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The commutators of multilinear Calderón–Zygmund operators on weighted Hardy spaces

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In this paper, we study the behaviours of the commutators $[\vec{b}, T]$ generated by multilinear Calderón–Zygmund operators T with $\vec{b} = (b_1, \ldots, b_m) \in L_{\text{loc}}(\mathbb{R}^n)$ on weighted Hardy spaces. We show that for some $p_i \in (0, 1]$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $\omega \in A_\infty$ and $b_i \in \mathcal{BMO}_{\omega,p_i}$ $(1 \leqslant i \leqslant m)$, which are a class of non-trivial subspaces of BMO, the commutators $[\vec{b}, T]$ are bounded from $H^{p_1}(\omega) \times \cdots \times H^{p_m}(\omega)$ to $L^p(\omega)$. Meanwhile, we also establish the corresponding results for a class of maximal truncated multilinear commutators $T^*_{\vec{b}}$.

Keywords: BMO spaces; commutators; multilinear Calderón–Zygmund operators; weighted Hardy spaces

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1. Introduction and main results

This paper is devoted to exploring the behaviours of the commutators of multilinear operators in weighted Hardy spaces. As well known, multilinear Calderón–Zygmund theory was introduced and first investigated by Coifman and Meyer [1, 2]. Later on, the topic was retaken by several authors: including Grafakos and Torres [10], Lerner *et al.* [15] and Cruz-Uribe *et al.* [4], etc. We first recall the definition of multilinear Calderón–Zygmund operators.

Definition 1.1. *Assume that $K(y_0, y_1, \dots, y_m)$ is a function defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, which satisfies the following estimates*

$$|\partial_{y_0}^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K(y_0, y_1, \dots, y_m)| \leqslant \frac{A_\alpha}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+|\alpha|}}, \tag{1.1}$$

for all $\alpha = (\alpha_0, \cdots, \alpha_m)$ such that $|\alpha| = |\alpha_0| + \cdots + |\alpha_m| \leqslant N$, where $|\alpha_j|$ is the order of each multi-index α_j, and N is a large integer to be determined later. An m-linear Calderón–Zygmund operator is a multilinear operator T that satisfies

$$T : L^{q_1} \times \cdots \times L^{q_m} \to L^q$$

for some $1 < q_1, \dots, q_m < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$, T has the integral representation

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j) \mathrm{d}y_j$$

whenever $f_i \in L_c^\infty$ and $x \notin \cap_i \mathrm{supp} f_i$.

It was shown in [9] that if T is an m-linear Calderón–Zygmund operator, $1/p_1 + \cdots + 1/p_m = 1/p$ and $p_0 = \min\{p_j, j = 1, \dots, m\} > 1$, then T is bounded from $L^{p_1}(\omega) \times \cdots \times L^{p_m}(\omega)$ into $L^p(\omega)$, provided that the weight ω is in the class A_{p_0} (see subsection 2.1 for the definition of A_{p_0}). In 2001, Grafakos and Kalton [8] discussed the boundedness of multilinear Calderón–Zygmund operators on the product of Hardy spaces. Later on, Cruz-Uribe *et al.* [4] generalized the results in [8] to the weighted Hardy spaces. Precisely,

Theorem A. (cf. [4]) *Let $0 < p_1, \dots, p_m < \infty$, $\omega_i \in A_\infty$, $1 \leqslant i \leqslant m$ and*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

*Suppose that T is an m-linear Calderón–*Zygmund *operator associated to a kernel K that satisfies (1.1) with*

$$N \geqslant \max\left\{ \left\lfloor mn\left(\frac{q_{\omega_i}}{p_i} - 1\right)\right\rfloor_+, 1 \leqslant i \leqslant m\right\} + (m-1)n.$$

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Then

$$\left\|T(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega_i)},$$

where $\nu_{\vec{\omega}} = \Pi_{i=1}^m \omega_i^{p/p_i}$, $q_\omega := \inf\{q > 1 : \omega \in A_q\}$.

In this paper, we will focus on the commutators of multilinear operators. For an m-linear Calderón–Zygmund operator T and a collection of locally integral functions $\vec{b} = (b_1, \ldots, b_m)$, the multilinear commutators generated by T and $\vec{b}$ are defined as follows:

$$[\vec{b}, T](f_1, \ldots, f_m) = \sum_{j=1}^{m} [b_j, T](f_1, \ldots, f_m),$$

where

$$[b_j, T](f_1, \ldots, f_m) := b_j T(f_1, \ldots, f_m) - T(f_1, \ldots, f_{j-1}, b_j f_j, f_{j+1}, \ldots, f_m).$$

The m-linear commutators were considered by Pérez and Torres in [**20**]. Lerner *et al.* [**15**] introduced the multiple weight $A_{\vec{P}}$ (see definition 3.5 in [**15**]), and they proved that when $\vec{b} \in (\mathrm{BMO})^m$, $[\vec{b}, T]$ is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ for $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{\vec{P}}$, the multiple Muckenhoupt class, where $1/p_1 + \cdots + 1/p_m = 1/p$ and $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. Moreover, inspired by the remarkable work of Lerner *et al.* [**16**], Kunwar and Ou [**14**] obtained the Bloom type two-weight inequalities of $[\vec{b}, T]$. Precisely, $1 < p_i < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$, λ_i, $\mu_i \in A_{p_i}$, $\nu_i = (\mu_i/\lambda_i)^{1/p_i}$, $\nu_{\vec{\lambda}} = \prod_{i=1}^m \lambda_i^{p/p_i}$, for $b \in \mathrm{BMO}_{\nu_i}$(see definition in [**14**]), $i = 1, \ldots, m$, it holds that

$$\|[\vec{b}, T](f_1, \ldots, f_m)\|_{L^p(\nu_{\vec{\lambda}})} \lesssim \left(\sum_{i=1}^{m} \|b_i\|_{BMO_{\nu_i}}\right) \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mu_i)}.$$

On the other hand, for $m = 1$, in the endpoint case, Harboure *et al.* [**11**] showed that for general $b \in \mathrm{BMO}(\mathbb{R}^n)$, the linear commutator $[b, T]$ cannot be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. However, Liang *et al.* [**19**] and Huy *et al.* [**13**] found out $\mathcal{BMO}_{\omega,p}$ (see subsection 2.2 for the definition and properties), a non-trivial subspace of $\mathrm{BMO}(\mathbb{R}^n)$ for some Muckenhoupt weights ω and $0 < p \leqslant 1$, such that $[b, T]$ is bounded from the weighted Hardy spaces $H^p(\omega)$ to the weighted Lebesgue spaces $L^p(\omega)$, when $b \in \mathcal{BMO}_{\omega,p}$. For the multilinear setting, He and Liang [**12**] recently proved that $[\vec{b}, T]$ is bounded from $H^1(\omega) \times \cdots \times H^1(\omega)$ to $L^{1/m}(\omega)$, when $\vec{b} \in (\mathcal{BMO}_{\omega,1})^m$.

Based on the results above, it is natural to ask the following question.

Question. Is $[\vec{b}, T]$ bounded from $H^{p_1}(\omega) \times \cdots \times H^{p_m}(\omega)$ to $L^p(\omega)$ for some $0 < p_i < 1,\ 1 \leqslant i \leqslant m$, when $b_i \in \mathcal{BMO}_{\omega,p_i}$, the non-trivial subspaces of $\mathrm{BMO}(\mathbb{R}^n)$?

One of the main purpose in this paper is to address the question above. Our result can be formulated as follows.

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THEOREM 1.2. *Let* $0 < p_i \leqslant 1$, $1 \leqslant i \leqslant m$, *and*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Suppose that $\omega \in A_\infty$ *with* $\int_{\mathbb{R}^n} \frac{\omega(x)}{(1+|x|)^{np_0}} < \infty$ *with* $p_0 = \min_{1\leqslant i\leqslant m} p_i$, T *is an* m*-linear Calderón–Zygmund operator with* K *that satisfies* (1.1) *with*

$$N \geqslant \max\left\{ \left\lfloor mn\left(\frac{q_\omega}{p_i} - 1\right)\right\rfloor_+, 1 \leqslant i \leqslant m \right\} + (m-1)n. \tag{1.2}$$

Then for $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \mathcal{BMO}_{\omega,p_i}$, $1 \leqslant i \leqslant m$,

$$\left\|[\vec{b}, T](\vec{f})\right\|_{L^p(\omega)} \lesssim \left(\sum_{j=1}^m \|b_j\|_{\mathcal{BMO}_{\omega,p_j}}\right) \prod_{i=1}^m \|f_i\|_{H^{p_i}(\omega)}.$$

Moreover, we consider the maximal truncated multilinear commutators. Let K satisfy (1.1), the maximal truncated multilinear operator is defined by

$$T^*(\vec{f})(x) := \sup_{\delta>0} |T_\delta(\vec{f})(x)| = \sup_{\delta>0}\left| \int_{\mathbb{R}^n} K_\delta(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) \mathrm{d}y_j \right|, \tag{1.3}$$

where $K_\delta(x, y_1, \ldots, y_m) = \phi(\sqrt{|x-y_1|^2 + \cdots + |x-y_m|^2}/2\delta) K(x, y_1, \ldots, y_m)$ and $\phi(x)$ is a smooth function on $\mathbb{R}^n$, which vanishes if $|x| \leqslant 1/4$ and is equal to 1 if $|x| > 1/2$. Given a collection of locally integral functions $\vec{b} = (b_1, \ldots, b_m)$, the maximal truncated multilinear commutators are defined by

$$T^*_{\vec{b}}(\vec{f})(x) := \sum_{i=1}^m T^*_{b_i}(\vec{f})(x),$$

where

$$T^*_{b_i}(\vec{f})(x) = \sup_{\delta>0}\left| \int_{\mathbb{R}^n} (b_i(x_i) - b_i(y_i)) K_\delta(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) \mathrm{d}y_j \right|. \tag{1.4}$$

The boundedness of T^* on the weighted Lebesgue spaces was first given by Grafakos and Torres [**9**]. Subsequently, Grafakos and Kalton [**8**] and Li *et al.* [**18**] successively discussed the boundedness of T^* on Hardy spaces and weighted Hardy spaces. Recently, Wen *et al.* [**21**] extended and improved the results of [**8**] and [**18**] as follows.

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Theorem B. (cf. [21]) Let $0 < p_1, \ldots, p_m < \infty$, $\omega_i \in A_\infty$, $1 \leqslant i \leqslant m$, and

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Suppose that T^* is defined as in (1.3) and K satisfies (1.1) with N as in theorem A. Then

$$\left\|T^*(\vec{f})\right\|_{L^p(\nu_{\vec{\omega}})} \lesssim \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega_i)},$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$.

Inspired by the results above, for the maximal truncated multilinear commutator $T^*_{\vec{b}}$, we can obtain the following theorem.

Theorem 1.3. *Let $0 < p_i \leqslant 1$, $1 \leqslant i \leqslant m$, and*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Suppose that $\omega \in A_\infty$ and satisfies $\int_{\mathbb{R}^n} \omega(x)/(1+|x|)^{np_0} < \infty$ with $p_0 = \min_{1\leqslant i\leqslant m} p_i$, $T^*_{\vec{b}}$ is defined as in (1.4) and K satisfies (1.1) with N as in theorem 1.2. Then for $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \mathcal{BMO}_{\omega,p_i}$, $1 \leqslant i \leqslant m$,

$$\left\|T^*_{\vec{b}}(\vec{f})\right\|_{L^p(\omega)} \lesssim \left(\sum_{j=1}^{m} \|b_j\|_{\mathcal{BMO}_{\omega,p_j}}\right) \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega)}.$$

Remark 1.4. (i) It is worth noting that for some $p_i > 1$, $i = 1, 2, \ldots, m$, the results of theorems 1.2 and 1.3 still hold. (ii) Moreover, theorem 1.2 extends the result in [12] for $p_i = 1$ to the cases for certain $0 < p_i < 1 (i = 1, \ldots, m)$. (iii) For the general different $\omega_i \in A_\infty$ with $\int_{\mathbb{R}^n} \omega(x)/(1+|x|)^{np_i} \mathrm{d}x < \infty$, $1 \leqslant i \leqslant m$, our method doesn't work. It would be interesting to know whether $[\vec{b}, T]$ or $T^*_{\vec{b}}$ with $b_i \in \mathcal{BMO}_{\omega_i,p_i}$ $(1 \leqslant i \leqslant m)$ are bounded from $H^{p_1}(\omega_1) \times \cdots \times H^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ for the different Muckenhoupt weights ω_i, $1 \leqslant i \leqslant m$, with $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$.

The rest of this paper is organized as follows. We will recall some definitions and known results about Muckenhoupt weights, $\mathcal{BMO}_{\omega,p}$ spaces and weighted Hardy spaces in § 2. The proof of theorem 1.2 will be given in § 3. Finally, we will prove theorem 1.3 in § 4. We remark that some ideas in our arguments are taken from [4, 13, 19, 21], in which the multilinear Calderón–Zygmund operators and the linear commutators of Calderón–Zygmund operators were dealt with.

Finally, we make some conventions on notation. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We denote $f \lesssim g$, $f \approx g$ if $f \leqslant Cg$ and $f \lesssim g \lesssim f$ respectively. For $1 \leqslant p \leqslant \infty$, p' is the conjugate index of p, and $1/p + 1/p' = 1$. $E^c = \mathbb{R}^n \backslash E$ is the complementary set of any measurable subset E of $\mathbb{R}^n$. Any cube $\tilde{Q}$ is denoted as $\tilde{Q} := 8\sqrt{n}Q$, where the cube is with the same centre and 8 times the side length of Q.

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2. Preliminaries

In this section, we recall some auxiliary facts and lemmas, which will be used in our arguments.

2.1. Muckenhoupt weights

A non-negative measurable function ω is said to be in the Muckenhoupt class A_p with $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$[\omega]_{A_p,Q} = \left(\frac{1}{|Q|}\int_Q \omega(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\mathrm{d}x\right)^{p-1} \leqslant C$$

for all cubes $Q \subset \mathbb{R}^n$, where $1/p + 1/p' = 1$. And we denote $[w]_{A_p} := \sup_Q [\omega]_{A_p,Q}$. When $p = 1$, a non-negative measurable function ω is said to belong A_1 if

$$\frac{1}{|Q|}\int_Q \omega(y)\mathrm{d}y \lesssim \operatorname{ess\,inf}_{x\in Q} \omega(x)$$

for all cubes $Q \subset \mathbb{R}^n$. We denote $A_\infty := \cup_{p\geqslant 1} A_p$ and by $q_\omega := \inf\{q > 1 : \omega \in A_q\}$ for $\omega \in A_\infty$. It is well known that if $\omega \in A_p$ for $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$ and $\omega \in A_q$ for some $1 \leqslant q < p$. Then we give some important results about A_p weight that will be used later on.

Lemma 2.1 [7]. *Let $\omega \in A_p$, $p \geqslant 1$. Then, for any cube Q and $\lambda > 1$,*

$$\omega(\lambda Q) \lesssim \lambda^{np}\omega(Q).$$

Lemma 2.2 [4]. *Let $\omega \in A_\infty$, $0 < p < \infty$ and $\max\{1, p\} < q < \infty$. Then for any collection of cubes $\{Q_k\}_{k=1}^\infty$ in $\mathbb{R}^n$ and non-negative integrable functions $\{f_k\}_{k=1}^\infty$ with $\operatorname{supp} f_k \subset Q_k$, we have*

$$\left\|\sum_{k=1}^\infty f_k\right\|_{L^p(\omega)} \lesssim \left\|\sum_{k=1}^\infty \left(\frac{1}{\omega(Q_k)}\int_{Q_k} f_k(x)^q \omega(x)\mathrm{d}x\right)^{1/q} \chi_{Q_k}\right\|_{L^p(\omega)}.$$

2.2. $\mathcal{BMO}_{\omega,p}$ spaces and basic facts

This subsection is concerning with the definition of $\mathcal{BMO}_{\omega,p}$ and its basic properties.

Definition of $\mathcal{BMO}_{\omega,p}$. Let $p \in (0, \infty)$, $\omega \in A_\infty$ and satisfy $\int_{\mathbb{R}^n} \omega(x)/(1 + |x|)^{np}$ $\mathrm{d}x < \infty$. A locally integrable function b is said to be in $\mathcal{BMO}_{\omega,p}$ if

$$\|b\|_{\mathcal{BMO}_{\omega,p}} := \sup_Q \left\{\left(\frac{1}{\omega(Q)}\int_{Q^c} \frac{\omega(x)}{|x - x_0|^{np}}\mathrm{d}x\right)^{1/p} \int_Q |b(y) - b_Q|\mathrm{d}y\right\} < \infty,$$

where the supremum is taken over all cubes $Q := Q(x_0, l) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $l \in (0, \infty)$. Here and hereafter,

$$\omega(Q) := \int_Q \omega(z)\mathrm{d}z \quad \text{and} \quad b_Q := \frac{1}{|Q|}\int_Q b(z)\mathrm{d}z.$$

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A locally integrable function b is said to be in BMO if

$$\|b\|_{\mathrm{BMO}} := \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(x)-b_Q|\mathrm{d}x < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Basic facts ([**13**, **19**]). (i) $\mathcal{BMO}_{\omega,p} \subset \mathrm{BMO}$, which is a proper inclusion.

(ii) Let $0 < p \leqslant 1$, $\omega \in A_\infty$ such that $\int_{\mathbb{R}^n} \omega(x)/(1+|x|)^{np}\mathrm{d}x < \infty$. Any Lipschitz function b with compact support belongs to $\mathcal{BMO}_{\omega,p}$.

Lemma 2.3 [**19**]. *Let $\omega \in A_\infty$ and $q \in [1, \infty)$. Then for $b \in \mathrm{BMO}$ and any cube $Q := Q(x_0, l) \subset \mathbb{R}^n$ with some $x_0 \in \mathbb{R}^n$ and $l \in (0, \infty)$,*

$$\left(\frac{1}{\omega(Q)}\int_Q |b(x)-b_Q|^q\omega(x)\mathrm{d}x\right)^{1/q} \lesssim \|b\|_{\mathrm{BMO}}.$$

2.3. Weighted Hardy spaces

Let $\mathscr{S}$ be the Schwartz class of smooth functions. For a large integer N_0, denote

$$\mathfrak{S}_{N_0} = \left\{\phi \in \mathscr{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n}(1+|x|)^{N_0}\left(\sum_{|\beta|\leqslant N_0}\left|\frac{\partial^\beta}{\partial x^\beta}\phi(x)\right|^2\right)\mathrm{d}x \leqslant 1\right\}.$$

Given $\omega \in A_\infty$ and $0 < p < \infty$, the weighted Hardy spaces $H^p(\omega)$ is defined by

$$H^p(\omega) = \{f \in \mathscr{S}'(\mathbb{R}^n) : \mathcal{M}_{N_0}(f) \in L^p(\omega)\}$$

with the quasi-norm

$$\|f\|_{H^p(\omega)} = \|\mathcal{M}_{N_0}(f)\|_{L^p(\omega)},$$

where $\mathcal{M}_{N_0}(f)$ is given by

$$\mathcal{M}_{N_0}(f)(x) = \sup_{\phi\in\mathfrak{S}_{N_0}}\sup_{t>0}|\phi_t * f(x)|.$$

Given an integer $N \geqslant 0$, we say that a function a is an $(H^p(\omega), \infty, N)$-atom if

$$\operatorname{supp} a_k \subset Q_k, \quad \|a_k\|_{L^\infty} \leqslant \big(\omega(Q_k)\big)^{-1/p}, \quad \int_{\mathbb{R}^n} x^\alpha a_k(x)\mathrm{d}x = 0, \quad |\alpha| \leqslant N.$$

For $\omega \in A_\infty$ and $0 < p < \infty$, denote $S_\omega := \lfloor n(q_\omega/p - 1)\rfloor_+$. Let $N \geqslant S_\omega$, we define

$$\mathcal{O}_N = \left\{f \in C_0^\infty : \int_{\mathbb{R}^n} x^\alpha f(x)\mathrm{d}x = 0, \quad 0 \leqslant |\alpha| \leqslant N\right\}.$$

Then $\mathcal{O}_N$ is dense in $H^p(\omega)$ (see [**4**, **5**]).

In addition, we have the following finite atomic decomposition which was given in [**5**].

Lemma 2.4 [5]. *Given* $0 < p < \infty$ *and* $\omega \in A_\infty$, $S_\omega := \lfloor n(q_\omega/p - 1)\rfloor_+$, *fix* $N \geqslant S_\omega$. *Then if* $f \in \mathcal{O}_N$, *there exists a finite sequence* $\{a_k\}_{k=1}^M$ *of* $(H^p(\omega), \infty, N)$*-atoms with supports* Q_k, *and a non-negative sequence* $\{\lambda_i\}_{i=1}^M$ *such that* $f = \sum_{k=1}^M \lambda_k a_k$ *and*

$$\sum_{k=1}^{M} \lambda_k^p \lesssim \|f\|_{H^p(\omega)}^p.$$

3. The proof of theorem 1.2

This section is devoted to proving theorem 1.2. First, we need to prove a weighted norm inequality for $[\vec{b}, T]$. To do so, we will make use of some recent developments in the theory of Harmonic analysis on the domination of multilinear operators by sparse operators. Next, we sketch the basic definitions.

A collection of cubes $\mathcal{S}$ is called a sparse family if each cube $Q \in \mathcal{S}$ contains measurable subset $E_Q \subset Q$ such that $|E_Q| \geqslant 1/2|Q|$ and the family $\{E_Q\}_{Q\in\mathcal{S}}$ is pairwise disjoint. Given a sparse family $\mathcal{S}$, the sparse operator $\mathcal{T}_{\mathcal{S},b}$ defined with a locally integrable function b by Lerner *et al.* in [16],

$$\mathcal{T}_{\mathcal{S},b}(f)(x) = \sum_{Q\in\mathcal{S}} |b(x) - b_Q| f_Q \chi_Q(x).$$

Let $\mathcal{T}^\star_{\mathcal{S},b}$ denote the adjoint operator to $\mathcal{T}_{\mathcal{S},b}$:

$$\mathcal{T}^\star_{\mathcal{S},b}(f)(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|}\int_Q |b(y) - b_Q| f(y)\mathrm{d}y\right)\chi_Q(x).$$

Proposition 3.1 [16]. *Let* $1 < p < \infty$ *and* $\omega \in A_p$, *then for* $b \in \mathrm{BMO}$, *given any sparse linear operators* $\mathcal{T}_{\mathcal{S},b}(f)$ *and* $\mathcal{T}^\star_{\mathcal{S},b}(f)$ *have*

$$\|\mathcal{T}_{\mathcal{S},b}(f)\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1,p'/p\}} \|b\|_{\mathrm{BMO}} \|f\|_{L^p(\omega)}$$

and

$$\|\mathcal{T}^\star_{\mathcal{S},b}(f)\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1,p'/p\}} \|b\|_{\mathrm{BMO}} \|f\|_{L^p(\omega)}.$$

In a similar way, for $b_l \in L^1_{loc}$, $l = 1, \ldots, m$, given a sparse family $\mathcal{S}$ we define the multilinear sparse operator:

$$\mathcal{T}_{\mathcal{S},b_l}(f_1, \ldots, f_m)(x) = \sum_{Q\in\mathcal{S}} |b_l(x) - b_{l,Q}| \prod_{i=1}^{m} f_{i,Q}\chi_Q(x).$$

Let $\mathcal{T}^\star_{\mathcal{S},b_l}$ denote the adjoint operator to $\mathcal{T}_{\mathcal{S},b_l}$:

$$\mathcal{T}^\star_{\mathcal{S},b_l}(f_1, \ldots, f_m)(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|}\int_Q |b_l(y) - b_{l,Q}| f_l(y)\mathrm{d}y\right) \prod_{i=1, i\neq l}^{m} f_{i,Q}\chi_Q(x).$$

The following pointwise sparse domination for the multilinear commutators of Calderón–Zygmund operators was proved by Kunwar and Ou [14]:

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PROPOSITION 3.2 [**14**]. *Let T be an m-linear Calderón–Zygmund operator with K satisfying (1.1) with N as in theorem 1.2. Given locally integral functions $\vec{b} = (b_1, \ldots, b_m)$ on $\mathbb{R}^n$. Then for any bounded functions $\vec{f} = (f_1, \ldots, f_m)$ with compact support, there exists 3^n sparse families $\mathcal{S}_j$ such that*

$$\left|[\vec{b}, T](f_1, \ldots, f_m)(x)\right| \lesssim \sum_{i=1}^{m} \left(\sum_{j=1}^{3^n} \left(\mathcal{T}_{\mathcal{S}_j, b_i}(|f_1|, \ldots, |f_m|)(x) \right.\right.$$
$$\left.\left. + \mathcal{T}^{\star}_{\mathcal{S}_j, b_i}(|f_1|, \ldots, |f_m|)(x) \right) \right).$$

Next, we prove the following weighted estimate for $[\vec{b}, T]$.

LEMMA 3.3. *Let T be an m-linear Calderón–Zygmund operator with K that satisfies (1.1) with N as in theorem 1.2. Fix $\omega \in A_p$, $1 < p < \infty$. Given functions $\vec{b} = (b_1, \ldots, b_m)$ which $b_i \in \mathrm{BMO}$, $i = 1, \ldots, m$. Then for any bounded functions $\vec{f} = (f_1, \ldots, f_m)$ with compact support, we have*

$$\left\|[\vec{b}, T](f_1, \ldots, f_m)\right\|_{L^p(\omega)} \lesssim \left(\sum_{i=1}^{m} \|b_i\|_{\mathrm{BMO}} \right) \|f_l\|_{L^p(\omega)} \prod_{j=1, j\neq l}^{m} \|f_j\|_{L^\infty}, \quad l = 1, 2, \ldots, m.$$

Proof. By linearity it is enough to consider the operator with only one symbol. For $1 \leqslant k \leqslant m$, fix $b_k \in \mathrm{BMO}$ and consider the operator $[b_k, T](f_1, \ldots, f_m)(x)$. By proposition 3.2, it suffices to prove this estimate for any multilinear sparse operators $\mathcal{T}_{\mathcal{S}, b_k}$, $\mathcal{T}^{\star}_{\mathcal{S}, b_k}$ and non-negative functions $f_1, \ldots, f_m$. By the definition of the sparse operator, we have

$$\mathcal{T}_{\mathcal{S}, b_k}(f_1, \ldots, f_m)(x) \leqslant \prod_{i=1, i\neq l}^{m} \|f_i\|_{L^\infty} \sum_{Q \in \mathcal{S}} |b_k(x) - b_{k,Q}| f_{l,Q} \chi_Q(x)$$
$$= \mathcal{T}_{\mathcal{S}, b_k}(f_l)(x) \prod_{i=1, i\neq l}^{m} \|f_i\|_{L^\infty}.$$

Then, by proposition 3.1, we obtain

$$\|\mathcal{T}_{\mathcal{S}, b_k}(f_1, \ldots, f_m)\|_{L^p(\omega)} \lesssim \|b_k\|_{\mathrm{BMO}} \|f_l\|_{L^p(\omega)} \prod_{i=1, i\neq l}^{m} \|f_i\|_{L^\infty},$$

Next, we estimate $\mathcal{T}^{\star}_{\mathcal{S}, b_k}$ in two different cases:

Case 1: $k = l$,

$$\mathcal{T}^{\star}_{\mathcal{S}, b_k}(f_1, \ldots, f_m)(x) \leqslant \prod_{i=1, i\neq l}^{m} \|f_i\|_{L^\infty} \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |b_k(y) - b_{k,Q}| |f_k(y)| \mathrm{d}y \right) \chi_Q(x)$$
$$= \mathcal{T}^{\star}_{\mathcal{S}, b_k}(f_k)(x) \prod_{i=1, l\neq l}^{m} \|f_i\|_{L^\infty}.$$

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Then, by proposition 3.1, we have that

$$\|\mathcal{T}^{\star}_{\mathcal{S},b_k}(f_1,\dots,f_m)\|_{L^p(\omega)} \lesssim \|b_k\|_{\mathrm{BMO}}\|f_l\|_{L^p(\omega)} \prod_{i=1,i\neq k}^{m} \|f_i\|_{L^\infty}.$$

Case 2: $k \neq l$,

$$\begin{aligned}\mathcal{T}^{\star}_{\mathcal{S},b_k}(f_1,\dots,f_m)(x) &\leqslant \prod_{i=1,i\neq l}^{m} \|f_i\|_{L^\infty} \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|}\int_Q |b_k(y)-b_{k,Q}|\mathrm{d}y\right) f_{l,Q}\chi_Q(x)\\ &\lesssim \|b_k\|_{\mathrm{BMO}} \prod_{i=1,i\neq l}^{m} \|f_i\|_{L^\infty} \sum_{Q\in\mathcal{S}} f_{l,Q}\chi_Q(x)\\ &=: \|b_k\|_{\mathrm{BMO}} \prod_{i=1,i\neq l}^{m} \|f_i\|_{L^\infty} \mathcal{T}_{\mathcal{S}}(f_l)(x),\end{aligned}$$

Recall the well-known bound for the sparse operator $\mathcal{T}_S$ (see [**3**]):

$$\|\mathcal{T}_{\mathcal{S}}(f_l)\|_{L^p(\omega)} \lesssim [\omega]_{A_p}^{\max\{1,p'/p\}} \|f_l\|_{L^p(\omega)}, \quad p\in(1,\infty).$$

Thus, we have

$$\|\mathcal{T}^{\star}_{\mathcal{S},b_k}(f_1,\dots,f_m)\|_{L^p(\omega)} \lesssim \|b_k\|_{\mathrm{BMO}}\|f_l\|_{L^p(\omega)} \prod_{i=1,i\neq l}^{m} \|f_i\|_{L^\infty},$$

which completes the proof of lemma 3.3. □

We also need the following lemma:

LEMMA 3.4 [**17**]. *Let T be an m-linear Calderón–Zygmund operator with K that satisfies (1.1) with N as in theorem 1.2. Let $0 < p_i \leqslant 1$, a_i be an $(H^{p_i}(\omega), \infty, N)$-atom supported in Q_k, and c_i be the centre of Q_i, l_i be the side length of Q_i, $i = 1, \dots, m$. Assume $\tilde{Q}_1 \cap \cdots \cap \tilde{Q}_m \neq \emptyset$. Then for any $x \in (\tilde{Q}_1 \cap \cdots \cap \tilde{Q}_m)^c$, we have*

$$|T(a_1,\dots,a_m)(x)| \lesssim \prod_{i=1}^{m} \frac{\big(\omega(Q_i)\big)^{-1/p_i}|Q_i|^{1+(N+1)/nm}}{(|x-c_i|+l_i)^{n+(N+1)/m}}.$$

Now, we are in the position to prove theorem 1.2.

Proof of theorem 1.2. By linearity, it is enough to consider the operator with only one symbol. For $1 \leqslant l \leqslant m$, fix then $b_l \in \mathcal{BMO}_{\omega,p_l}$ and consider the operator $[b_l, T](f_1, \dots, f_m)(x)$. By lemma 2.4, we will work with finite sums of weighted Hardy atoms and obtain estimates independent of the number of terms in each

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sum. We write f_i as a finite sum of atoms,

$$f_i = \sum_{k_i=1}^{M} \lambda_{i,k_i} a_{i,k_i}, \quad i = 1, 2, \ldots, m,$$

where $\lambda_{i,k_i} \geqslant 0$ and a_{i,k_i} are $(H^{p_i}(\omega),\, \infty,\, N)$-atoms. They are supported in cubes Q_{i,k_i}, $\|a_{i,k_i}\|_{L^\infty} \leqslant (\omega(Q_{i,k_i}))^{-1/p_i}$, $\int_{Q_{i,k_i}} x^\beta a_{i,k_i}(x)\mathrm{d}x = 0$ for all $|\beta| \leqslant N$, and

$$\sum_{k_i} \lambda_{i,k_i}^{p_i} \lesssim \|f_i\|_{H^{p_i}(\omega)}^{p_i}.$$

Denote the centre of Q_{i,k_i} by c_{i,k_i} and the side length of Q_{i,k_i} by l_{i,k_i}. Using multilinearity we write

$$[b_l, T](f_1, \ldots, f_m)(x) = \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} [b_l, T](a_{1,k_1}, \ldots, a_{m,k_m})(x).$$

Then, we decompose $[b_l,\, T](f_1,\, \ldots,\, f_m)(x)$ into two parts, for $x \in \mathbb{R}^n$

$$\big|[b_l, T](f_1, \ldots, f_m)(x)\big| \leqslant I_1(x) + I_2(x),$$

where

$$I_1(x) = \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} \big|[b_l, T](a_{1,k_1}, \ldots, a_{m,k_m})(x)\big| \chi_{\tilde{Q}_{1,k_1} \cap \cdots \cap \tilde{Q}_{m,k_m}},$$

$$I_2(x) = \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} \big|[b_l, T](a_{1,k_1}, \ldots, a_{m,k_m})(x)\big| \chi_{\tilde{Q}_{1,k_1}^c \cup \cdots \cup \tilde{Q}_{m,k_m}^c}.$$

Now, let us begin to discuss $\|I_1\|_{L^p(\omega)}$. For fixed $k_1,\, \ldots,\, k_m$, assume that

$$\tilde{Q}_{1,k_1} \cap \cdots \cap \tilde{Q}_{m,k_m} \neq \emptyset,$$

since otherwise there is nothing needed to be proved. Suppose that $\omega(\tilde{Q}_{1,k_1})$ has the smallest value among $\omega(\tilde{Q}_{i,k_i})$, $i = 1,\, 2,\, \ldots,\, m$. For $q \in (q_\omega,\, \infty)$, by lemma 3.3, we have

$$\begin{aligned}
&\left(\frac{1}{\omega(\tilde{Q}_{1,k_1})} \int_{\tilde{Q}_{1,k_1}} \big|[b_l, T](a_{1,k_1}, \ldots, a_{m,k_m})(x)\big|^q \omega(x)\mathrm{d}x\right)^{1/q} \\
&\quad \leqslant \big(\omega(\tilde{Q}_{1,k_1})\big)^{-1/q} \big\|[b_l, T](a_{1,k_1}, \ldots, a_{m,k_m})\big\|_{L^q(\omega)} \\
&\quad \lesssim \|b_l\|_{\mathrm{BMO}} \big(\omega(\tilde{Q}_{1,k_1})\big)^{-1/q} \|a_{1,k_1}\|_{L^q(\omega)} \prod_{i=2}^{m} \|a_{i,k_i}\|_{L^\infty} \\
&\quad \lesssim \|b_l\|_{\mathrm{BMO}} \big(\omega(\tilde{Q}_{1,k_1})\big)^{-1/q} \big(\omega(Q_{1,k_1})\big)^{1/q-1/p_1} \prod_{i=2}^{m} \big(\omega(Q_{i,k_i})\big)^{-1/p_i} \\
&\quad \lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^{m} \big(\omega(Q_{i,k_i})\big)^{-1/p_i}.
\end{aligned}$$

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By lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned}\|I_1\|_{L^p(\omega)} &\lesssim \|b_l\|_{\mathrm{BMO}} \left\| \sum_{k_1,\ldots,k_m} \lambda_{1,k_1}\cdots\lambda_{m,k_m} \prod_{i=1}^m \big(\omega(Q_{i,k_i})\big)^{-1/p_i} \chi_{\tilde{Q}_{1,k_1}} \right\|_{L^p(\omega)} \\ &\lesssim \|b_l\|_{\mathrm{BMO}} \left\| \prod_{i=1}^m \left(\sum_{k_i} \lambda_{i,k_i} \big(\omega(Q_{i,k_i})\big)^{-1/p_i} \chi_{\tilde{Q}_{1,k_1}} \right) \right\|_{L^p(\omega)} \\ &\lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^m \left\| \sum_{k_i} \lambda_{i,k_i} \big(\omega(Q_{i,k_i})\big)^{-1/p_i} \omega(\cdot)^{1/p_i} \chi_{\tilde{Q}_{1,k_1}} \right\|_{L^{p_i}} \\ &\lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^m \left(\sum_{k_i} \lambda_{i,k_i}^{p_i} \right)^{1/p_i} \lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{H^{p_i}(\omega)}.\end{aligned}$$

Thus,

$$\|I_1\|_{L^p(\omega)} \lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{H^{p_i}(\omega)}.$$

Next, we estimate $\|I_2\|_{L^p(\omega)}$, we split it again

$$\begin{aligned}\|I_2\|_{L^p(\omega)} \lesssim &\left\| \sum_{k_1,\ldots,k_m} \lambda_{1,k_1}\cdots\lambda_{m,k_m} |b_l - b_{l,Q_{l,k_l}}| \right. \\ &\left. \times |T(a_{1,k_1},\ldots,a_{l,k_l},\ldots,a_{m,k_m})| \chi_{\tilde{Q}^c_{1,k_1}\cup\cdots\cup\tilde{Q}^c_{m,k_m}} \right\|_{L^p(\omega)} \\ &+ \left\| \sum_{k_1,\ldots,k_m} \lambda_{1,k_1}\cdots\lambda_{m,k_m} \right. \\ &\left. \times \left| T\big(a_{1,k_1},\ldots,(b_l - b_{l,Q_{l,k_l}})a_{l,k_l},\ldots,a_{m,k_m}\big)\right| \chi_{\tilde{Q}^c_{1,k_1}\cup\cdots\cup\tilde{Q}^c_{m,k_m}} \right\|_{L^p(\omega)} \\ =: &\ \|I_{21}\|_{L^p(\omega)} + \|I_{22}\|_{L^p(\omega)}.\end{aligned}$$

For $\|I_{21}\|_{L^p(\omega)}$, using the Hölder inequality and lemma 3.4, we get

$$\begin{aligned}\|I_{21}\|_{L^p(\omega)} \lesssim &\left\| \sum_{k_1,\ldots,k_m} \lambda_{1,k_1}\cdots\lambda_{m,k_m} |b_l - b_{l,Q_{l,k_l}}| \prod_{i=1}^m \right. \\ &\left. \times \frac{\big(\omega(Q_{i,k_i})\big)^{-1/p_i} |Q_{i,k_i}|^{1+(N+1)/nm}}{(l_{i,k_i} + |\cdot - c_{i,k_i}|)^{n+(N+1)/m}} \right\|_{L^p(\omega)}\end{aligned}$$

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$$
\begin{aligned}
&\lesssim \left\| \left(\sum_{k_l} \frac{\lambda_{l,k_l}\big(\omega(Q_{l,k_l})\big)^{-1/p_l}|b_l - b_{l,Q_{l,k_l}}| l_{l,k_l}^{n+(N+1)/m}}{(l_{l,k_l} + |\cdot - c_{l,k_l}|)^{n+(N+1)/m}} \right) \right. \\
&\qquad \left. \times \prod_{i=1, i\neq l}^{m} \left(\sum_{k_i} \frac{\lambda_{i,k_i}\big(\omega(Q_{i,k_i})\big)^{-1/p_i} l_{i,k_i}^{n+(N+1)/m}}{(l_{i,k_i} + |\cdot - c_{i,k_i}|)^{n+(N+1)/m}} \right) \right\|_{L^p(\omega)} \\
&\lesssim \left\| \sum_{k_l} \frac{\lambda_{l,k_l}\big(\omega(Q_{l,k_l})\big)^{-1/p_l}|b_l - b_{l,Q_{l,k_l}}| l_{l,k_l}^{n+(N+1)/m}}{(l_{l,k_l} + |\cdot - c_{l,k_l}|)^{n+(N+1)/m}} \right\|_{L^{p_l}(\omega)} \\
&\qquad \times \prod_{i=1, i\neq l}^{m} \left\| \sum_{k_i} \frac{\lambda_{i,k_i}\big(\omega(Q_{i,k_i})\big)^{-1/p_i} l_{i,k_i}^{n+(N+1)/m}}{(l_{i,k_i} + |\cdot - c_{i,k_i}|)^{n+(N+1)/m}} \right\|_{L^{p_i}(\omega)} =: \quad J_1 \cdot J_2.
\end{aligned}
$$

For J_2, by (1.2) and lemma 2.1, we have

$$
\begin{aligned}
&\left\| \sum_{k_i} \frac{\lambda_{i,k_i}\big(\omega(Q_{i,k_i})\big)^{-1/p_i} l_{i,k_i}^{n+(N+1)/m}}{(l_{i,k_i} + |\cdot - c_{i,k_i}|)^{n+(N+1)/m}} \right\|_{L^{p_i}(\omega)}^{p_i} \\
&\quad \leqslant \sum_{k_i} \lambda_{i,k_i}^{p_i} \left(\int_{Q_{i,k_i}} \frac{\big(\omega(Q_{i,k_i})\big)^{-1} l_{i,k_i}^{p_i n + p_i(N+1)/m} \omega(x)}{(l_{i,k_i} + |x - c_{i,k_i}|)^{p_i n + p_i(N+1)/m}} \mathrm{d}x \right. \\
&\qquad \left. + \sum_{j=1}^{\infty} \int_{2^j Q_{i,k_i} \setminus 2^{j-1} Q_{i,k_i}} \frac{\big(\omega(Q_{i,k_i})\big)^{-1} l_{i,k_i}^{p_i n + p_i(N+1)/m} \omega(x)}{(l_{i,k_i} + |x - c_{i,k_i}|)^{p_i n + p_i(N+1)/m}} \mathrm{d}x \right) \\
&\quad \lesssim \sum_{k_i} \lambda_{i,k_i}^{p_i} \big(\omega(Q_{i,k_i})\big)^{-1} \left(\sum_{j=0}^{\infty} \frac{\omega(2^j Q_{i,k_i})}{2^{j(p_i n + p_i(N+1)/m)}} \right) \\
&\quad \lesssim \sum_{k_i} \lambda_{i,k_i}^{p_i} \big(\omega(Q_{i,k_i})\big)^{-1} \left(\sum_{j=1}^{\infty} \frac{\omega(Q_{i,k_i})}{2^{j(p_i n + p_i(N+1)/m - n q_\omega)}} \right) \\
&\quad \lesssim \sum_{k_i} \lambda_{i,k_i}^{p_i} \lesssim \|f_i\|_{H^{p_i}(\omega)}^{p_i}.
\end{aligned}
$$

For J_1, by (1.2) and lemmas 2.1 and 2.3, we obtain

$$
\begin{aligned}
&\left\| \sum_{k_l} \frac{\lambda_{l,k_l}\big(\omega(Q_{l,k_l})\big)^{-1/p_l}|b_l - b_{l,Q_{l,k_l}}| l_{l,k_l}^{n+(N+1)/m}}{(l_{l,k_l} + |\cdot - c_{l,k_l}|)^{n+(N+1)/m}} \right\|_{L^{p_l}(\omega)}^{p_l} \\
&\quad \lesssim \sum_{k_l} \lambda_{l,k_l}^{p_l} \big(\omega(Q_{l,k_l})\big)^{-1} \left(\int_{Q_{l,k_l}} \frac{|b_l(x) - b_{l,Q_{l,k_l}}|^{p_l} l_{l,k_l}^{p_l n + p_l(N+1)/m} \omega(x)}{(l_{l,k_l} + |x - c_{l,k_l}|)^{p_l n + p_l(N+1)/m}} \mathrm{d}x \right. \\
&\qquad \left. + \sum_{j=1}^{\infty} \int_{2^{j+1} Q_{l,k_l} \setminus 2^j Q_{l,k_l}} \frac{|b_l(x) - b_{l,Q_{l,k_l}}|^{p_l} l_{l,k_l}^{p_l n + p_l(N+1)/m} \omega(x)}{(l_{l,k_l} + |x - c_{l,k_l}|)^{p_l n + p_l(N+1)/m}} \mathrm{d}x \right) \\
&\quad \lesssim \|b_l\|_{\mathrm{BMO}}^{p_l} \|f_l\|_{H^{p_l}(\omega)}^{p_l}.
\end{aligned}
$$

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Thus,

$$\|I_{21}\|_{L^p(\omega)} \lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega)}.$$

To estimate $\|I_{22}\|_{L^p(\omega)}$, we write

$$\|I_{22}\|_{L^p(\omega)} = \left\| T\left(f_1, \ldots, \sum_{k_l} \lambda_{l,k_l}(b_l - b_{l,Q_{l,k_l}})a_{l,k_l}, \ldots, f_m\right)\right\|_{L^p(\omega)}.$$

By the boundedness of T from $H^{p_1}(\omega) \times \cdots \times H^{p_m}(\omega)$ to $L^p(\omega)$, we only need to show

$$\left\| \sum_{k_l} \lambda_{l,k_l}(b_l - b_{l,Q_{l,k_l}})a_{l,k_l} \right\|_{H^{p_l}(\omega)} \lesssim \|f_l\|_{H^{p_l}(\omega)} \|b_l\|_{\mathcal{BMO}_{\omega,p_l}},$$

that is,

$$\left\| \sum_{k_l} \lambda_{l,k_l} \mathcal{M}_N\big((b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big) \right\|_{L^{p_l}(\omega)} \lesssim \|f_l\|_{H^{p_l}(\omega)} \|b_l\|_{\mathcal{BMO}_{\omega,p_l}}. \tag{3.1}$$

We write

$$\begin{aligned}
&\left\| \sum_{k_l} \lambda_{l,k_l} \mathcal{M}_N\big((b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big) \right\|_{L^{p_l}(\omega)}^{p_l} \\
&\quad \leqslant \sum_{k_l} \lambda_{l,k_l}^{p_l} \int_{2Q_{l,k_l}} \big|\mathcal{M}_N\big((b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big)(x)\big|^{p_l} \omega(x)\mathrm{d}x \\
&\qquad + \sum_{k_l} \lambda_{l,k_l}^{p_l} \int_{(2Q_{l,k_l})^c} \big|\mathcal{M}_N\big((b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big)(x)\big|^{p_l} \omega(x)\mathrm{d}x =: L_1 + L_2.
\end{aligned}$$

For L_1, by Hölder's inequality for t/p_l ($q_\omega < t < \infty$), lemma 2.3 and the boundedness of $\mathcal{M}_N$ on $L^t(\omega)$, we obtain

$$\begin{aligned}
L_1 &= \sum_{k_l} \lambda_{l,k_l}^{p_1} \int_{2Q_{l,k_l}} \big|\mathcal{M}_N\big((b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big)(x)\big|^{p_l} \omega(x)\mathrm{d}x \\
&\leqslant \sum_{k_l} \lambda_{l,k_l}^{p_l} \big\|\mathcal{M}_N\big((b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big)\big\|_{L^t(\omega)}^{p_l} \left(\int_{2Q_{l,k_l}} \omega(x)\mathrm{d}x \right)^{1-p_l/t} \\
&\lesssim \sum_{k_l} \lambda_{l,k_l}^{p_l} \big\|(b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big\|_{L^t(\omega)}^{p_l} \big(\omega(Q_{l,k_l})\big)^{1-p_l/t} \\
&\lesssim \sum_{k_l} \lambda_{l,k_l}^{p_l} \|b_l\|_{\mathrm{BMO}}^{p_l} \lesssim \|f_l\|_{H^{p_l}(\omega)}^{p_l} \|b_l\|_{\mathcal{BMO}_{\omega,p_l}}^{p_l}.
\end{aligned}$$

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For L_2, note that for $x \in (2Q_{l,k_l})^c$ and $y \in Q_{l,k_l}$, $|x-y| \approx |x-c_{l,k_l}|$. Then, for $\phi \in \mathfrak{S}_N$, $t>0$, we have

$$\begin{aligned}
&\frac{1}{t^n}\left|\int_{Q_{l,k_l}}(b_l(y)-b_{l,Q_{l,k_l}})a_{l,k_l}(y)\phi(\frac{x-y}{t})\mathrm{d}y\right| \\
&\lesssim \frac{1}{|x-c_{l,k_l}|^n}\int_{Q_{l,k_l}}|b_l(y)-b_{l,Q_{l,k_l}}||a_{l,k_l}(y)|\mathrm{d}y \\
&\lesssim \frac{1}{|x-c_{l,k_l}|^n\big(\omega(Q_{l,k_l})\big)^{1/p_l}}\int_{Q_{l,k_l}}|b_l(y)-b_{l,Q_{l,k_l}}|\mathrm{d}y.
\end{aligned}$$

This, together with the definition of $\mathcal{BMO}_{\omega,p_l}$, deduces that

$$L_2 \lesssim \sum_{k_l}\lambda_{l,k_l}^{p_l}\|b_l\|_{\mathcal{BMO}_{\omega,p_l}}^{p_l} \lesssim \|f_l\|_{H^{p_l}(\omega)}^{p_l}\|b_l\|_{\mathcal{BMO}_{\omega,p_l}}^{p_l}.$$

Summing up the estimates of L_1 and L_2, we obtain

$$\|I_{22}\|_{L^p(\omega)} \lesssim \|b_l\|_{\mathcal{BMO}_{\omega,p_l}}\prod_{i=1}^{m}\|f_i\|_{H^{p_i}(\omega)}.$$

Combining the estimates in both cases, there is

$$\big\|[\vec{b},T](\vec{f})\big\|_{L^p(\omega)} \lesssim \left(\sum_{j=1}^{m}\|b_j\|_{\mathcal{BMO}_{\omega,p_j}}\right)\prod_{i=1}^{m}\|f_i\|_{H^{p_i}(\omega)},$$

which completes the proof of theorem 1.2. □

4. The proof of theorem 1.3

Before proving theorem 1.3, we need to prove a weighted norm inequality for $T_{\vec{b}}^{*}$. We first recall some definitions and results. Given $\vec{f}=(f_1,\ldots,f_m)$, we define the multilinear maximal operator $\mathcal{M}$ by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q\ni x}\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q|f_i(y_i)|\mathrm{d}y_i,$$

where the supremum is taken over all cubes Q containing x.

For $\rho>0$, let M_ρ be the maximal function

$$M_\rho(f)(x) = M(|f|^\rho)^{1/\rho}(x) = \left(\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|^\rho\mathrm{d}y\right)^{1/\rho}.$$

Also, let $M^\sharp$ be the sharp maximal function of Fefferman-Stein [6],

$$M^\sharp(f)(x) = \sup_{Q\ni x}\inf_{c}\frac{1}{|Q|}\int_Q|f(y)-c|\mathrm{d}y \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|\mathrm{d}y,$$

and

$$M_\rho^\sharp(f)(x) = \big(M^\sharp(|f|^\rho)(x)\big)^{1/\rho} = \left(\sup_{Q\ni x}\inf_c \frac{1}{|Q|}\int_Q \big||f(y)|^\rho - c\big|\mathrm{d}y\right)^{1/\rho}.$$

The maximal function $\mathcal{M}_{L(\log L)}(\vec{f})(x)$ is defined by

$$\mathcal{M}_{L(\log L)}(\vec{f})(x) = \sup_{Q\ni x}\prod_{i=1}^m \|f_i\|_{L(\log L),Q},$$

and $\mathcal{M}_{L(\log L)}(\vec{f})$ is pointwise controlled by a multiple of $\prod_{j=1}^m M^2(f_j)(x)$.

We will use the following form of classical result of Fefferman and Stein [6]. Let $0 < p,\ \rho < \infty$ and $\omega \in A_\infty$. Then

$$\int_{\mathbb{R}^n}\big(M_\rho(f)(x)\big)^p\omega(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n}\big(M_\rho^\sharp(f)(x)\big)^p\omega(x)\mathrm{d}x,$$

for all functions f for which the left-hand side is finite.

Lemma 4.1. *Let $T^*_{\vec{b}}$ be defined as in (1.4) and K satisfies (1.1) with N as in theorem 1.2. Fix $\omega \in A_p$, $1 < p < \infty$. Given functions $\vec{b} = (b_1, \ldots, b_m)$ which $b_i \in \mathrm{BMO}$, $i = 1, \ldots, m$. Then for any bounded functions $\vec{f} = (f_1, \ldots, f_m)$ with compact support, we have*

$$\big\|T^*_{\vec{b}}(f_1, \ldots, f_m)\big\|_{L^p(\omega)} \lesssim \left(\sum_{i=1}^m \|b_i\|_{\mathrm{BMO}}\right)\|f_l\|_{L^p(\omega)}\prod_{j=1, j\neq l}^m \|f_j\|_{L^\infty}, \quad l = 1, 2, \ldots, m.$$

Proof. By sublinearity, it is enough to consider the operator with only one symbol. For $1 \leqslant i \leqslant m$, fix $b_i \in \mathrm{BMO}$ and consider the operator $T^*_{b_i}(\vec{f})(x)$. Let $0 < \delta < \varepsilon$ with $0 < \delta < 1/m$, Xue [22] proved:

$$M_\delta^\sharp(T^*_{b_i}(\vec{f}))(x) \lesssim \|b_i\|_{\mathrm{BMO}}\left(\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(T^*(\vec{f}))(x)\right), \tag{4.1}$$

and

$$M_\delta^\sharp(T^*(\vec{f}))(x) \lesssim \mathcal{M}(\vec{f})(x). \tag{4.2}$$

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Taking $0 < \delta < \varepsilon < 1/m$, using (4.1) and (4.2) and the Fefferman–Stein inequality, we have

$$\begin{aligned}
\|T^*_{b_i}(\vec{f})\|_{L^p(\omega)} &\leqslant \|M_\delta(T^*_{b_i}(\vec{f}))\|_{L^p(\omega)} \lesssim \|M^\sharp_\delta(T^*_{b_i}(\vec{f}))\|_{L^p(\omega)} \\
&\lesssim \|b_i\|_{\mathrm{BMO}}\big(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)} + \|M_\varepsilon(T^*(\vec{f}))\|_{L^p(\omega)}\big) \\
&\lesssim \|b_i\|_{\mathrm{BMO}}\big(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)} + \|M^\sharp_\varepsilon(T^*(\vec{f}))\|_{L^p(\omega)}\big) \\
&\lesssim \|b_i\|_{\mathrm{BMO}}\big(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)} + \|\mathcal{M}(\vec{f})\|_{L^p(\omega)}\big) \\
&\lesssim \|b_i\|_{\mathrm{BMO}}\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)} \lesssim \|b_i\|_{\mathrm{BMO}}\Bigg\|\prod_{j=1}^m M^2(f_j)\Bigg\|_{L^p(\omega)} \\
&\lesssim \|b_i\|_{\mathrm{BMO}} \prod_{j=1, j\neq l}^m \|M^2(f_j)\|_{L^\infty}\|M^2(f_l)\|_{L^p(\omega)} \\
&\lesssim \|b_i\|_{\mathrm{BMO}}\|f_l\|_{L^p(\omega)} \prod_{j=1, j\neq l}^m \|f_j\|_{L^\infty}.
\end{aligned}$$

To apply the Fefferman–Stein inequality in the above computations, we need to check that $\|M_\delta(T^*_{b_i})(\vec{f})\|_{L^p(\omega)}$ and $\|M_\varepsilon(T^*(\vec{f}))\|_{L^p(\omega)}$ are finite. Note that $\omega \in A_p$, ω is also in A_{p_0} with $pm < p_0 < \infty$. So with $\varepsilon < p/p_0 < 1/m$ and the boundedness of Hardy–Littlewood maximal function, we have

$$\begin{aligned}
\|M_\varepsilon(T^*(\vec{f}))\|_{L^p(\omega)} &\leqslant \|M_{p/p_0}(T^*(\vec{f}))\|_{L^p(\omega)} = \|M(T^*(\vec{f})^{p/p_0})\|^{p_0/p}_{L^{p_0}(\omega)} \\
&\lesssim \|T^*(\vec{f})^{p/p_0}\|^{p_0/p}_{L^{p_0}(\omega)} = \|T^*(\vec{f})\|_{L^p(\omega)}.
\end{aligned}$$

Then it is enough to prove $\|T^*(\vec{f})\|_{L^p(\omega)}$ is finite for each family $\vec{f}$ of bounded functions with compact support for which $\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)}$ is finite. The arguments are as follows.

Without loss of generality, we assume $\operatorname{supp} f_i \subset Q(0, l)$ for $i = 1, \ldots, m$. The weight ω is also in L^r_{loc} for r sufficiently close to 1 such that its dual exponent r' satisfies $1/m < pr' < \infty$. Thus, it follows from Hölder's inequality and the boundedness of T^*

$$\begin{aligned}
\|T^*(\vec{f})\chi_{2Q}\|_{L^p(\omega)} &\leqslant \left(\int_{2Q} |T^*(\vec{f})(x)|^{pr'}\mathrm{d}x\right)^{1/pr'} \left(\int_{2Q} \omega(x)^r dx\right)^{1/pr} \\
&\lesssim \|T^*(\vec{f})\|_{L^{pr'}} \lesssim \prod_{i=1}^m \|f_i\|_{L^{s_i}} < \infty, \qquad (4.3)
\end{aligned}$$

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where $1/pr' = \sum_{i=1}^{m} 1/s_i$. For $x \in (2Q)^c$, $y_i \in Q$, we have $|x - y_i| \approx |x|$, $i = 1, \ldots, m$,

$$|T^*(\vec{f})(x)| \lesssim \int_{(Q(0,l))^m} \frac{\prod_{i=1}^{m} |f_i(y_i)|}{(\sum_{i=1}^{m} |x-y_i|)^{mn}} \mathrm{d}y_i$$
$$\lesssim \prod_{i=1}^{m} \frac{1}{|x|^n} \int_{Q(0,|x|)} |f_i(y_i)| \mathrm{d}y_i \lesssim \mathcal{M}(\vec{f})(x) \lesssim \mathcal{M}_{L(\log L)}(\vec{f})(x). \tag{4.4}$$

Fom the assumption $\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)}$ is finite, we have

$$\|T^*(\vec{f})\chi_{(2Q)^c}\|_{L^p(\omega)} \lesssim \|\mathcal{M}_{L(\log L)}(\vec{f})\chi_{(2Q)^c}\|_{L^p(\omega)} < \infty.$$

Thus, we obtain $\|M_\varepsilon(T^*(\vec{f}))\|_{L^p(\omega)}$ is finite.

Next, we show $\|M_\delta(T^*_{b_i})(\vec{f})\|_{L^p(\omega)}$ is finite. It suffices to prove $\|T^*_{b_i}(\vec{f})\|_{L^p(\omega)}$ is finite. First, we assume b_i is bounded,

$$T^*_{b_i}(\vec{f})(x) = \sup_{\delta>0} \left| \int_{(\mathbb{R}^n)^m} (b_i(x) - b_i(y_j)) K_\delta(x, y_1, \ldots, y_m) \prod_{i=1}^{m} f_i(y_i) \mathrm{d}y_i \right|$$
$$\leqslant |b_i(x)| T^*(\vec{f})(x) + T^*(f_1, \ldots, b_i f_i, \ldots, f_m)(x)$$
$$\lesssim T^*(\vec{f})(x) + T^*(f_1, \ldots, b_i f_i, \ldots, f_m)(x).$$

Thus, following the similar arguments as (4.3), we have

$$\|T^*_{b_i}(\vec{f})\chi_{2Q}\|_{L^p(\omega)} \lesssim \|T^*(\vec{f})\chi_{2Q}\|_{L^p(\omega)} + \|T^*(f_1, \ldots, b_i f_i, \ldots, f_m)\chi_{2Q}\|_{L^p(\omega)}$$
$$\lesssim \prod_{i=1}^{m} \|f_i\|_{L^{s_i}(\omega)} < \infty.$$

On the other hand, for $x \in (2Q)^c$, note that b is bounded, then similar to the arguments of (4.4), we have

$$T^*_{b_i}(\vec{f})(x) \lesssim \mathcal{M}_{L(\log L)}(\vec{f})(x).$$

From the assumption, we obtain

$$\|T^*_{b_i}(\vec{f})\chi_{(2Q)^c}\|_{L^p(\omega)} \lesssim \|M_{L(\log L)}(\vec{f})\chi_{(2Q)^c}\|_{L^p(\omega)} < \infty.$$

Thus, we proved $\|T^*_{b_i}(\vec{f})\|_{L^p(\omega)}$ is finite when b_i is bounded.

For general b, we use the limiting argument as in [**16**]. Let $\{b_{i,j}\}$ be a sequence of functions such that

$$b_{i,j}(x) = \begin{cases} j, & b_i(x) > j \\ b_i(x), & |b_i(x)| \leqslant j, \\ -j, & b_i(x) < -j. \end{cases}$$

Note that the sequence converges pointwise to b_i almost everywhere, and $\|b_{i,j}\|_{\mathrm{BMO}} \lesssim \|b_i\|_{\mathrm{BMO}}$.

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Since the family $\vec{f}$ is bounded with compact support and T^* is bounded, we have that $T^*_{b_{i,j}}(\vec{f})$ convergence to $T^*_{b_i}(\vec{f})$ in L^p is for every $1 < p < \infty$. It follows that for a subsequence $\{b_{i,j'}\} \subset \{b_{i,j}\}$, $T^*_{b_{i,j'}}(\vec{f})$ convergence to $T^*_{b_i}(\vec{f})$ is almost everywhere. Then by Fatou's lemma, we get the required estimate. Thus, we complete the proof of lemma 4.1. □

LEMMA 4.2 [**18**, **21**]. *Let T^* be defined as in (1.3) and K satisfies (1.1) with N as in theorem 1.2. For $0 < p_i \leqslant 1$, let a_i be an $(H^{p_i}(\omega), \infty, N)$-atom supported in Q_k, and c_i be the centre of Q_i, l_i be the side length of Q_i, $i = 1, \ldots, m$. Assume $\tilde{Q}_1 \cap \cdots \cap \tilde{Q}_m \neq \emptyset$, then for any $x \in (\tilde{Q}_1 \cap \cdots \cap \tilde{Q}_m)^c$, we have*

$$|T^*(a_1, \ldots, a_m)(x)| \lesssim \prod_{i=1}^{m} \frac{\big(\omega(Q_i)\big)^{-1/p_i} |Q_i|^{1+(N+1)/nm}}{(|x - c_i| + l_i)^{n+(N+1)/m}}.$$

Now, we are in the position to prove theorem 1.3.

Proof of theorem 1.3. We use the same arguments as in proving theorem 1.2. By sublinearity, it is enough to consider the operator with only one symbol. For $1 \leqslant l \leqslant m$, fix then $b_l \in \mathcal{BMO}_{\omega,p_l}$ and consider the operator $T^*_{b_l}(f_1, \ldots, f_m)(x)$. By lemma 2.4, we will work with finite sums of weighted Hardy atoms and obtain estimates independent of the number of terms in each sum. We write f_i as a finite sum of atoms,

$$f_i = \sum_{k_i=1}^{M} \lambda_{i,k_i} a_{i,k_i}, \quad i = 1, 2, \ldots, m,$$

where $\lambda_{i,k_i} \geqslant 0$ and a_{i,k_i} are $(H^{p_i}(\omega), \infty, N)$-atoms. They are supported in cubes Q_{i,k_i}, $\|a_{i,k_i}\|_{L^\infty} \leqslant (\omega(Q_{i,k_i}))^{-1/p_i}$, $\int_{Q_{i,k_i}} x^\beta a_{i,k_i}(x) \mathrm{d}x = 0$ for all $|\beta| \leqslant N$, and

$$\sum_{k_i} \lambda_{i,k_i}^{p_i} \lesssim \|f_i\|^{p_i}_{H^{p_i}(\omega)}.$$

Denote the centre of Q_{i,k_i} by c_{i,k_i} and the side length of Q_{i,k_i} by l_{i,k_i}. Using multi-sublinearity, we write

$$T^*_{b_l}(f_1, \ldots, f_m)(x) \leqslant \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T^*_{b_l}(a_{1,k_1}, \ldots, a_{m,k_m})(x).$$

Then, we decompose $T^*_{b_l}(f_1, \ldots, f_m)(x)$ into two parts, for $x \in \mathbb{R}^n$

$$T^*_{b_l}(f_1, \ldots, f_m)(x) \leqslant I(x) + II(x),$$

where

$$I(x) := \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T^*_{b_l}(a_{1,k_1}, \ldots, a_{m,k_m})(x) \chi_{\tilde{Q}_{1,k_1} \cap \cdots \cap \tilde{Q}_{m,k_m}},$$

$$II(x) := \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T^*_{b_l}(a_{1,k_1}, \ldots, a_{m,k_m})(x) \chi_{\tilde{Q}^c_{1,k_1} \cup \cdots \cup \tilde{Q}^c_{m,k_m}}.$$

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By lemmas 2.2 and 4.1 and the same arguments as estimating I_1 in the proof of theorem 1.2, we have

$$\|I\|_{L^p(\omega)} \lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega)}.$$

Next, we estimate $\|II\|_{L^p(\omega)}$, we split it again

$$\begin{aligned}\|II\|_{L^p(\omega)} \lesssim & \Bigg\| \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} \big|b_l - b_{l,Q_{l,k_l}}\big| \\ & \times T^*(a_{1,k_1},\ldots,a_{l,k_l},\ldots,a_{m,k_m}) \chi_{\tilde{Q}^c_{1,k_1}\cup\cdots\cup\tilde{Q}^c_{m,k_m}} \Bigg\|_{L^p(\omega)} \\ & + \Bigg\| \sum_{k_1,\ldots,k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} \\ & \times T^*\big(a_{1,k_1},\ldots,(b_l - b_{l,Q_{l,k_l}})a_{l,k_l},\ldots,a_{m,k_m}\big) \chi_{\tilde{Q}^c_{1,k_1}\cup\cdots\cup\tilde{Q}^c_{m,k_m}} \Bigg\|_{L^p(\omega)} \\ & =: \|II_1\|_{L^p(\omega)} + \|II_2\|_{L^p(\omega)}.\end{aligned}$$

Using lemmas 2.3 and 4.2 and the same arguments as estimating I_{21} in the proof of theorem 1.2, we can obtain

$$\|II_1\|_{L^p(\omega)} \lesssim \|b_l\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega)}.$$

To estimate $\|II_2\|_{L^p(\omega)}$, for any $k_i \in \{1, 2, \ldots, M\}$, $i = 1, \ldots, m$, we only need to show

$$\begin{aligned}&\Big\| T^*\big(\lambda_{1,k_1} a_{1,k_1},\ldots,\lambda_{l,k_l}(b_l - b_{l,Q_{l,k_l}})a_{l,k_l},\ldots,\lambda_{m,k_m} a_{m,k_m}\big)\Big\|_{L^p(\omega)} \\ &\quad \lesssim \|b_l\|_{\mathcal{BMO}_{\omega,p_l}} \prod_{i=1}^{m} \|f_i\|_{H^{p_i}(\omega)}.\end{aligned}$$

By the boundedness of T^* from $H^{p_1}(\omega) \times \cdots \times H^{p_m}(\omega)$ to $L^p(\omega)$, we need to show

$$\big\|\lambda_{i,k_i} a_{i,k_i}\big\|_{H^{p_i}(\omega)} \lesssim \|f_i\|_{H^{p_i}(\omega)}, \quad k_i \in \{1,\ldots,M\}, \quad i \in \{1,\ldots,m\}\backslash l,$$

and

$$\big\|\lambda_{l,k_l}(b_l - b_{l,Q_{l,k_l}})a_{l,k_l}\big\|_{H^{p_l}(\omega)} \lesssim \|f_l\|_{H^{p_l}(\omega)} \|b_l\|_{\mathcal{BMO}_{\omega,p_l}}, \quad k_l \in \{1,\ldots,M\}.$$

Using the same argument as (3.1), we can obtain

$$\big\|\lambda_{i,k_i} \mathcal{M}_N(a_{i,k_i})\big\|_{L^{p_i}(\omega)} \lesssim \|f_i\|_{H^{p_i}(\omega)}, \quad k_i \in \{1,\ldots,M\}, \quad i \in \{1,\ldots,m\}\backslash l,$$

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and

$$\left\|\lambda_{l,k_l}\mathcal{M}_N\left((b_l-b_{l,Q_{l,k_l}})a_{l,k_l}\right)\right\|_{L^{p_l}(\omega)}\lesssim\|f_l\|_{H^{p_l}(\omega)}\|b_l\|_{\mathcal{BMO}_{\omega,p_l}},\quad k_l\in\{1,\ldots,M\}.$$

Thus,

$$\|II_2\|_{L^p(\omega)}\lesssim\|b_l\|_{\mathcal{BMO}_{\omega,p_l}}\prod_{i=1}^m\|f_i\|_{H^{p_i}(\omega)}.$$

Combining the estimates in both cases, there is

$$\left\|T^*_{\vec{b}}(\vec{f})\right\|_{L^p(\omega)}\lesssim\left(\sum_{j=1}^m\|b_j\|_{\mathcal{BMO}_{\omega,p_j}}\right)\prod_{i=1}^m\|f_i\|_{H^{p_i}(\omega)},$$

which completes the proof of theorem 1.3. □

Data availability statement
No datasets were generated or analysed during the current study.

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Conflict of interest
None.

Ethical standards
Compliance with ethical standard.

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