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AXIOMS FOR AN n-METRIC STRUCTURE

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1. Introduction. From Euclid to Hilbert, and beyond, the primitive terms of geometry have been taken as "point," "line," etc., while "distance" plays a secondary role. The reversal of this situation is a modern development. Frechet [**4**], in 1906 first considered the properties of distance which should be formalized. The most significant contributions to the geometric properties of metric spaces have been by Menger [**10**] and Blumenthal [**2**; **3**].

Menger devoted one section of his article to a logical generalization of metric spaces; viz., n-metric spaces. Just as a metric structure is a set with a function defined on pairs of points, an n-metric structure is a set with a function defined on $(n + 1)$-tuples of points. The generalization has lain dormant until quite recently [**5**; **6**; **7**; **8**; **11**], and the generalizations have not always been consistent with each other.

2. The problem. For reference, let us state:

Definition 1. A metric structure is a pair, (M, f), where M is a set and f a function on $M \times M$ to the reals (whose image at (p, q) we denote by pq) such that for all $p, q, r \in M$

(a) $pq \geqq 0$;
(b) $pq = qp$;
(c) if $p = q$ then $pq = 0$;
(d) if $p \neq q$ then $pq \neq 0$;
(e) $pq \leqq pr + qr$.

This is, of course, not the most concise form in which to state the definition, but is most serviceable for the problem of generalization. Two closely related structures are the pseudo-metric (omitting property (d)) and the semi-metric (omitting property (e)).

In generalizing a metric to an n-metric structure, one obviously requires a set, M, and a function, f_n, on M^{n+1} to the reals. The difficulty lies in choosing restrictions on f_n which are sufficient to generate a useful structure, but not so stringent as to restrict greatly the class of examples.

Generalizing (a), non-negativeness, and (b), symmetry, is straightforward and intuitively desirable. And, when one contrasts the paucity of results in a semi-metric space with the wealth of results in a metric space, the simplex inequality, a generalization of (e), the triangle inequality, is most desirable.

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The generalization of properties (c) and (d) is a more difficult problem. Do we, for example, generalize property (c) by saying: "If 2 of the points in an $(n + 1)$-tuple are identical, the n-measure is 0," or by: "If all points are identical, the n-measure is 0"? The example of Euclidean geometry suggests that we adopt the former.

Another problem in the process of generalization occurs when we consider a space with several n-metrics, for different values of n. In particular, if a 2-metric is sought for a space already possessing a metric, one might reasonably expect or require a connection between the two. Namely, since colinearity is an easily defined metric property, should we not demand that the 2-measure of a colinear triple be 0, as it is with the usual 2-measure (area) in E_2?

The answer to this last question must be no, for otherwise we would eliminate a class of metric spaces from consideration as 2-metrics, and hold ourselves too rigorously to Euclidean properties. For there are many well-known metric spaces in which there exist convex tripods, i.e. quadruples with one point metrically between each pair of the other three points. The above suggestion would thus assign 2-measure 0 to three of the triples in the quadruple, and the simplex inequality would require 2-measure 0 of the remaining triple. In the metric space derived from a normed lattice, for example, for every non-colinear triple there exists a fourth point forming a convex tripod with these three; this suggested restriction, then, would require us to assign 2-measure 0 to every triple of the space.

Furthermore, for $n \geqq 3$, n-measure cannot be characterized in terms of the $(n - 1)$-metric in E_n [**2**]. Thus an axiomatic restriction of the sort suggested would eliminate E_n as an example of an n-metric space for $n \geqq 3$.

3. The answer. The solution comes, formalistically, by separation, first generalizing axioms for a pseudo-metric, and then generalizing (1d). For the former, we characterize a pseudo-metric in a form which generalizes naturally, thanks to a heuristic lead furnished by Birkhoff [**1**, section 2, p. 466]. One easily verifies:

THEOREM 2. *A pseudo-metric structure is a pair, (M, f) where M is a set and f a function on $M \times M$ to the reals such that for all $p, q, r \in M$,*

(a) $pq \leqq qr + rp$;

(b) *if* $r = p$ *or* $r = q$, *then* $pq = qr + rp$.

The only problem in generalizing this characterization is notational. Thus, we adopt the following notational abbreviations throughout the remainder of this paper: x_0 or y_0 for $p_1p_2 \ldots p_{n+1}$ (the "n-measure" or "pseudo-n-measure" of $p_1, p_2, \ldots p_{n+1}$); x_1 or y_1 for $p_2p_3 \ldots p_{n+2}$; x_j for $p_{j+1}p_{j+2} \ldots p_{n+2}p_1 \ldots p_{j-1}$ $(2 \leqq j \leqq n + 1)$; and y_j for $p_1p_2 \ldots p_{j-1}p_{j+1} \ldots p_{n+2}$ $(2 \leqq j \leqq n + 1)$. With this notation, we facilitate:

Definition 3. A pseudo-n-metric structure is a pair, (M, f_n) where M is a set and f_n a function on M^{n+1} to the reals such that for all $p_1, p_2 \ldots p_{n+2} \in M$,

(a) $x_0 \leqq x_1 + x_2 + \ldots + x_{n+1}$;

(b) if $p_{n+2} = p_1$ or $p_{n+2} = p_2$ or ... or $p_{n+2} = p_{n+1}$, then $x_0 = x_1 + x_2 \ldots + x_{n+1}$.

THEOREM 4. *Let $p_1, p_2 \ldots p_{n+1} \in M$, a pseudo-n-metric space. If at least m of these points are identical* $(2 \leqq m \leqq n + 1)$, *then* $x_0 = 0$.

Proof. (by reverse induction) (a) If $m = n + 1$ (all $n + 1$ points are identical), let $p_{n+2} = p_1 = \ldots = p_{n+1}$. Then, $x_0 = x_1 = \ldots = x_{n+1}$, and, by (3b), $x_0 = x_1 + x_2 + \ldots + x_{n+1}$. Therefore $x_0 = 0$.

(b) Assuming the theorem true for $m = k + 1 (2 \leqq k \leqq n)$, we show it true for $m = k$. Let $p_1, p_2 \ldots p_{n+1}$ be any $n + 1$ points, at least k of which are identical; let the subscripts of these k identical points be $i_1, i_2 \ldots i_k$; let $p_{n+2} = p_{i_1} = \ldots = p_{i_k}$. Then, by the inductive hypothesis, $x_j = 0$ if $j \neq i_h$, and thus $x_0 = x_{i_1} + x_{i_2} + \ldots + x_{i_k}$ and

$$x_{i_h} = x_{i_{h+1}} + \ldots + x_{i_k} + x_0 + \ldots + x_{i_{h-1}}$$

$(1 \leqq h \leqq k)$. These $k + 1$ equations in $k + 1$ indeterminates clearly have the solution $x_0 = x_{i_1} = \ldots = x_{i_k} = 0$, and this solution is seen to be unique by observing that the matrix of coefficients is non-singular.

This theorem, for $m = 2$, is the generalization of (1c) consistent with Euclidean intuition. We can use this theorem to obtain the generalization of (1b):

THEOREM 5. *The pseudo-n-measure of $n + 1$ points is unchanged by any permutation of the points.*

Proof. Let $p_1, p_2 \ldots p_{n+1} \in M$; let $p_{n+2} = p_i$ for some i, $1 \leqq i \leqq n + 1$. Then, by (4), $x_j = 0$ if $j \neq 0$ or $j \neq i$; and thus $x_0 = x_i$, by (3b). For $i = 1$, then, $p_1p_2 \ldots p_{n+1} = p_2p_3 \ldots p_{n+1}p_1$, which implies that the pseudo-n-measure is unchanged by cyclic permutation. And, for $i = 2$,

$$p_1p_2 \ldots p_{n+1} = p_3p_4 \ldots p_{n+1}p_2p_1 = p_2p_1p_3 \ldots p_{n+1},$$

thus allowing transposition of the first two points. Obviously every permutation of the $n + 1$ points is a finite combination of these two.

As a corollary, we obtain the generalization of (1e):

COROLLARY 6. *For all $p_1, p_2 \ldots p_{n+2} \in M$, a pseudo-n-metric space,*

$$y_0 \leqq y_{n+1} + y_n + \ldots + y_1.$$

Proof. By (5), $x_j = y_j$, $0 \leqq j \leqq n + 1$.

The generalization of (1a) now follows easily:

THEOREM 7. *For all $q_1, q_2 \ldots q_{n+1} \in M$, a pseudo-n-metric space,* $q_1q_2 \ldots q_{n+1} \geqq 0$.

Proof. Let $p_1 = q_1$, and $p_j = q_{j-1}$, $2 \leqq j \leqq n + 2$. Then, by (4), $y_0 = y_3 =$

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$y_4 = \ldots = y_{n+1} = 0$; and, by (6), $0 \leqq y_1 + y_2$. But $y_1 = p_2p_3 \ldots p_{n+2} = q_1q_2 \ldots q_{n+1} = p_1p_3 \ldots p_{n+2} = y_2$; thus $q_1q_2 \ldots q_{n+1} \geqq 0$.

Definition 3, then, gives us all the desired properties of a pseudo-n-metric as generalized from Definition 1. There remains, then, the key problem of the generalization of (1d).

As background to this final problem, we observe:

THEOREM 8. *For all $p_1, p_2 \ldots p_{n+2} \in M$, a pseudo-n-metric space, if $y_2 = y_3 = \ldots = y_{n+1} = 0$, then $y_0 = y_1$.*

Proof. By (6), $y_0 \leqq y_{n+1} + y_n + \ldots + y_2 + y_1 = 0 + \ldots + 0 + y_1 = y_1$. Similarly, by (5) and (6), $y_1 \leqq y_0$, and therefore $y_0 = y_1$.

We may paraphrase this theorem by saying that if n of the $n + 2$ $(n + 1)$-tuples of an $(n + 2)$-tuple have pseudo-n-measure 0, then the remaining two have equal pseudo-n-measure. As an obvious corollary, we have

COROLLARY 9. *For all $p_1, p_2 \ldots p_{n+2} \in M$, a pseudo-n-metric space, if $n + 1$ of the $n + 2$ $(n + 1)$-tuples have pseudo-n-measure 0, then the $(n + 2)$nd also has pseudo-n-measure 0.*

We now give

Definition 10. A trivial n-metric space, M, is one in which the pseudo-n-measure of every $(n + 1)$-tuple is 0.

Then from (4) follows

THEOREM 11. *If M is a non-trivial pseudo-n metric, $|M| \geqq n + 1$.*

We also obtain

THEOREM 12. *For all $q_1 \in M$, a non-trivial pseudo-n-metric space, there exist $q_2, q_3 \ldots q_{n+1} \in M$ such that $q_1q_2 \ldots q_{n+1} \neq 0$.*

Proof. M non-trivial implies there exist $p_1, p_2 \ldots p_{n+1} \in M$ such that $y_0 \neq 0$. Then, letting $p_{n+2} = q_1$, by (6) and (7), $0 < y_0 \leqq y_{n+1} + y_n + \ldots + y_1$, which implies at least one of $y_1, y_2 \ldots y_{n+1}$ is non-zero; e.g.,

$$y_i = p_1p_2 \ldots p_{i-1}p_{i+1} \ldots p_{n+2} \neq 0.$$

Then, letting $q_j = p_{j-1}$ $(j = 2, 3 \ldots i)$ and $q_j = p_j$ $(j = i + 1, \ldots n + 1)$, the theorem follows from (5).

Finally, we are ready to complete the generalization by stating:

Definition 13. A pseudo-n-metric space, M, is an n-metric space if and only if: if $|M| \geqq n + 1$ and if p_1, p_2 are distinct elements of M, then there exists a set of $n - 1$ points, $\{p_3, p_4 \ldots p_{n+1}\} \subseteq M$, such that $p_1p_2 \ldots p_{n+1} \neq 0$.

The conditional requirement on cardinality allows one to consider trivial examples of n-metric spaces. The other condition excludes the trivial as an example of n-metric space with cardinality greater than n. It follows from (12) that the "almost trivial" is also excluded.

4. Other answers and their interrelations. The fact that the condition of the above definition is a valid generalization of (1d) is obvious. Also obvious is the fact that this is not the most immediate generalization which comes to mind or has been used. Rather, the following two definitions suggest themselves, the first on logical grounds, the second on Euclidean grounds.

Definition 14. A pseudo-n-metric space, M, is an L-n-metric space if and only if: for all $p_1p_2 \ldots p_{n+1} \in M$, if $x_0 = 0$, then there exist i, j, $(1 \leqq i < j \leqq n + 1)$ such that $p_i = p_j$.

Definition 15. A pseudo-n-metric space, M, is an E-n-metric space if and only if: for all $p_1, p_2 \ldots p_{n+2} \in M$, if $p_1 \neq p_2$ and if $y_0 = y_{n+1} = y_n \ldots = y_3 = 0$, then $y_2 = y_1 = 0$.

We can simplify the problem of comparing the three conditions by observing:

THEOREM 16. *Every L-n-metric space is an E-n-metric space.*

Proof. Let $p_1, p_2 \ldots p_{n+2} \in M$ such that $p_1 \neq p_2$ and $y_0 = y_{n+1} = y_n \ldots = y_3 = 0$. $y_0 = 0 \Rightarrow$ there exist i, j $(1 \leqq i < j \leqq n + 1)$ such that $p_i = p_j$.

(a) $i = 1$: $p_1 \neq p_2 \Rightarrow j \geqq 3 \Rightarrow y_2 = 0$ by (4), $\Rightarrow y_1 = 0$ by (9).

(b) $i = 2$: $\Rightarrow y_1 = 0$ by (4), $\Rightarrow y_2 = 0$ by (9).

(c) $i \geqq 3$: $\Rightarrow y_1 = y_2 = 0$ by (4).

And the relationship between (13) and (14) is obvious from the definitions; viz.,

THEOREM 17. *Every L-n-metric space is an n-metric space.*

That the converses of these two theorems are false is easily seen in the example of E_2 with the usual 2-metric (area of a triangle), which is a 2-metric and an E-2-metric, but not an L-2-metric.

In order to compare (13) and (15), we must also consider cardinality and triviality. From definitions (10) and (15), we have:

THEOREM 18. *Every trivial pseudo-n-metric space is an E-n-metric space.* (We should perhaps note that the analogous theorem for L-n-metric holds if and only if $|M| \leqq n$.)

If we then consider any set M such that $|M| \geqq n + 1$ and assign the trivial pseudo-n-metric, we observe, by (13) and (18),

COROLLARY 19. *An E-n-metric space is not necessarily an n-metric space.*

A partial answer to the question of relationship is given in

THEOREM 20. *If M is an n-metric space such that $|M| \leqq n + 2$, then M is an E-n-metric space.*

Proof. (a) $|M| \leqq n$: the result is immediate from (11) and (18).

(b) $|M| = n + 1$: the (unique) $(n + 1)$-tuple of $n + 1$ distinct points has non-zero n-measure, and every other $(n + 1)$-tuple has n-measure zero. M is therefore an L-n-metric space and, by (16), an E-n-metric space.

(c) $|M| = n + 2$: if we suppose M is not an E-n-metric space, then there exist $p_1, p_2 \ldots p_{n+2} \in M$ such that $p_1 \neq p_2$ and $y_0 = y_{n+1} = y_n = \ldots = y_3 = 0$, but by (8) and (15), $y_1 = y_2 \neq 0$. From (4), then, $p_1, p_2 \ldots p_{n+2}$ are pairwise distinct, and must be the $n + 2$ points of M. But, then there does not exist a set $\{q_3, q_4 \ldots q_{n+1}\} \subseteq M$ such that $p_1p_2q_3 \ldots q_{n+1} \neq 0$; i.e., M is not an n-metric space.

(From the first two parts of the proof, we see that the analogous theorem for L-n-metric holds for $|M| \leqq n + 1$. A counterexample for $|M| \geqq n + 2$ is easily constructed with 4 points for $n = 2$.) The failure of Theorem 20 for $|M| = n + 3$ is given in the proof of the following, simpler relationship:

THEOREM 21. *An n-metric space is either an E-n-metric space or non-trivial, but not necessarily both.*

Proof. If $|M| \leqq n + 2$, then M is an E-n-metric, by (20). If $|M| \geqq n + 1$, then M is non-trivial, by (13). To show that both are not necessarily true, we need merely choose M such that $|M| \leqq n$. We can also exemplify a non-trivial n-metric space which is not an E-n-metric; viz., $M = \{p_1, p_2 \ldots p_{n+3}\}$, $y_0 = y_{n+1} = y_n = \ldots = y_3 = 0$, and the n-measure of every other $(n + 1)$-tuple is either 1 or 0, as the points are pairwise distinct or not, respectively.

The counterexamples of (19) and (21) show that there is no necessary logical connection between an E-n-metric space and an n-metric space. It may be argued that the counterexample of (19) is trivial (in the technical and non-technical sense). This is so, and necessarily, as the two structures are logically related if we require the pseudo-n-metric structure to be non-trivial.

THEOREM 22. *Every non-trivial E-n-metric space is an n-metric space.*

Proof. Since M is non-trivial, $|M| \geqq n + 1$, by (11). Let p_1, p_2 be distinct elements of M. By (12), there exist $p_3, p_4 \ldots p_{n+2}$ such that $y_1 \neq 0$. Then since M is an E-n-metric, one of $y_0, y_{n+1}, y_n \ldots y_3$ is non-zero, and thus M is an n-metric space.

5. Conclusion. As a final theorem, we formalize the following, obvious from the definitions alluded to:

THEOREM 23. *A set M is a metric space* (1) *if and only if M is a* 1*-metric space* (13) *if and only if M is an L*-1*-metric space* (14) *if and only if M is an E*-1-*metric space* (15).

In other words, each of (13), (14), or (15) may validly be termed a generalization of a metric structure. Why, then, do we choose one rather than another?

The answer is in Theorems (16), (17) and (22) which show that for all examples, the L-n-metric is a special case of both the n-metric and the E-n-metric and for non-trivial examples, our usual interest, the E-n-metric is a special case of the n-metric. It seems reasonable, then, to choose definition (13), the "most general" generalization.

This does not preclude the option of studying properties of E-n-metric and L-n-metric spaces. But it seems unreasonable to restrict oneself to one of these, thus eliminating consideration of the more general, yet accurate, generalization of a metric structure.

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