Equivariant cobordisms between freely periodic knots

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Abstract. We consider free symmetries on cobordisms between knots, which is equivalent to cobordisms between knots in lens spaces. We classify which freely periodic knots bound equivariant surfaces in the 4-ball in terms of corresponding homology classes in lens spaces. We give a numerical condition determining the free periods for which torus knots bound equivariant surfaces in the 4-ball.

1 Introduction and main results

A knot $K \subset S^3$ is *freely (p, q)-periodic* if there is a free $\mathbb{Z}/p\mathbb{Z}$-action on S^3 with quotient $L(p, q)$ which leaves K invariant. An example is shown in Figure 1. Thinking of S^3 as the unit sphere in $\mathbb{C}^2$, this symmetry is conjugate to $(z, w) \mapsto (\alpha z, \alpha^q w)$, where $\alpha = e^{2\pi i/p}$. The goal of this paper is to understand when a freely periodic knot bounds an equivariant orientable surface in B^4. For convenience, we will work in the smooth category, although all results in this paper can easily be seen to hold in a more general topologically locally linear category.

Definition 1.1 A freely (p, q)-periodic knot K is an *equivariant boundary* if there is a smooth order p extension $\rho : B^4 \to B^4$ of the $\mathbb{Z}/p\mathbb{Z}$ symmetry on S^3 and an orientable surface S properly smoothly embedded in B^4 with $\rho(S) = S$ and $\partial S = K$.

It is interesting to compare the case of periodic and strongly invertible knots, which always equivariantly bound an orientable surface in S^3 (and hence in B^4) (see, for example, [BI21a, Proposition 1]). On the other hand, freely periodic knots never bound equivariant orientable surfaces in S^3, since such a surface would necessarily contain a fixed point, but the symmetry acts freely on S^3.

In this paper, we characterize which freely periodic knots equivariantly bound in B^4.

Theorem 1.2 *Let $K \subset S^3$ be a freely (p, q)-periodic knot, and let $\pi : S^3 \to L(p, q)$ be the quotient map. Then K is an equivariant boundary if and only if $\pi(K)$ represents a simple homology class (see Definition 2.1) in $H_1(L(p, q); \mathbb{Z})$.*
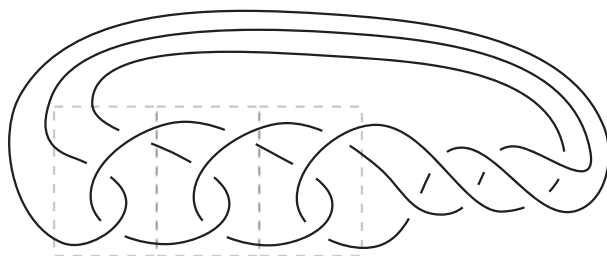
Figure 1: A freely $(3,1)$-periodic diagram for $T(2,7)$. This diagram consists of three identical tangles and one full twist.

Furthermore, we prove that a simple count of strands in a freely periodic diagram determines if the knot equivariantly bounds.

Theorem 1.3   *Let D be an oriented freely (p,q)-periodic diagram (see Definition 2.8) for a knot K. Let m be the signed count of strands in D. Then K represents a simple homology class in the quotient $L(p,q)$ if and only if $m \equiv \pm 1$ or $m \equiv \pm q^{-1}$ (mod p).*

In the case of torus knots, this reduces to a numerical condition on the torus knot parameters and the order of the symmetry.

Corollary 1.4 (Corollary of Theorem 1.3)   *A freely (p,q)-periodic symmetry of $T(r,s)$ equivariantly bounds if and only if p is a divisor of $r+1, r-1, s+1,$ or $s-1$.*

Throughout the paper, all surfaces are orientable unless otherwise stated.

2   Definitions and proofs

Our results are based on considering the homology class which a freely periodic knot represents in the quotient lens space $L(p,q)$. We begin by distinguishing some elements of $H_1(L(p,q))$.

Definition 2.1   *A* simple *homology class in $H_1(L(p,q);\mathbb{Z})$ is one which is represented by the core of a handlebody in a genus 1 Heegaard splitting.*

In fact, there is a unique genus 1 Heegaard splitting for $L(p,q)$ (see [Bon83]), so that there are at most four simple homology classes in $H_1(L(p,q))$ coming from the two orientations on the cores of the two handlebodies.

The following proposition is a restatement of [BE12, Lemma 5.2], noting that the lift in S^3 of a knot $\overline{K}$ in $L(p,q)$ is an unknot if and only if $\overline{K}$ is a rational unknot.

Proposition 2.2 [BE12, Lemma 5.2]   *Let U be a freely (p,q)-periodic unknot. Then the quotient $\overline{U}$ represents a simple homology class in $H_1(L(p,q))$. Conversely, every simple homology class is represented by the quotient of an unknot.*

We now define our main object of study: equivariant cobordisms between freely periodic knots. As we will see in Lemma 2.7, studying equivariant cobordisms from K to the unknot is equivalent to studying equivariant surfaces with boundary K.

Definition 2.3   An *equivariant cobordism* $(S^3 \times I, S, \overline{\rho})$ between a freely (p, q)-periodic knot (K, ρ) and a freely (p, q')-periodic knot (K', ρ') is a proper smooth embedding of a surface $S \to S^3 \times I$ such that $S \cap (S^3 \times \{0\}) = K$ and $S \cap (S^3 \times \{1\}) = K'$, along with a free smooth $\mathbb{Z}/p\mathbb{Z}$ action $\overline{\rho}$ on $S^3 \times I$ which restricts to the freely periodic symmetries ρ on $S^3 \times \{0\}$ and ρ' on $S^3 \times \{1\}$, and leaves S invariant.

To study these cobordisms, it will be convenient to first take the quotient by the free symmetry.

Lemma 2.4   *Let $W = S^3 \times I$, let $\overline{\rho}$ be a finite order diffeomorphism acting freely on W, and let $\overline{\rho}|_{S^3 \times \{0\}} = \rho$ and $\overline{\rho}|_{S^3 \times \{1\}} = \rho'$, for example, an equivariant cobordism between freely periodic knots. Then the quotient $\overline{W} = W/\overline{\rho}$ is a homology cobordism between lens spaces.*

Proof   Since $S^3 \times I$ is simply connected, $\pi_1(\overline{W}) = \mathbb{Z}/p\mathbb{Z} = H_1(\overline{W})$, and the maps

$$H_1(S^3/\rho) \to H_1(\overline{W}) \text{ and } H_1(S^3/\rho') \to H_1(\overline{W})$$

induced by inclusion are isomorphisms. Since W is connected, the same is true for H_0 so that $\overline{W}$ is a homology cobordism.   ∎

We are interested in the maps induced on homology from these homology cobordisms. The following theorem, which is a consequence of the work in [AB68], first appeared in [Mil66]. The theorem also follows from an analysis of the d-invariants of lens spaces (see [DW15]).

Theorem 2.5 [Mil66, Corollary 12.2]   *Let $\overline{W} : L(p, q) \to L(p, q')$ be a homology cobordism. Then $L(p, q)$ is homeomorphic to $L(p, q')$ and $\overline{W}$ induces $\pm\mathrm{Id}$ on $H_1(L(p, q); \mathbb{Z})$.*

Corollary 2.6   *Let K be a freely (p, q)-periodic knot, and let K' be a freely (p, q')-periodic knot. If there is an equivariant cobordism between K and K', then $q' = \pm q^{\pm 1} \in \mathbb{Z}/p\mathbb{Z}$. In particular, K' is a freely (p, q)-periodic knot.*

Proof   Apply Theorem 2.5 to the quotient of the equivariant cobordism, and use the homeomorphism classification of lens spaces [Bro60].   ∎

We now relate equivariant cobordisms to equivariant surfaces for freely periodic knots.

Lemma 2.7   *A freely (p, q)-periodic knot K is an equivariant boundary if and only if there is an equivariant cobordism between K and a freely (p, q)-periodic unknot.*

Proof   Let S be an equivariant surface for K with respect to an order p diffeomorphism $\rho : B^4 \to B^4$. By classical Smith theory [Smi41], ρ has a contractible fixed-point set F. Furthermore, since the fixed set of a self-diffeomorphism is a submanifold, F is a single point. Similarly, the fixed-point set of $\rho|_S$ is also the single point F. Removing an equivariant neighborhood $N(F)$ from B^4 leaves an S^3 boundary component containing an unknot U. Hence, we have $(S - N(F)) \subset S^3 \times I$, which is preserved by the free $\mathbb{Z}/p\mathbb{Z}$ action and $\partial(S - N(F)) = K \cup U$. That is an equivariant cobordism between K and U. By Corollary 2.6, U comes with a freely (p,q)-periodic symmetry.

On the other hand, suppose that we have an equivariant cobordism between K and U. By Proposition 2.2, U can be taken to be the lift of a core of a handlebody in a genus 1 Heegaard decomposition of $L(p,q)$, and in particular, the cone of (S^3, U) is a smooth equivariant disk in B^4. Gluing this to the cobordism gives an equivariant surface for K. ∎

We now have all of the tools we need in order to prove Theorem 1.2.

Proof   Suppose $K \subset S^3$ is a freely (p,q)-periodic knot which bounds a surface $S \subset B^4$ which is invariant under an order p diffeomorphism $\rho : B^4 \to B^4$ with $\rho|_{S^3}$ the free $\mathbb{Z}/p\mathbb{Z}$ action preserving K. Then, by Lemma 2.7, there is an equivariant cobordism from K to a freely (p,q)-periodic unknot $U_{p,q}$. By Lemma 2.4, the quotient of this cobordism is a cobordism of the quotient knots $\overline{K}$ and $\overline{U}_{p,q}$ in a homology cobordism of lens spaces. In particular, there is a map $f : H_1(L(p,q)) \to H_1(L(p,q))$ with $[f(\overline{K})] = [\overline{U}_{p,q}]$ which is induced by a homology cobordism of lens spaces. By Theorem 2.5, $f = \pm\mathrm{Id}$. Hence, $\overline{K}$ represents the same class as an unknot in $H_1(L(p,q))$. Then, by Proposition 2.2, we have that $[\overline{K}]$ is simple.

On the other hand, suppose that $[\overline{K}]$ is simple. Then, by Proposition 2.2, $\overline{K}$ is in the same homology class as the core $\overline{U}$ of a handlebody in the genus 1 Heegaard decomposition of $L(p,q)$, which lifts to a freely periodic unknot U. Hence, there is a surface $\overline{S}$ in $L(p,q)$ with boundary $\overline{K} \cup \overline{U}$, and lifting $\overline{S}$ to S^3 gives us an equivariant surface S with boundary $K \cup U$. We can then take the cone of (S^3, U) with the free symmetry to obtain a smooth equivariant disk in B^4. Gluing this to S gives us an equivariant surface for K. ∎

We now turn to understanding simple homology classes in terms of freely periodic knot diagrams, the standard way in which we expect to present a freely periodic knot.

Definition 2.8   A freely (p,q)-periodic diagram is the closure of a tangle consisting of the concatenation of p identical tangles and q full twists. See Figures 1–3 for examples.

Theorem 2.9 [Chb97, Theorem 1]   *Every freely (p,q)-periodic knot has a freely (p,q)-periodic diagram.*

The proof of Theorem 1.3 is now straightforward.

Proof   It is clear that a tangle with one strand is homologous to the core c of a genus 1 Heegaard splitting for $L(p,q)$. It follows immediately from the definition of a lens
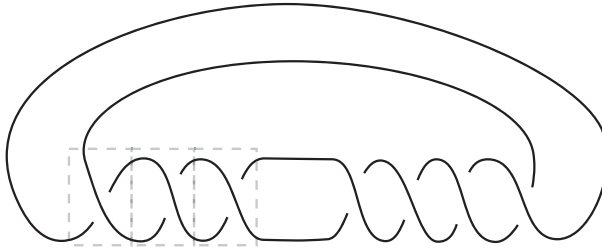
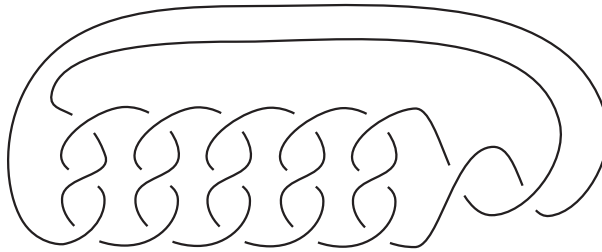*Figure 2*: A freely $(3, 2)$-periodic diagram for $T(2, 7)$.

Figure 3: A freely $(5, 1)$-periodic knot.

space that the other core is $q^{-1} \cdot c \in H_1(L(p, q))$ so that a diagram with q^{-1} strands is homologous to the other core. ∎

Remark 2.10   Theorem 1.3 depends on our convention for freely (p, q)-periodic diagrams. We could instead switch the role of the two cores in our diagrams so that a freely (p, q)-periodic diagram would have $q^{-1} \pmod{p}$ full twists. In this case, the simple homology classes would be represented by diagrams with ± 1 or $\pm q$ strands.

Remark 2.11   If we instead consider surfaces which need not be orientable, then a version of Theorem 1.2 using homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients implies that every freely periodic knot bounds a smooth equivariant surface in B^4.

We are also interested in the special case of torus knots. We first describe their free symmetries.

Theorem 2.12   *There is a freely (p, q)-periodic symmetry of $T(r, s)$ if and only if $gcd(p, rs) = 1$), and $\pm q^{\pm 1} \equiv sr^{-1} \pmod{p}$. With respect to this symmetry, there is a freely (p, q)-periodic diagram on r strands if and only if $\pm q \equiv sr^{-1} \pmod{p}$. Finally, if ρ and ρ' are two free periods of the same order, then the subgroups of the orientation-preserving diffeomorphism group of the exterior of $T(r, s)$ generated by ρ and ρ' are conjugate.*

Proof   By [MS86, Theorem 2.2] and [Sch24] (see also [Kaw96, Exercise 10.6.4]), the freely periodic symmetry preserves a Seifert fibered structure on the exterior of $T(r, s)$, which is a circle bundle over $D(r, s)$, the disk with cone points of orders r

and s. For $p > 2$, this orbifold has no order p symmetries. For the involution reflecting across an arc containing the two cone points, we must also reflect the fibers to get an orientation-preserving symmetry of the exterior of $T(r, s)$, but this has global fixed points. Thus, our free symmetry acts as identity on the base orbifold, and therefore as a rotation on the fibers. In particular, there is a unique symmetry for each p.

When p has a common factor with r or s, then the order $\gcd(p, rs)$ subgroup has a fixed circle in S^3, and so there is no free symmetry of order p. On the other hand, the symmetry is free when $\gcd(p, rs) = 1$. We will describe these free symmetries explicitly. Consider the standard tangle diagram for $T(r, s)$ with r strands. Then there are s total twists, which can be grouped as $\pm q$ full twists and p concatenated tangles of n twists each (see Figure 2 for an example). Hence, $s = \pm q \cdot r + n \cdot p$ from which we deduce that $\pm q \equiv sr^{-1} \pmod{p}$. However, a free (p, q)-symmetry can also be regarded as a free (p, q^{-1})-symmetry (cf. the homeomorphism classification of lens spaces). In particular, the diagram with s strands has $\pm q \equiv rs^{-1} \pmod{p}$. ∎

From Theorem 1.3, we can now deduce Corollary 1.4.

Proof   Consider the freely (p, q)-periodic diagram for $T(r, s)$ with r strands, which consists of s total twists: $\pm q$ full twists and p tangles of n twists each. By Theorems 1.2 and 1.3, $T(r, s)$ is an equivariant boundary if and only if $r \equiv \pm 1$ or $r \equiv \pm q^{-1} \pmod{p}$. By Theorem 2.12, $\pm q \equiv sr^{-1} \pmod{p}$ so that p is a divisor of $r - 1$, $r + 1$, $s - 1$, or $s + 1$. ∎

Remark 2.13   Note that a freely $(2, q)$ or $(3, q)$-periodic knot is always an equivariant boundary by Theorem 1.3.

Our final result is the observation that the genus of an equivariant surface for a freely periodic knot must be a multiple of p.

Theorem 2.14   Let S be an equivariant surface for a freely (p, q)-periodic knot. Then the genus of S is a multiple of p.

Proof   Let n be the genus of S so that $\chi(S) = 1 - 2n$. Since S has exactly one fixed point F (see the proof of Lemma 2.7), $\chi(S - F) = -2n$. Quotienting $S - F$ by the free $\mathbb{Z}/p\mathbb{Z}$ action leaves a surface $\overline{S}$ with $\chi(\overline{S}) = -2n/p$. However, since $\overline{S}$ has exactly two boundary components, it must have an even Euler characteristic and so $p | n$. ∎

3   Examples and questions

We conclude with some examples and questions.

Example 3.1   Consider the torus knot $T(2, 7)$. It has a unique free period of order 3, which is shown in Figure 2. By Corollary 1.4, $T(2, 7)$ bounds an invariant surface in B^4 with respect to its free $(3, 2)$-period. This surface can be seen directly from Figure 1, where changing the sign of the crossing in the bottom of each of the three tangle boxes gives an equivariant genus 3 cobordism to the unknot, which then bounds an equivariant disk in B^4. Note that, by Theorem 2.14, this is the minimum possible genus for such a surface, since $T(2, 7)$ is not slice.

Example 3.2   Consider the torus knot $T(2,3)$. Since there are no primes which divide $1, 3, 2$, or 4 and are relatively prime to 2 and 3, Corollary 1.4 implies that $T(2,3)$ does not equivariantly bound with respect to any of its infinitely many free symmetries.

Example 3.3   Consider the freely $(5,1)$-periodic knot K shown in Figure 3. It has a signed count of two strands, so, by Theorems 1.2 and 1.3, K does not bound an equivariant surface in B^4.

In principle, one could use Theorems 1.2 and 1.3 to show that some freely periodic slice knots do not bound equivariant surfaces (let alone disks). However, every freely periodic slice knot we know of represents a simple homology class in the quotient lens space.

Question 3.4   What tools can give a lower bound on the genus of an equivariant surface for a freely periodic knot?

Question 3.5   Does there exist a freely periodic slice knot which does not bound any (smooth) equivariant disk?

Note that these questions have been answered for several other types of symmetric knots. In particular, examples of obstructions to slice symmetric knots being equivariantly slice can be found for periodic knots in [BI21a, CK99, DN06, Nai97], for strongly invertible knots in [BI21a, DHM22, Sak86], and for strongly negative amphichiral knots in [BI21b].

References

[AB68]   M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*. Ann. Math. **88**(1968), 451–491.

[BE12]   K. Baker and J. Etnyre, *Rational linking and contact geometry*. In: Perspectives in analysis, geometry, and topology, Progress in Mathematics, 296, Birkhäuser/Springer, New York, 2012, pp. 19–37.

[Bon83]   F. Bonahon, *Difféotopies des espaces lenticulaires*. Topology **22**(1983), no. 3, 305–314.

[BI21a]   K. Boyle and A. Issa, *Equivariant 4-genera of strongly invertible and periodic knots*. Preprint, 2021. https://arxiv.org/abs/2101.05413

[BI21b]   K. Boyle and A. Issa, *Equivariantly slicing strongly negative amphichiral knots*. Preprint, 2021. https://arxiv.org/abs/2109.01198

[Bro60]   E. J. Brody, *The topological classification of the lens spaces*. Ann. Math. **71**(1960), 163–184.

[CK99]   J. C. Cha and K. H. Ko, *On equivariant slice knots*. Proc. Amer. Math. Soc. **127**(1999), no. 7, 2175–2182.

[Chb97]   N. Chbili, *On the invariants of lens knots*. In: KNOTS'96 (Tokyo), World Scientific Publishing, River Edge, NJ, 1997, pp. 365–375.

[DHM22]   I. Dai, M. Hedden, and A. Mallick, *Corks, involutions, and Heegaard Floer homology*. J. Eur. Math. Soc.. Published online 21 May 2022.

[DN06]   J. F. Davis and S. Naik, *Alexander polynomials of equivariant slice and ribbon knots in S^3*. Trans. Amer. Math. Soc. **358**(2006), no. 7, 2949–2964.

[DW15]   M. Doig and S. Wehrli, *A combinatorial proof of the homology cobordism classification of lens spaces*. Preprint, 2015.   https://arxiv.org/abs/1505.06970

[Kaw96] A. Kawauchi, *A survey of knot theory*, Birkhäuser, Basel, 1996, Translated and revised from the 1990 Japanese original by the author.

[MS86]   W. H. Meeks III and P. Scott, *Finite group actions on 3-manifolds*. Invent. Math. **86**(1986), no. 2, 287–346.

[Mil66] J. Milnor, *Whitehead torsion*. Bull. Amer. Math. Soc. **72**(1966), 358–426.

[Nai97] S. Naik, *Equivariant concordance of knots in S^3*. In: KNOTS'96 (Tokyo), World Scientific Publishing, River Edge, NJ, 1997, pp. 81–89.

[Sak86] M. Sakuma, *On strongly invertible knots*. In: Algebraic and topological theories (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, pp. 176–196.

[Sch24] O. Schreier, *Über die gruppen $A^a B^b$ = 1*. Abh. Math. Sem. Univ. Hamburg **3**(1924), no. 1, 167–169.

[Smi41] P. A. Smith, *Fixed-point theorems for periodic transformations*. Amer. J. Math. **63**(1941), 1–8.

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