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A COUNTEREXAMPLE TO A RESULT OF JABERI AND MAHMOODI

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Abstract

We show that $\ell^1(\mathbb{N}_\wedge)$ is φ-amenable for each multiplicative linear functional $\varphi : \ell^1(\mathbb{N}_\wedge) \to \mathbb{C}$. This is a counterexample to the final corollary of Jaberi and Mahmoodi ['On φ-amenability of dual Banach algebras', *Bull. Aust. Math. Soc.* **105** (2022), 303–313] and shows that the final theorem in that paper is not valid.

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1. Introduction and preliminaries

The cohomological notion of amenability was introduced and studied in the pioneering work of Johnson [5]. A Banach algebra $\mathcal{A}$ is amenable if every continuous derivation from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule E^* is inner. A modification of amenability depending on multiplicative linear functionals was introduced and studied by Kaniuth *et al.* [6]. A Banach algebra $\mathcal{A}$ is called φ-amenable if there exists an element m in $\mathcal{A}^{**}$ such that $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for every $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where φ is a multiplicative linear functional on $\mathcal{A}$. For a locally compact group G, the Fourier algebra $\mathcal{A}(G)$ is always φ-amenable. Moreover, the Segal algebra $S^1(G)$ is φ-amenable if and only if G is amenable (see [1, 6]).

Jaberi and Mahmoodi [4] introduced the new concept of φ-injectivity for the category of dual Banach algebras, where φ is a wk^*-continuous multiplicative linear functional on $\mathcal{A}$. A dual Banach algebra $\mathcal{A}$ is φ-injective if whenever $\pi : \mathcal{A} \to \mathcal{L}(E)$ is a wk^*-continuous unital representation on a reflexive Banach space E, then there is a projection $Q : \mathcal{L}(E) \to \pi(\mathcal{A})^\varphi$ such that $Q(STU) = SQ(T)U$ for $S, U \in \pi(\mathcal{A})^c$ and $T \in \mathcal{L}(E)$, where $\pi(\mathcal{A})^\varphi = \{T \in \mathcal{L}(E) : \ \pi(a)T = \varphi(a)T \quad (a \in \mathcal{A})\}$. They proved that φ-injectivity is equivalent to φ-amenability [4, Theorem 3.6].

There is an important category of dual Banach algebras, called enveloping dual Banach algebras. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule.

An element $x \in E$ is called weakly almost periodic if the module maps $\mathcal{A} \to E$ given by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weakly compact. The set of all weakly almost periodic elements of E is denoted by $WAP(E)$ [7, Definition 4.1]. Runde observed that $WAP(\mathcal{A}^*)^*$ is a canonical dual Banach algebra associated to an arbitrary Banach algebra $\mathcal{A}$ [7, Theorem 4.10]. By means of the new notion of φ-injectivity, Jaberi and Mahmoodi investigated φ-amenability of the enveloping dual Banach algebra $WAP(\mathcal{A}^*)^*$ [4, Theorem 4.8]. In a short final section of the paper, they claimed that $WAP(\ell^1(\mathbb{N}_\wedge)^*)^*$ is not $\tilde{\varphi}$-amenable, where $\tilde{\varphi}$ is the unique extension of the augmentation character φ on the semigroup algebra $\ell^1(\mathbb{N}_\wedge)$ [4, Theorem 5.4]. From this result, they concluded that $\ell^1(\mathbb{N}_\wedge)$ is not φ-amenable, where φ is the augmentation character [4, Corollary 5.5].

On the contrary, we show that $\ell^1(\mathbb{N}_\wedge)$ is φ-amenable for each multiplicative linear functional $\varphi : \ell^1(\mathbb{N}_\wedge) \to \mathbb{C}$ and comment on the reason for this counterexample to the result stated in [4].

2. φ-amenability of $\ell^1(\mathbb{N}_\wedge)$

Let $S = \mathbb{N}$. With the semigroup product $m \wedge n = \min\{m, n\}$, for $m, n \in S$, the set S becomes a semigroup. It is known that $\Delta(\ell^1(S))$ consists of all the functions $\varphi_n : \ell^1(S) \to \mathbb{C}$ given by $\varphi_n(\sum_{i=1}^{\infty} \alpha_i \delta_i) = \sum_{i=n}^{\infty} \alpha_i$, for $n \in S$ (see [2, page 32]). Suppose that $m = \delta_1$. Then $\varphi_1(m) = \varphi_1(\delta_1) = 1$ and

$$am = a\delta_1 = \Big(\sum_{i=1}^{\infty} a_i\Big)\delta_1 = \varphi_1(a)\delta_1 = \varphi_1(a)m, \quad \text{where } a = \sum_{i=1}^{\infty} a_i\delta_i \in \ell^1(S).$$

It follows that $\ell^1(S)$ is φ_1-amenable. For $n > 1$, define $m_n = \delta_n - \delta_{n-1}$. Then,

$$\varphi_n(m_n) = \varphi_n(\delta_n - \delta_{n-1}) = 1 - 0 = 1$$

and

$$am_n = a(\delta_n - \delta_{n-1}) = \sum_{i=n}^{\infty} a_i(\delta_n - \delta_{n-1}) = \varphi_n(a)(\delta_n - \delta_{n-1}) = \varphi_n(a)m_n,$$

where $a = \sum_{i=1}^{\infty} a_i\delta_i \in \ell^1(S)$. It follows that $\ell^1(S)$ is φ-amenable with respect to each multiplicative linear functional $\varphi : \ell^1(S) \to \mathbb{C}$. Thus, [4, Corollary 5.5] is not true.

This counterexample to [4, Corollary 5.5] shows that [4, Theorem 5.4] is also not true. The mistake is the assertion in the second sentence of the proof of Theorem 5.4 that 'there is an isometric isomorphism Θ from $\rho(\ell^1(\mathbb{N}_\wedge)^c$ onto $\rho(\ell^1(\mathbb{N}_\wedge))^{\varphi}$'. An example showing that Θ cannot be isometric can be constructed using [3, Theorem 7.6]. Take $\| \sum_{n=1}^{\infty} a_n\delta_n\| = \sup_F \| \sum_{n\in F} a_n\delta_n\|$, where F is a finite subset of $\mathbb{N}$. Take indices 1 and $2n + 1$ so that the corresponding basis elements belong to distinct summands. Set A to be the diagonal matrix having ones at indices 1 and $2n + 1$ and zero otherwise. Set B to have ones at indices 1 and $2n + 1$ in the first row and zeros otherwise. Then $B = \Theta(A)$ and

$$\|A\| = \sup\Big\{\|a_1\delta_n + a_{2n+1}\delta_{2n+1}\| : \Big\|\sum a_n\delta_n\Big\| = 1\Big\}$$
$$= \sum\{(a_1^2 + a_{2n+1}^2)^{1/2} : (a_1^2 + a_{2n+1}^2)^{1/2} = 1\} = 1,$$

while

$$\|B\| = \sup\Big\{|a_1 + a_{2n+1}| : \Big\|\sum a_n\delta_n\Big\| = 1\Big\}$$
$$= \sum\{|a_1 + a_{2n+1}| : (a_1^2 + a_{2n+1}^2)^{1/2} = 1\} = \sqrt{2}.$$

Consequently, Θ is not isometric. By taking k summands in a similar way, it can be shown that Θ is unbounded on diagonal elements of finite support.

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