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SUPREMUM AND INFIMUM OF SUBHARMONIC FUNCTIONS OF ORDER BETWEEN 1 AND 2

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Abstract For functions u, subharmonic in the plane, let

$$A(r,u) = \inf_{|z|=r} u(z),$$
$$B(r,u) = \sup_{|z|=r} u(z)$$

and let $N(r,u)$ be the integrated counting function. Suppose that $\mathcal{N}\colon [0,\infty) \to \mathbb{R}$ is a non-negative non-decreasing convex function of $\log r$ for which $\mathcal{N}(r) = 0$ for all small r and $\limsup_{r\to\infty} \log \mathcal{N}(r)/\log r = \rho$, where $1 < \rho < 2$, and define

$$\mathcal{A}(r,\mathcal{N}) = \inf\{A(r,u)\colon N(r,u) = \mathcal{N}(r)\},$$
$$\mathcal{B}(r,\mathcal{N}) = \sup\{B(r,u)\colon N(r,u) = \mathcal{N}(r)\}.$$

A sharp upper bound is obtained for $\liminf_{r\to\infty} \mathcal{B}(r,\mathcal{N})/\mathcal{N}(r)$ and a sharp lower bound is obtained for $\limsup_{r\to\infty} \mathcal{A}(r,\mathcal{N})/\mathcal{N}(r)$.

Keywords: subharmonic; supremum; infimum

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1. Introduction

In [**1**, **2**] Rossi and Fenton showed that a method of Beurling is effective in approaching questions on the supremum and infimum of subharmonic and delta-subharmonic functions of order less than 1. The intention here is to apply Beurling's method to subharmonic functions u of order between 1 and 2.

It involves no loss of generality in our results to assume that u is harmonic at the origin. For such functions there is, from the Riesz representation theorem, a Borel measure μ such that

$$u(z) = \alpha + \mathrm{Re}(\beta z) + \int_{|\zeta|<\infty} \log|E(z/\zeta)|\,\mathrm{d}\mu(\zeta), \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are constants and $E(z) = \mathrm{e}^z(1-z)$. We define

$$A(r,u) = \inf_{|z|=r} u(z),$$

$$B(r,u) = \sup_{|z|=r} u(z)$$

and

$$N(r,u) = \int_0^r \frac{\mu^*(t)}{t}\,\mathrm{d}t, \tag{1.2}$$

where $\mu^*(r) = \mu(\{|z| < r\})$. Since $\mu^*(t) = 0$ for all small t (u being harmonic at 0), N is well defined. If u has order ρ, $1 < \rho < 2$, then

$$\limsup_{r\to\infty} \frac{\log N(r,u)}{\log r} = \rho,$$

and, conversely, if $\mathcal{N}\colon [0,\infty) \to \mathbb{R}$ is a non-negative non-decreasing convex function of $\log r$, for which $\mathcal{N}(r) = 0$ for all small r and

$$\limsup_{r\to\infty} \frac{\log \mathcal{N}(r)}{\log r} = \rho, \tag{1.3}$$

and if μ is a Borel measure for which μ^* is given by (1.2), then u given by (1.1) is subharmonic in the plane and has order ρ [**3**, p. 146].

Our main result concerns functions that have the same N. With

$$\mathcal{A}(r,\mathcal{N}) = \inf\{A(r,u)\colon N(r,u) = \mathcal{N}(r)\}, \tag{1.4}$$

$$\mathcal{B}(r,\mathcal{N}) = \sup\{B(r,u)\colon N(r,u) = \mathcal{N}(r)\}, \tag{1.5}$$

we have the following result.

Theorem 1.1. *Suppose that $\mathcal{N}\colon [0,\infty) \to \mathbb{R}$ is a non-negative non-decreasing convex function of $\log r$ for which $\mathcal{N}(r) = 0$ for all small r and (1.3) holds, where $1 < \rho < 2$. Then*

$$\limsup_{r\to\infty} \frac{\mathcal{A}(r,\mathcal{N})}{\mathcal{N}(r)} \geqslant c(\rho), \tag{1.6}$$

where

$$c(\rho) = \rho\biggl(\pi\cot(\pi(\rho-1)) - \frac{2T^{1-\rho}}{\rho-1} - \int_0^{1/T} \frac{2t^\rho}{1-t^2}\,\mathrm{d}t\biggr) \tag{1.7}$$

and $T \approx 1.2$ is the positive solution of the equation

$$2T = \log\left|\frac{T+1}{T-1}\right|; \tag{1.8}$$

and

$$\liminf_{r\to\infty} \frac{\mathcal{B}(r,\mathcal{N})}{\mathcal{N}(r)} \leqslant C(\rho), \tag{1.9}$$

where

$$C(\rho) = \frac{2^{1-\rho}}{\rho-1} + \frac{2^{3-\rho}}{2-\rho} - \int_0^{1/2} \frac{t^\rho}{(1-t)^2}\,\mathrm{d}t. \tag{1.10}$$

The constants $C(\rho)$ and $c(\rho)$ are best possible.

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In fact we will show that there is a sequence $r_j \to \infty$ such that

$$\mathcal{A}(r_j, \mathcal{N}) > (c(\rho) + o(1))\mathcal{N}(r_j) \quad \text{and} \quad \mathcal{B}(r_j, \mathcal{N}) < (C(\rho) + o(1))\mathcal{N}(r_j) \quad \text{as } j \to \infty.$$

Evidently, $(\rho - 1)c(\rho) \to -1$ and $(\rho - 1)C(\rho) \to 1$ as $\rho \to 1^+$. If we also denote the best possible values of the left-hand sides of (1.6) and (1.9) by $c(\rho)$ and $C(\rho)$ for $0 < \rho < 1$, we have $(1 - \rho)c(\rho) \to -1$ and $(1 - \rho)C(\rho) \to 1$ as $\rho \to 1^-$ (see, for example, [**1**, Theorem 3]).

It is perhaps worth stating explicitly that Theorem 1.1 has nothing to say on the key question in this context: that of finding the best lower bound for

$$\limsup_{r \to \infty} \frac{A(r, u)}{B(r, u)} \quad \text{for } \rho > 1.$$

2. Preliminaries

Let

$$\Phi(r) = \max_{0 \leqslant \theta \leqslant 2\pi} |E(re^{i\theta})|, \qquad \Psi(r) = \min_{0 \leqslant \theta \leqslant 2\pi} |E(re^{i\theta})|. \tag{2.1}$$

We have the following.

Lemma 2.1.

$$\Phi(r) = \begin{cases} r^2/2, & 0 \leqslant r \leqslant 2, \\ r + \log(r - 1), & r \geqslant 2, \end{cases}$$

and

$$\Psi(r) = \begin{cases} r + \log|r - 1|, & 0 \leqslant r \leqslant T, \\ -r + \log(r + 1), & r \geqslant T, \end{cases}$$

where T is given by (1.8).

With

$$H(r, \theta) = \log|E(re^{i\theta})| = r\cos\theta + \tfrac{1}{2}\log(r^2 - 2r\cos\theta + 1)$$

for $0 \leqslant \theta \leqslant \pi$, we have

$$\frac{\partial H}{\partial \theta} = -2r^2 \sin\theta \frac{r/2 - \cos\theta}{r^2 - 2r\cos\theta + 1},$$

which is 0 when

$$\theta = \begin{cases} 0,\ \pi \text{ or } \cos^{-1}(r/2), & 0 \leqslant r \leqslant 2, \\ 0 \text{ or } \pi, & r \geqslant 2. \end{cases}$$

The critical values of H are thus $r + \log|r - 1|$, $-r + \log(r + 1)$ and $r^2/2$ for $0 \leqslant r \leqslant 2$, and $r + \log(r - 1)$ and $-r + \log(r + 1)$ for $r \geqslant 2$. Since

$$\frac{d}{dr}\left(-2r + \log\left|\frac{r + 1}{r - 1}\right|\right) = \frac{2r^2}{1 - r^2},$$

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which is positive for $0 < r < 1$ and negative for $r > 1$, we have

$$-r + \log(r+1) \geqslant r + \log|r-1|, \quad 0 \leqslant r \leqslant T,$$
$$-r + \log(r+1) \leqslant r + \log(r-1), \quad r \geqslant T,$$

where T is given by (1.8). Also,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{2} - r - \log|r-1|\right) = r - 1 - \frac{1}{r-1},$$

which is positive for $0 < r < 1$ and negative for $1 < r < 2$, and thus

$$\tfrac{1}{2}r^2 \geqslant r + \log|r-1|, \qquad 0 \leqslant r \leqslant 2.$$

A similar argument shows that

$$\tfrac{1}{2}r^2 \geqslant -r + \log(r+1), \quad 0 \leqslant r \leqslant 2,$$

and Lemma 2.1 follows.

In proving Theorem 1.1 there is evidently no loss of generality in assuming that $\alpha = \beta = 0$ in (1.1). With this assumption we have

$$\mathcal{B}(r, \mathcal{N}) = \int_0^\infty \Phi\left(\frac{r}{t}\right) \mathrm{d}\mu^*(t),$$

and, from Lemma 2.1,

$$\begin{aligned}\int_0^\infty \Phi\left(\frac{r}{t}\right) \mathrm{d}\mu^*(t) &= \left[\Phi\left(\frac{r}{t}\right)\mu^*(t)\right]_{t=0}^{\infty} + \int_0^\infty \frac{r}{t^2}\Phi'\left(\frac{r}{t}\right)\mu^*(t)\,\mathrm{d}t \\ &= \int_0^{r/2} \frac{r^2}{t^2(r-t)}\mu^*(t)\,\mathrm{d}t + \int_{r/2}^\infty \frac{r^2}{t^3}\mu^*(t)\,\mathrm{d}t. \qquad (2.2)\end{aligned}$$

Similarly,

$$\mathcal{A}(r, \mathcal{N}) = \int_0^\infty \Psi\left(\frac{r}{t}\right) \mathrm{d}\mu^*(t)$$

and

$$\int_0^\infty \Psi\left(\frac{r}{t}\right) \mathrm{d}\mu^*(t) = -\int_0^{r/T} \frac{r^2}{t^2(r+t)}\mu^*(t)\,\mathrm{d}t + \int_{r/T}^\infty \frac{r^2}{t^2(r-t)}\mu^*(t)\,\mathrm{d}t. \qquad (2.3)$$

All of what follows is concerned with estimating the integrals in (2.2) and (2.3). If τ satisfies $\rho < \tau < 2$, then, by the hypotheses of Theorem 1.1,

$$\frac{\mathcal{N}(r)}{r^\tau} \to 0 \quad \text{as } r \to \infty.$$

Thus, if σ satisfies $1 < \sigma < \rho$ and η is any positive number,

$$\frac{\mathcal{N}(r)}{r^\sigma} - \eta r^{\tau-\sigma} \to -\infty$$

as $r \to \infty$. We define

$$a_\eta = \max_{r \geqslant 0} \left(\frac{\mathcal{N}(r)}{r^\sigma} - \eta r^{\tau - \sigma} \right) \tag{2.4}$$

and we let r_η be any value of r at which the maximum in (2.4) is attained. Since $\mathcal{N}(r)/r^\sigma$ is unbounded as $r \to \infty$,

$$a_\eta \to \infty \quad \text{and} \quad r_\eta \to \infty \tag{2.5}$$

as $\eta \to 0^+$. Also

$$\mathcal{N}(r) \leqslant \eta r^\tau + a_\eta r^\sigma \tag{2.6}$$

for all r and

$$\mathcal{N}(r_\eta) = \eta r_\eta^\tau + a_\eta r_\eta^\sigma. \tag{2.7}$$

(Were we to follow [**1**] precisely, we would consider $\max_{r \geqslant 0}(\mathcal{N}(r) - \eta r^\tau)$ instead of the right-hand side of (2.4), but this leads to divergent integrals and the method breaks down.) Arguing as in [**1**, Lemma 4], we have the following.

Lemma 2.2. *$\mathcal{N}(r)$ is differentiable at $r = r_\eta$ and, at $r = r_\eta$,*

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\mathcal{N}(r)}{r^\sigma} \right) = (\tau - \sigma) \eta r^{\tau - \sigma - 1}.$$

3. Estimates for $\mathcal{A}(r_\eta, \mathcal{N})$ and $\mathcal{B}(r_\eta, \mathcal{N})$

We shall prove the following result.

Lemma 3.1. *For all $\eta > 0$,*

$$\mathcal{B}(r_\eta, \mathcal{N}) \leqslant C(\tau) \mathcal{N}(r_\eta) + a_\eta r_\eta^\sigma (C(\sigma) - C(\tau)), \tag{3.1}$$

where C is given by (1.10).

With (2.2) in mind, write

$$I_1 = \int_0^{r/2} \frac{r^2}{t^2(r-t)} \mu^*(t) \, \mathrm{d}t, \qquad I_2 = \int_{r/2}^\infty \frac{r^2}{t^3} \mu^*(t) \, \mathrm{d}t. \tag{3.2}$$

Integrating by parts, we have

$$\begin{aligned} I_1 &= \left[\frac{r^2}{t(r-t)} \mathcal{N}(t) \right]_{t=0}^{r/2} + \int_0^{r/2} \frac{r^2(r-2t)}{t^2(r-t)^2} \mathcal{N}(t) \, \mathrm{d}t \\ &= 4\mathcal{N}(\tfrac{1}{2}r) + \int_0^{r/2} \frac{r^2(r-2t)}{t^2(r-t)^2} \mathcal{N}(t) \, \mathrm{d}t \\ &\leqslant 4\mathcal{N}(\tfrac{1}{2}r) + \int_0^{r/2} \frac{r^2(r-2t)}{t^2(r-t)^2} (\eta t^\tau + a_\eta t^\sigma) \, \mathrm{d}t, \end{aligned} \tag{3.3}$$

using (2.6). Also

$$\int_0^{r/2} \frac{r^2(r-2t)}{(r-t)^2} t^{\mu-2} \, \mathrm{d}t = C_1(\mu) r^\mu,$$

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where

$$C_1(\mu) = \int_0^{1/2} \frac{1-2t}{(1-t)^2} t^{\mu-2}\,\mathrm{d}t = \frac{2^{1-\mu}}{\mu-1} - \int_0^{1/2} \frac{t^\mu}{(1-t)^2}\,\mathrm{d}t,$$

and thus

$$I_1 \leqslant 4\mathcal{N}(\tfrac{1}{2}r) + \eta C_1(\tau) r^\tau + a_\eta C_1(\sigma) r^\sigma. \tag{3.4}$$

Similarly,

$$\begin{aligned} I_2 &= \left[\frac{r^2}{t^2}\mathcal{N}(t)\right]_{t=r/2}^{\infty} + \int_{r/2}^{\infty} \frac{2r^2}{t^3}\mathcal{N}(t)\,\mathrm{d}t \\ &= -4\mathcal{N}(\tfrac{1}{2}r) + \int_{r/2}^{\infty} \frac{2r^2}{t^3}\mathcal{N}(t)\,\mathrm{d}t \\ &\leqslant -4\mathcal{N}(\tfrac{1}{2}r) + \int_{r/2}^{\infty} \frac{2r^2}{t^3}(\eta t^\tau + a_\eta t^\sigma)\,\mathrm{d}t \\ &= -4\mathcal{N}(\tfrac{1}{2}r) + \eta \frac{2^{3-\tau}}{2-\tau} r^\tau + a_\eta \frac{2^{3-\sigma}}{2-\sigma} r^\sigma, \end{aligned} \tag{3.5}$$

using (2.6). Combining (3.4) and (3.5), we obtain

$$\mathcal{B}(r,\mathcal{N}) \leqslant \left(C_1(\tau) + \frac{2^{3-\tau}}{2-\tau}\right)\eta r^\tau + \left(C_1(\sigma) + \frac{2^{3-\sigma}}{2-\sigma}\right) a_\eta r^\sigma. \tag{3.6}$$

Evaluating this at $r = r_\eta$ and using (2.7), we obtain (3.1).

Lemma 3.2. *For all* $\eta > 0$,

$$\mathcal{A}(r_\eta,\mathcal{N}) \geqslant c(\tau)\mathcal{N}(r_\eta) + (c(\sigma) - c(\tau)) a_\eta r_\eta^\sigma, \tag{3.7}$$

where c *is given by (1.7).*

With (2.3) in mind, write

$$J_1 = \int_0^{r/T} \frac{r^2}{t^2(r+t)} \mu^*(t)\,\mathrm{d}t, \qquad J_2 = \int_{r/T}^{\infty} \frac{r^2}{t^2(r-t)} \mu^*(t)\,\mathrm{d}t. \tag{3.8}$$

Considering the second of these integrals first, we have

$$J_2 = \lim_{\varepsilon \to 0^+} (J_2' + J_2''), \tag{3.9}$$

where

$$J_2' = \int_{r/T}^{r-\varepsilon} \frac{r^2}{t^2(r-t)} \mu^*(t)\,\mathrm{d}t, \qquad J_2'' = \int_{r+\varepsilon}^{\infty} \frac{r^2}{t^2(r-t)} \mu^*(t)\,\mathrm{d}t. \tag{3.10}$$

Integrating by parts, we have

$$J_2' = \left[\frac{r^2}{t(r-t)}\mathcal{N}(t)\right]_{t=r/T}^{r-\varepsilon} - \int_{r/T}^{r-\varepsilon} r\left(\frac{1}{(r-t)^2} - \frac{1}{t^2}\right)\mathcal{N}(t)\,\mathrm{d}t.$$

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Also, since $T \approx 1.2$ we have $r/T > r/2$ and therefore $(r-t)^{-2} > t^{-2}$ for $r/T \leqslant t \leqslant r$. Thus, using (2.6) and integrating by parts again,

$$\begin{aligned} J_2' &\geqslant \left[\frac{r^2}{t(r-t)}\mathcal{N}(t)\right]_{t=r/T}^{r-\varepsilon} - \int_{r/T}^{r-\varepsilon} r\left(\frac{1}{(r-t)^2} - \frac{1}{t^2}\right)(\eta t^\tau + a_\eta t^\sigma)\,\mathrm{d}t \\ &= \left[\frac{r^2}{t(r-t)}(\mathcal{N}(t) - \eta t^\tau - a_\eta t^\sigma)\right]_{t=r/T}^{r-\varepsilon} + \int_{r/T}^{r-\varepsilon} \frac{r^2}{r-t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t. \quad (3.11) \end{aligned}$$

Also, using (2.7) and Lemma 2.2,

$$\begin{aligned} &\mathcal{N}(r_\eta - \varepsilon) - \eta(r_\eta - \varepsilon)^\tau - a_\eta(r_\eta - \varepsilon)^\sigma \\ &\qquad = (r_\eta - \varepsilon)^\sigma\left(\frac{\mathcal{N}(r_\eta - \varepsilon)}{(r_\eta - \varepsilon)^\sigma} - \eta(r_\eta - \varepsilon)^{\tau-\sigma} - a_\eta\right) \\ &\qquad = (r_\eta - \varepsilon)^\sigma\left(\frac{\mathcal{N}(r_\eta - \varepsilon)}{(r_\eta - \varepsilon)^\sigma} - \frac{\mathcal{N}(r_\eta)}{r_\eta^\sigma} + \eta(r_\eta^{\tau-\sigma} - (r_\eta - \varepsilon)^{\tau-\sigma})\right) \\ &\qquad = o(\varepsilon) \qquad (3.12) \end{aligned}$$

as $\varepsilon \to 0^+$. Thus, taking $r = r_\eta$ in (3.11),

$$\begin{aligned} J_2' &\geqslant -\frac{T^2}{T-1}(\mathcal{N}(r_\eta/T) - \eta(r_\eta/T)^\tau - a_\eta(r_\eta/T)^\sigma) \\ &\qquad + \int_{r_\eta/T}^{r_\eta-\varepsilon} \frac{r_\eta^2}{r_\eta - t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t + o(1) \\ &\geqslant \int_{r_\eta/T}^{r_\eta-\varepsilon} \frac{r_\eta^2}{r_\eta - t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t + o(1) \qquad (3.13) \end{aligned}$$

as $\varepsilon \to 0^+$, from (2.6).

Similarly, integrating by parts twice,

$$\begin{aligned} J_2'' &= \left[\frac{r^2}{t(r-t)}\mathcal{N}(t)\right]_{r+\varepsilon}^{\infty} - \int_{r+\varepsilon}^{\infty} r\left(\frac{1}{(r-t)^2} - \frac{1}{t^2}\right)\mathcal{N}(t)\,\mathrm{d}t \\ &\geqslant \left[\frac{r^2}{t(r-t)}\mathcal{N}(t)\right]_{t=r+\varepsilon}^{\infty} - \int_{r+\varepsilon}^{\infty} r\left(\frac{1}{(r-t)^2} - \frac{1}{t^2}\right)(\eta t^\tau + a_\eta t^\sigma)\,\mathrm{d}t \\ &= \left[\frac{r^2}{t(r-t)}(\mathcal{N}(t) - \eta t^\tau - a_\eta t^\sigma)\right]_{t=r+\varepsilon}^{\infty} + \int_{r+\varepsilon}^{\infty} \frac{r^2}{r-t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t \\ &= \frac{r^2}{\varepsilon(r+\varepsilon)}(\mathcal{N}(r+\varepsilon) - \eta(r+\varepsilon)^\tau - a_\eta(r+\varepsilon)^\sigma) + \int_{r+\varepsilon}^{\infty} \frac{r^2}{r-t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t. \end{aligned}$$

As in (3.12), $\mathcal{N}(r_\eta + \varepsilon) - \eta(r_\eta + \varepsilon)^\tau - a_\eta(r_\eta + \varepsilon)^\sigma = o(\varepsilon)$ as $\varepsilon \to 0^+$ and we obtain

$$J_2'' \geqslant \int_{r_\eta+\varepsilon}^{\infty} \frac{r_\eta^2}{r_\eta - t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t + o(1). \qquad (3.14)$$

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Combining (3.9), (3.13) and (3.14), we have

$$\begin{aligned} J_2 &\geqslant \int_{r_\eta/T}^{\infty} \frac{r_\eta^2}{r_\eta - t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t \\ &= \tau\eta r_\eta^\tau \int_{1/T}^{\infty} \frac{t^{\tau-2}}{1-t}\,\mathrm{d}t + \sigma a_\eta r_\eta^\sigma \int_{1/T}^{\infty} \frac{t^{\sigma-2}}{1-t}\,\mathrm{d}t. \end{aligned} \tag{3.15}$$

Turning to the other integral in (3.8), we have, arguing similarly (but there are fewer complications),

$$\begin{aligned} J_1 &= \left[\frac{r^2}{t(r+t)}\mathcal{N}(t)\right]_{t=0}^{r/T} - \int_0^{r/T} r\left(\frac{1}{(r+t)^2} - \frac{1}{t^2}\right)\mathcal{N}(t)\,\mathrm{d}t \\ &\leqslant \left[\frac{r^2}{t(r+t)}\mathcal{N}(t)\right]_{t=0}^{r/T} - \int_0^{r/T} r\left(\frac{1}{(r+t)^2} - \frac{1}{t^2}\right)(\eta t^\tau + a_\eta t^\sigma)\,\mathrm{d}t \\ &= \left[\frac{r^2}{t(r+t)}(\mathcal{N}(t) - \eta t^\tau - a_\eta t^\sigma)\right]_{t=0}^{r/T} + \int_0^{r/T} \frac{r^2}{r+t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t \\ &\leqslant \int_0^{r/T} \frac{r^2}{r+t}(\tau\eta t^{\tau-2} + \sigma a_\eta t^{\sigma-2})\,\mathrm{d}t \\ &= \tau\eta r^\tau \int_0^{1/T} \frac{t^{\tau-2}}{1+t}\,\mathrm{d}t + \sigma a_\eta r^\sigma \int_0^{1/T} \frac{t^{\sigma-2}}{1+t}\,\mathrm{d}t, \end{aligned} \tag{3.16}$$

using (2.6). Combining (2.3), (3.15) and (3.16), we obtain

$$\mathcal{A}(r_\eta, \mathcal{N}) \geqslant c(\tau)\eta r_\eta^\tau + c(\sigma) a_\eta r_\eta^\sigma, \tag{3.17}$$

where c is given by (1.7), since

$$\begin{aligned} \int_{1/T}^{\infty} \frac{t^{\mu-2}}{1-t}\,\mathrm{d}t - \int_0^{1/T} \frac{t^{\mu-2}}{1+t}\,\mathrm{d}t &= \int_0^{\infty} \frac{t^{\mu-2}}{1-t}\,\mathrm{d}t - \int_0^{1/T} \frac{2t^{\mu-2}}{1-t^2}\,\mathrm{d}t \\ &= \pi\cot(\pi(\mu-1)) - \int_0^{1/T} \frac{2(1-t^2+t^2)t^{\mu-2}}{1-t^2}\,\mathrm{d}t \\ &= \pi\cot(\pi(\mu-1)) - \frac{2T^{1-\mu}}{\mu-1} - \int_0^{1/T} \frac{2t^\mu}{1-t^2}\,\mathrm{d}t \\ &= c(\mu)/\mu. \end{aligned}$$

Then (3.7) follows from (3.17) and (2.7).

4. Proof of Theorem 1.1

From (2.7) we have $a_\eta r_\eta^\sigma \leqslant \mathcal{N}(r_\eta)$ and thus, from the second part of (2.5) and Lemmas 3.1 and 3.2,

$$\liminf_{r\to\infty} \frac{\mathcal{B}(r,\mathcal{N})}{\mathcal{N}(r)} \leqslant C(\tau) + |C(\sigma) - C(\tau)|$$

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and

$$\limsup_{r\to\infty} \frac{\mathcal{A}(r,\mathcal{N})}{\mathcal{N}(r)} \geqslant c(\tau) - |c(\sigma) - c(\tau)|.$$

This proves (1.9) and (1.6), since $\sigma < \rho$ and $\tau > \rho$ are arbitrary.

Finally, as an examination of the proof of Lemmas 3.1 and 3.2 shows (taking $\tau = \rho$, $\eta = 1$ and $a_\eta = 0$ in the calculations), when $\mathcal{N}(r) = r^\rho$ we have $\mathcal{A}(r,\mathcal{N}) = c(\rho)\mathcal{N}(r)$ and $\mathcal{B}(r,\mathcal{N}) = C(\rho)\mathcal{N}(r)$, and thus the constants $c(\rho)$ and $C(\rho)$ are best possible.

References

1. P. C. Fenton and J. Rossi, Phragmén–Lindelöf theorems, *Proc. Am. Math. Soc.* **132** (2003), 761–768.
2. P. C. Fenton and J. Rossi, $\cos\pi\rho$ theorems for δ-subharmonic functions, *J. Analyse Math.* **92** (2004), 385–396.
3. W. K. Hayman, *Subharmonic functions*, Volume 1 (Academic Press, London, 1976).

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