 14	3 5 27	4 7 22	44 48 13	30 35 46
Adder	0	1	2	13	42	35
		19 25 45	33 40 50	9 17 36	15 24 38	
		19	12	7	51	
$v = 75$						
Starter	∞ 0 37	1 2 17	3 5 34	4 7 24	6 10 38	8 13 31
Adder		1	2	4	10	11
		16 22 49	44 51 65	54 62 14	33 42 55	18 28 53
		35	8	47	7	52
		15 26 45	27 39 63			
		15	56			

PROOF. Let $v = 12m + 3$. By Lemma 3.4, there exist $KS_3(12m + 3; 1, 2)$ for $m = 0, 1, 2, 3$ and 4. All of these arrays contain a $KS_3(3; 1, 2)$ as a subarray.

Let $N_1 = \{10, 14, 16, 18, 22, 24, 26, 30, 34, 38, 42, 46\}$, $N_2 = \{24, 39, 51, 123\}$, $N_3 = \{20, 28, 32, 44, 52\}$ and $N_4 = \{6, 43, 47, 59, 118\}$. Let $N = \bigcup_{i=1}^{4} N_i$.

Since there exist $(1, 2; 3, m, 3)$-frames for $m \geqslant 5$, $m \notin N$ (Theorem 3.1, Lemma 3.3), we can apply Theorem 2.2. We first use it with $h = 3$, $w = 3$ and $n = 4$. Since there exist three mutually orthogonal Latin squares of side 4 and a $KS_3(15; 1, 2)$ with a $KS_3(3; 1, 2)$ as a subarray, there exist $KS_3(12m + 3; 1, 2)$ for $m \geqslant 5$ and $m \notin N$.

Since there exists a $KS_3(27; 1, 2)$ with a $KS_3(3; 1, 2)$ as a subarray and three mutually orthogonal Latin squares of side 8, we apply Theorem 2.2 with $h = 3$, $w = 3$ and $n = 8$ to construct $KS_3(24m + 3; 1, 2)$ for $m \geqslant 5$, $m \notin N$. This will construct $KS_3(12m + 3; 1, 2)$ for $m \in N_1$. Similarly, we can apply Theorem 2.2 with $h = 3$, $w = 3$ and $n = 12$ to construct $KS_3(36m + 3; 1, 2)$ for $m \geqslant 5$, $m \notin N$. This will construct $KS_3(12m + 3; 1, 2)$ for $m \in N_2$. Applying Theorem 2.2 again with $h = w = 3$ and $n = 16$ will construct $KS_3(48m + 3; 1, 2)$ for $m \geqslant 5$, $m \notin N$. This will provide $KS_3(12m + 3; 1, 2)$ for $m \in N_3$.

There are now five values of m left to consider, $m \in N_4 = \{6, 43, 47, 59, 118\}$. By Lemma 3.6, there exists a $KS_3(12 \cdot 6 + 3; 1, 2)$. We construct a $KS_3(12 \cdot 43 + 3; 1, 2)$ by applying Theorem 2.1 with $m = 37$ and $n = 14$ since $12 \cdot 43 + 3 = 37 \cdot 14 + 1$. (A $(1, 2; 3, 37, 1)$-frame is constructed in [13].) A

$KS_3(12 \cdot 118 + 3; 1, 2)$ can be constructed by applying Theorem 2.2 with $m = 59$, $h = 6$, $w = 3$ and $n = 4$.

We use the indirect product (Theorem 2.3) for the two remaining values of m. There exist a $(1, 2; 3, 10, 1)$-frame [1], a $KS(63; 1, 2)$ which contains as a subarray a $KS_3(87; 1, 2)$ and an $IA(56, 8, 5)$. By applying Theorem 2.3 with the parameters $v = 62$, $w = 14$, $u = 6$, $h = 1$ and $m = 10$, we construct a $KS_3(12 \cdot 47 + 3; 1, 2)$. Since there exists a $(1, 2; 3, 13, 6)$-frame (Theorem 3.2), a $KS_3(63; 1, 2)$ which contains a $KS_3(15; 1, 2)$, a $KS_3(87; 1, 2)$ and an $IA(9, 1, 5)$, we can apply Theorem 2.3 again with $m = 13$, $h = 6$, $v = 62$, $w = 14$ and $u = 8$ to construct a $KS_3(13(54) + 9; 1, 2)$. This is a $KS_3(12 \cdot 59 + 3; 1, 2)$.

Note that each of the arrays that we have constructed contains as a subarray a $KS_3(3; 1, 2)$. □

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