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LOW RANK SPECIALISATIONS OF ELLIPTIC SURFACES

MENTZELOS MELISTAS

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Abstract

Let $E/\mathbb{Q}(T)$ be a nonisotrivial elliptic curve of rank r. A theorem due to Silverman ['Heights and the specialization map for families of abelian varieties', *J. reine angew. Math.* **342** (1983), 197–211] implies that the rank r_t of the specialisation $E_t/\mathbb{Q}$ is at least r for all but finitely many $t \in \mathbb{Q}$. Moreover, it is conjectured that $r_t \leq r+2$, except for a set of density 0. When $E/\mathbb{Q}(T)$ has a torsion point of order 2, under an assumption on the discriminant of a Weierstrass equation for $E/\mathbb{Q}(T)$, we produce an upper bound for r_t that is valid for infinitely many t. We also present two examples of nonisotrivial elliptic curves $E/\mathbb{Q}(T)$ such that $r_t \leq r+1$ for infinitely many t.

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1. Introduction

Let $\pi : \mathcal{E} \to \mathbb{P}^1$ be a nonconstant elliptic surface defined over $\mathbb{Q}$. By this, we mean a two-dimensional projective variety $\mathcal{E}$ endowed with a morphism π as above such that all but finitely many fibres of π are curves of genus one and such that there exists a section σ to π. Let $E/\mathbb{Q}(T)$ be the generic fibre of $\mathcal{E}$, where T is a coordinate of $\mathbb{P}^1_{\mathbb{Q}}$. The Mordell–Weil theorem for function fields (see [16, page 230]) asserts that $E(\mathbb{Q}(T))$ is a finitely generated group.

Denote by r the rank of the generic fibre $E/\mathbb{Q}(T)$ of $\mathcal{E}$ and by r_t the rank of the specialisation $E_t/\mathbb{Q}$ of $\mathcal{E}$ at $T = t$, provided that $E_t/\mathbb{Q}$ is an elliptic curve. It follows from a theorem of Silverman (see [14, Theorem C] or [16, Theorem III.11.4]) that $r \leq r_t$ for all but finitely many t. A natural question to ask is how far the above inequality is from being an equality. Assume from now on that $\pi : \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}}$ is nonisotrivial. Let

$$\mathcal{N}(E) = \{t \in \mathbb{P}^1(\mathbb{Q}) : E_t/\mathbb{Q} \text{ is an elliptic curve and } r_t = r\}$$

and

$$\mathcal{F}(E) = \{t \in \mathbb{P}^1(\mathbb{Q}) : E_t/\mathbb{Q} \text{ is an elliptic curve and } r_t \geq 2 + r\}.$$

The density conjecture (see [15, page 556] or [4, Appendix A]) predicts that $\mathcal{N}(\mathcal{E})$ is infinite while $\mathcal{F}(\mathcal{E})$ has density zero.

Proving either of these two statements at the moment seems to be out of reach. Moreover, not a single (unconditional) example of a nonisotrivial elliptic surface for which $\mathcal{N}(E)$ is infinite is known. However, conditional examples relying on standard conjectures in analytic number theory have been found. For instance, under the assumption that there are infinitely many Mersenne primes, Caro and Pasten in [3] found an elliptic curve $E/\mathbb{Q}(T)$ of rank 0 and infinitely many primes q such that $E_q/\mathbb{Q}$ has rank 0 as well. Moreover, work of Neuman and Setzer (see [11, 13]) on elliptic curves with prime conductor combined with a conjecture of Bouniakowsky [2] provides another such example.

For every $i \geq 1$, we let

$$\mathcal{I}_i(E) = \{t \in \mathbb{P}^1(\mathbb{Q}) : E_t \text{ is an elliptic curve and } r_t \leq r + i\}.$$

In this article, we are interested in providing examples of elliptic surfaces and explicit positive integers i such that $\mathcal{I}_i(E)$ is infinite. Our first result is the following theorem (see Theorems 2.3 and 2.4 below).

THEOREM 1.1. *Let $E/\mathbb{Q}(T)$ be either the elliptic curve given by the Weierstrass equation*

$$y^2 = x^3 + Tx^2 - x$$

or the elliptic curve given by the Weierstrass equation

$$y^2 + xy = x^3 + \frac{T-1}{4}x^2 - x.$$

Then, there exist infinitely many integers n such that $\operatorname{rk} E_n(\mathbb{Q}) \leq 1$. In particular, the set $\mathcal{I}_1(E)$ is infinite.

Before we state our next theorem, we need to introduce some notation. If $F(x)$ is an irreducible polynomial with integer coefficients, then we write $\rho_F(p)$ for the number of solutions of the congruence

$$F(x) \equiv 0 \pmod{p}.$$

THEOREM 1.2. *Let $E/\mathbb{Q}(T)$ be a nonisotrivial elliptic curve whose Mordell–Weil group contains a point of order 2. Fix an integral Weierstrass equation for $E/\mathbb{Q}(T)$ and denote by $\Delta(T) \in \mathbb{Z}[T]$ its discriminant.*

(1) *Assume that $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} f(T)^k$, where $m \geq 0$, $k, a_1, \ldots, a_m > 0$, and $f(T)$ is an irreducible polynomial with integral coefficients such that $\rho_f(p) < p$ for every prime p. Then, there exist infinitely many positive integers n such that*

$$\operatorname{rk} E_n(\mathbb{Q}) \leq 2\deg(\Delta) + 2m + 1.$$

In particular, the set $\mathcal{I}_{2\deg(\Delta)+2m+1}(E)$ is infinite.

(2) *Assume that* $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} T^{k_1} f(T)^{k_2}$, *where* $m \geq 0$, $k_1, k_2, a_1, \ldots, a_m > 0$, *and* $f(T) \neq \pm T$ *is an irreducible polynomial with* $\rho_f(p) < p$ *for every prime* p. *Assume also that* $\rho_f(p) < p - 1$ *for every prime* $p \leq \deg(f) + 1$ *such that* $p \nmid f(0)$. *Then, there exist infinitely many positive integers* n *such that*

$$\operatorname{rk} E_n(\mathbb{Q}) \leq 4 \deg(f) + 2m + 2.$$

In particular, the set $\mathcal{I}_{4\deg(f)+2m+2}(E)$ *is infinite.*

In fact, we prove a more general theorem where we also treat the general case where $\Delta(T)$ factors into a product of any number of irreducible polynomials. To prove our results, we combine a bound on ranks of elliptic curves over $\mathbb{Q}$ that depends on their discriminants coming from 2-descent (see [1, 3]) with results on almost prime values of polynomials that are derived from sieve methods in analytic number theory (see, for example, [5] or [6]).

2. Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. In fact, we will prove more general versions of the theorems stated in the introduction. Before we begin our proofs, let us recall a theorem that provides an upper bound for the rank of elliptic curves with a torsion point of order 2.

THEOREM 2.1 (See [1, Proposition 1.1] and [3, Theorem 2.3]). *Let* $E/\mathbb{Q}$ *be an elliptic curve that has a point of order* 2.

(1) *If* $E/\mathbb{Q}$ *has an integral Weierstrass equation of the form* $y^2 = x^3 + Ax^2 + Bx$, *then*

$$\operatorname{rk} E(\mathbb{Q}) \leq \nu(A^2 - 4B) + \nu(B) - 1,$$

where $\nu(n)$ *denotes the number of positive prime divisors of a nonzero integer* n.

(2) *Let* α *and* μ *be the number of places of additive and of multiplicative reduction of* $E/\mathbb{Q}$, *respectively. Then,*

$$\operatorname{rk} E(\mathbb{Q}) \leq 2\alpha + \mu - 1.$$

REMARK 2.2. Elliptic curves for which the inequality of part (1) of Theorem 2.1 is an equality are called elliptic curves of maximal Mordell–Weil rank. Examples of such curves have been exhibited by Aguirre *et al.* in [1].

Throughout the rest of this section, we will denote by P_r the set of positive integers with at most r prime divisors, counted with multiplicity. We are now ready to proceed to the proof of Theorem 1.1 for one of the two elliptic curves.

THEOREM 2.3. *Consider the elliptic curve* $E/\mathbb{Q}(T)$ *given by* $y^2 = x^3 + Tx^2 - x$. *Then, there exist infinitely many integers* n *such that* $E_n/\mathbb{Q}$ *has Mordell–Weil rank at most* 1. *Moreover, there exists a positive constant* C *such that, if* X *is sufficiently large,*

$$\#\{n : n \leq X \text{ and } \operatorname{rk} E_n(\mathbb{Q}) \leq 1\} \geq C \frac{X}{\log X}.$$

PROOF. For an integer n, consider the elliptic curve $E_n/\mathbb{Q}$ given by $y^2 = x^3 + nx^2 - x$. We first show that there exist infinitely many n such that $E_n/\mathbb{Q}$ has Mordell–Weil rank at most 1. Since $E/\mathbb{Q}(T)$ has a torsion point of order 2 and the torsion subgroup of $E(\mathbb{Q}(T))$ injects in $E_n(\mathbb{Q})$ when $E_n/\mathbb{Q}$ is nonsingular, $E_n/\mathbb{Q}$ has a point of order 2 for all but finitely many n. Therefore, it follows from part (1) of Theorem 2.1 that

$$\operatorname{rk} E_n(\mathbb{Q}) \leq \nu(n^2 + 4) + \nu(-1) - 1 = \nu(n^2 + 4) - 1,$$

where $\nu(N)$ denotes the number of positive prime divisors of a nonzero integer N.

If we can find infinitely many n such that $n^2 + 4 \in P_2$, then we are done. In contrast, for every prime p, we have $\rho_{n^2+4}(p) \leq 2$ and $\rho_{n^2+4}(2) = 1$, so that

$$\Gamma_{n^2+4} = \prod_{p \text{ prime}} \frac{1 - \rho_{n^2+4}(p)/p}{1 - 1/p} > 0.$$

Therefore, it follows from [10] that there exist infinitely many positive integers n such that $n^2 + 4$ has at most two prime divisors, counted with multiplicity. This proves that there exist infinitely many positive integers n such that $E_n/\mathbb{Q}$ has Mordell–Weil rank at most 1.

We now show the inequality of the theorem. Consider the polynomial

$$h(n') = (n' + 1)^2 + 4 = n'^2 + 2n' + 5.$$

According to [9, Theorem 1] (see also [8, page 172] and [10]), if X is sufficiently large,

$$\#\{n' : n' \leq X \text{ and } h(n') \in P_2\} \geq \frac{1}{144} \prod_{p \text{ prime}} \frac{1 - \rho_h(p)/p}{1 - 1/p} \frac{X}{\log X}.$$

Further,

$$\#\{n : n \leq X \text{ and } n^2 + 4 \in P_2\} \geq \#\{n' : n' \leq X \text{ and } (n' + 1)^2 + 4 \in P_2\} - 1.$$

Therefore,

$$\#\{n : n \leq X \text{ and } n^2 + 4 \in P_2\} \geq \frac{1}{144} \prod_{p \text{ prime}} \frac{1 - \rho_h(p)/p}{1 - 1/p} \frac{X}{\log X} - 1 \geq C \frac{X}{\log X}$$

for all X sufficiently large (picking an appropriate constant C). This proves our theorem. □

Consider now the elliptic curve $E/\mathbb{Q}(T)$ given by the Weierstrass equation

$$y^2 + xy = x^3 + \frac{T - 1}{4} x^2 - x.$$

Specialisations of this curve have been studied by Neumann [11] and Setzer [13]. More precisely, it is proved in [13, Theorem 2] that if $p \neq 2, 3, 17$ is a prime and $E/\mathbb{Q}$ is an elliptic curve of conductor p with a torsion point of order 2, then $p = b^2 + 64$ for some integer $b \equiv 1 \pmod 4$. In this case, $E/\mathbb{Q}$ is isomorphic to either the curve $E_b/\mathbb{Q}$ or to a curve which is isogenous to $E_b/\mathbb{Q}$.

It follows from part (2) of Theorem 2.1 that if p is a prime of the form $p = b^2 + 64$ for some integer $b \equiv 1 \pmod 4$, then the specialisation $E_b/\mathbb{Q}$ has rank equal to 0. According to a conjecture of Bouniakowsky [2, page 328], there are infinitely many such numbers b. Without relying on any conjectures, we show below that we can find infinitely many integers n such that $n^2 + 64 \in P_2$, which forces the rank of the corresponding curve $E_n/\mathbb{Q}$ to be at most 1.

THEOREM 2.4. *Consider the elliptic curve $E/\mathbb{Q}(T)$ given by*

$$y^2 + xy = x^3 + \frac{T-1}{4}x^2 - x.$$

Then, there exist infinitely many integers n such that $E_n/\mathbb{Q}$ has Mordell–Weil rank at most 1.

PROOF. The proof is similar to the proof of Theorem 2.3. For every $n \in \mathbb{Z}$, consider the elliptic curve $E_n/\mathbb{Q}$ given by

$$y^2 + xy = x^3 + \frac{n-1}{4}x^2 - x.$$

The discriminant of $E_n/\mathbb{Q}$ is $\Delta(n) = n^2 + 64$ and the c_4-invariant is $c_4(n) = n^2 + 48$. Since $E/\mathbb{Q}(T)$ has a torsion point of order 2, we find that $E_n/\mathbb{Q}$ has a point of order 2 for all but finitely many $E_n/\mathbb{Q}$. The strategy that we will follow for the rest of the proof is to try to control the primes of bad reduction of $E_n/\mathbb{Q}$ for sufficiently many integers n and apply Theorem 2.1.

CLAIM 2.5. If $n^2 + 64 \in P_2$, then n is odd. □

PROOF OF THE CLAIM. If $n^2 + 64$ is odd, then n must be odd. Therefore, assume that

$$n^2 + 64 = 2q,$$

where q is a prime (not necessarily distinct from 2). This means that n is even. If we write $n = 2n'$, then

$$2q = n^2 + 64 = 4n'^2 + 64$$

and we see that q is even, so $q = 2$. However, then $n^2 + 64 = 4$, which is impossible. Therefore, n must be odd. □

If $n^2 + 64 \in P_2$, then $n^2 + 64$ is odd and, hence, n is odd. Moreover, by replacing n with $-n$ if necessary, we can also arrange that $n \equiv 1 \pmod 4$. Thus, the given Weierstrass equation for $E_n/\mathbb{Q}$ is an integral Weierstrass equation and by looking at the corresponding c_4-invariant, we see that when $n^2 + 64 \in P_2$, every divisor of $\Delta(n)$ does not divide $c_4(n)$. This proves that if $n^2 + 64 \in P_2$, then the curve $E_n/\mathbb{Q}$ has at most two primes of multiplicative reduction and no primes of additive reduction. Therefore, it follows from part (2) of Theorem 2.1 that

$$\operatorname{rk} E_n(\mathbb{Q}) \leq 2 \cdot 0 + \mu_n - 1 = \mu_n - 1,$$

where $\mu_n \leq 2$ is the number of primes of multiplicative reduction of $E_n/\mathbb{Q}$. This shows that when $n^2 + 64 \in P_2$, we have $\operatorname{rk} E_n(\mathbb{Q}) \leq 1$.

Since $\rho_{n^2+64}(p) \leq 2$ and $\rho_{n^2+64}(2) = 1$ for every prime p,

$$\Gamma_{n^2+64} = \prod_{p \text{ prime}} \frac{1 - \rho_{n^2+64}(p)/p}{1 - 1/p} > 0.$$

Therefore, it follows from [10] that there exist infinitely many n such that $n^2 + 64 \in P_2$. Hence, there exist infinitely many integers n such that $\operatorname{rk} E_n(\mathbb{Q}) \leq 1$.

Before we proceed to the proof of a slightly more general version of Theorem 1.2, we need to recall two theorems on almost-prime values of polynomials that are derived from analytic number theory and will be needed in our proofs.

THEOREM 2.6 ([12, Theorem 6] and [12, Theorem 7]). *Let $F(x)$ be an irreducible polynomial of degree $g \geq 1$ with integral coefficients. Assume that $\rho_F(p) < p$ for every prime p.*

(1) *Then, there exists a constant $X_0(F)$ that depends on F such that for every $X \geq X_0(F)$,*

$$\#\{n : 1 \leq n \leq X, F(n) \in P_{g+1}\} \geq \frac{2}{3} \prod_{p} \frac{1 - \rho_F(p)/p}{1 - 1/p} \frac{X}{\log(X)}.$$

In particular, there exist infinitely many integers n such that $F(n)$ has at most $g + 1$ prime factors.

(2) *Assume in addition that $\rho_F(p) < p - 1$ for every prime $p \leq \deg(F) + 1$ with $p \nmid F(0)$. Then, there exist positive constants $C(F)$ and $X_0(F)$ that depend on F such that for every $X \geq X_0(F)$,*

$$\#\{p \text{ prime } : 1 \leq p \leq X, F(p) \in P_{2g+1}\} \geq C(F) \frac{X}{\log^2(X)}.$$

In particular, there exist infinitely many prime numbers p such that $F(p)$ has at most $2g + 1$ prime factors.

THEOREM 2.7 [6, Theorem 10.4]. *Let $F_1(x), F_2(x), \ldots, F_g(x)$ be distinct irreducible polynomials with integral coefficients and write $F(x) = F_1(x)F_2(x) \cdots F_g(x)$ for their product. Assume that $\rho_F(p) < p$ for every prime p. Then, there exists a positive integer s that can be explicitly computed and depends on F, and a positive constant $C(F)$ that depends on F such that for all X sufficiently large,*

$$\#\{n : 1 \leq n \leq X, F(n) \in P_s\} \geq C(F) \frac{X}{\log^g(X)}.$$

We are now ready to proceed to the proof of a slightly more general version of Theorem 1.2.

THEOREM 2.8. *Let $E/\mathbb{Q}(T)$ be a nonisotrivial elliptic curve whose Mordell–Weil group contains a point of order 2. Fix an integral Weierstrass equation for $E/\mathbb{Q}(T)$ and denote its discriminant by $\Delta(T) \in \mathbb{Z}[T]$.*

(1) *Assume that $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} f(T)^k$, where $m \geq 0$, $k, a_1, \ldots, a_m > 0$, and $f(T)$ is an irreducible polynomial with integral coefficients such that $\rho_f(p) < p$ for every prime p. Then, there exists a constant $X_0(f)$ that depends only on f such that for every $X \geq X_0(f)$,*

$$\#\{n : 1 \leq n \leq X, \operatorname{rk} E_n(\mathbb{Q}) \leq 2\deg(\Delta) + 2m + 1\} \geq \frac{2}{3} \prod_p \frac{1 - \rho_f(p)/p}{1 - 1/p} \frac{X}{\log(X)}.$$

In particular, there exist infinitely many positive integers n such that

$$\operatorname{rk} E_n(\mathbb{Q}) \leq 2\deg(\Delta) + 2m + 1.$$

(2) *Assume that $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} T^{k_1} f(T)^{k_2}$, where $m \geq 0$, $k_1, k_2, a_1, \ldots, a_m > 0$, and $f(T) \neq \pm T$ is an irreducible polynomial with $\rho_f(p) < p$ for every prime p. Assume also that $\rho_f(p) < p - 1$ for every prime $p \leq \deg(f) + 1$ with $p \nmid f(0)$. Then, there exist constants $C(f)$ and $X_1(f)$ that depend only on f such that for every $X \geq X_1(f)$,*

$$\#\{n : 1 \leq n \leq X, \operatorname{rk} E_n(\mathbb{Q}) \leq 4\deg(f) + 2m + 2\} \geq C(f) \frac{X}{\log^2(X)}.$$

In particular, there exist infinitely many positive integers n such that

$$\operatorname{rk} E_n(\mathbb{Q}) \leq 4\deg(f) + 2m + 2.$$

(3) *More generally, write $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} f_1(T)^{h_1} \cdots f_g(T)^{h_g}$, where $p_1, \ldots, p_m$ are distinct primes for some $m \geq 0$ and $f_1(T), \ldots, f_g(T)$ are distinct irreducible polynomials with integral coefficients for some $g \geq 1$. Assume that $\rho_{f_1(T)\cdots f_g(T)}(p) < p$ for every prime p. Then, there exists a positive constant $C(f_1, \ldots, f_g)$ that depends on $f_1, \ldots, f_g$ such that for all X sufficiently large,*

$$\#\{n : 1 \leq n \leq X, \operatorname{rk} E_n(\mathbb{Q}) \leq 2(m + s) - 1\} \geq C(f_1, \ldots, f_g) \frac{X}{\log^g(X)},$$

where s is a positive integer that can be explicitly computed and depends on $\deg \Delta(T)$ *and g.*

PROOF OF PART (1). Let $E/\mathbb{Q}(T)$ be an elliptic curve as in part (1). Since $E/\mathbb{Q}(T)$ has a torsion point of order 2 and the torsion subgroup of $E(\mathbb{Q}(T))$ injects in $E_t(\mathbb{Q})$ when $E_t/\mathbb{Q}$ is nonsingular, $E_t/\mathbb{Q}$ contains a point of order 2 for all but finitely many $E_t/\mathbb{Q}$. Moreover, applying part (1) of Theorem 2.6 to the polynomial $f(T)$ yields a constant $X_0(f)$ that depends on f such that for every $X \geq X_0(f)$,

$$\#\{n : 1 \leq n \leq X, f(n) \in P_{\deg(f)+1}\} \geq \frac{2}{3} \prod_p \frac{1 - \rho_f(p)/p}{1 - 1/p} \frac{X}{\log(X)}.$$

Recall that $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} f(T)^k$ and note that for each $n \in \mathbb{N}$, the primes of bad reduction of $E_n/\mathbb{Q}$ are a subset of the primes that divide $\Delta(n)$. Therefore, by applying part (2) of Theorem 2.1 to each $E_n/\mathbb{Q}$, we find that if $f(n) \in P_{\deg(f)+1}$, then the rank of $E_n/\mathbb{Q}$ is at most $2(\deg(f) + 1 + m) - 1 = 2\deg(f) + 2m + 1$. Since $\deg(f) = \deg(\Delta)$, the proof of part (1) is complete. □

PROOF OF PART (2). Now, let $E/\mathbb{Q}(T)$ be an elliptic curve as in part (2). As in the previous part, $E_t/\mathbb{Q}$ contains a point of order 2 for all but finitely many $E_t/\mathbb{Q}$. Applying part (2) of Theorem 2.6 to the polynomial $f(T)$ yields positive constants $C(f)$ and $X_0(f)$ that depend on f such that for every $X \geq X_0(f)$,

$$\#\{p \text{ prime } : 1 \leq p \leq X, f(p) \in P_{2\deg(f)+1}\} \geq C(f)\frac{X}{\log^2(X)}.$$

Recall $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} T^{k_1} f(T)^{k_2}$ and note that for each $n \in \mathbb{N}$, the primes of bad reduction of $E_n/\mathbb{Q}$ are a subset of the primes that divide $\Delta(n)$. Therefore, by applying part (2) of Theorem 2.1 to each $E_n/\mathbb{Q}$, we find that if n is prime with $f(n) \in P_{2\deg(f)+1}$, then the rank of $E_n/\mathbb{Q}$ is at most $2(2\deg(f) + 2 + m) - 1 = 4\deg(f) + 2m + 2$. □

PROOF OF PART (3). Now, let $E/\mathbb{Q}(T)$ be an elliptic curve as in part (3). As in the previous parts, $E_t/\mathbb{Q}$ contains a point of order 2 for all but finitely many $E_t/\mathbb{Q}$. Applying Theorem 2.7 to the polynomial $f_1(T) \cdots f_g(T)$, we find that there exists a positive integer s that can be explicitly computed, and depends on $f_1(T), \ldots, f_g(T)$ and a positive constant $C(f_1, \ldots, f_g)$ that depends on $f_1(T), \ldots, f_g(T)$ such that for all X sufficiently large,

$$\#\{n : 1 \leq n \leq X, f_1(n) \cdots f_g(T) \in P_s\} \geq C(F)\frac{X}{\log^g(X)}.$$

Recall that $\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} f_1(T)^{h_1} \cdots f_g(T)^{h_g}$ and note that for each $n \in \mathbb{N}$, the primes of bad reduction of $E_n/\mathbb{Q}$ are a subset of the primes that divide $\Delta(n)$. Therefore, by applying part (2) of Theorem 2.1 to each $E_n/\mathbb{Q}$, we find that if n is prime with $f(n) \in P_{2\deg(f)+1}$, then the rank of $E_n/\mathbb{Q}$ is at most $2(s + m) - 1 = 2s + 2m - 1$. This completes the proof of our theorem. □

We end this article by presenting some examples where Theorem 2.8 can be applied to find explicit bounds for the ranks of infinitely many specialisations.

EXAMPLE 2.9. Consider the elliptic curve $E/\mathbb{Q}(T)$ given $y^2 = x^3 + g(T)x^2 - \lambda x$, where $g(T) \in \mathbb{Z}[T]$ and λ is a positive integer. This elliptic curve has discriminant

$$\Delta(T) = 16(-\lambda)^2(g(T)^2 + 4\lambda).$$

When $g(T)^2 + 4\lambda$ is an irreducible polynomial such that $\rho_g(p) < p$ for every prime p, part (1) of Theorem 2.8 shows that there exist infinitely many positive integers n such that $\mathrm{rk}E_n(\mathbb{Q}) \leq 4\deg(g) + 2\nu(\lambda) + 3$, where $\nu(\lambda)$ is the number of positive prime divisors of λ.

EXAMPLE 2.10. Consider the elliptic curve $E/\mathbb{Q}(T)$ given by $y^2 + xy - Ty = x^3 - Tx^2$, which has discriminant $\Delta(T) = T^4(1 + 16T)$. It is well known (see [7, Section 4.4]) that if $E'/\mathbb{Q}$ is any elliptic curve with a torsion point of order 4, then there exists $\lambda \in \mathbb{Q}$ such that $E'/\mathbb{Q}$ is isomorphic to $E_\lambda/\mathbb{Q}$. According to part (2) of Theorem 2.8, there exist infinitely many integers n such that the rank of $E_n/\mathbb{Q}$ is at most 6. It is easy to check that the discriminant of $E/\mathbb{Q}(T)$ satisfies the hypotheses of part (2) of Theorem 2.8.

REMARK 2.11. Keep the same notation as in Theorem 2.8. Given an elliptic curve that satisfies the conditions of part (3) of Theorem 2.8, the corresponding number s can be explicitly computed. A formula for s can be found on [6, page 283]. In the simple case where $g = 2$, that is, when

$$\Delta(T) = p_1^{a_1} \cdots p_m^{a_m} f_1(T)^{h_1} f_2(T)^{h_2},$$

such an s can be computed based only on the degree of $\Delta(T)$. For example, when $\deg(\Delta(T))$ is equal to $3, 4, 5$ or 6, then s is equal to $7, 9, 10$ or 11, respectively (see [6, page 287]).

EXAMPLE 2.12. Consider the elliptic curve $E/\mathbb{Q}(T)$ given by $y^2 = x^3 + (T + 1)x^2 - (T^2 + 1)x$ which has discriminant

$$\Delta(T) = 16(T^2 + 1)^2((T + 1)^2 + 4(T^2 + 1)) = 16(T^2 + 1)^2(5T^2 + 2T + 5).$$

By part (3) of Theorem 2.8 combined with Remark 2.11, there exist infinitely many n such that the rank of $E_n/\mathbb{Q}$ is at most 9.

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MENTZELOS MELISTAS, Department of Applied Mathematics,
University of Twente, Drienerlolaan 5, 7522 NB Enschede, The Netherlands
e-mail: mentzmel@gmail.com