A NOTE ON PRIME ESSENTIAL RINGS

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It is proved that the upper radical determined by the class of prime essential rings is an N-radical.

Several examples are known of hereditary supernilpotent radicals which are not special. Recently Gardner and Stewart [2] showed how one can construct many such examples (including most of those already known) using prime essential rings, which were introduced for other reasons by Rowen in [5]. In particular Gardner and Stewart showed that the upper radical determined by the class of these rings is not special. The aim of this note is to prove that this upper radical is also an N-radical. Note that the first example of an N-radical which is not special was found by Jaegermann and Sands [3].

All rings in this paper are associative, not necessarily with 1. To denote that I is an ideal (prime ideal, left ideal) of a ring R we write $I \triangleleft R$ ($I \triangleleft' R$, $I < R$). An ideal I of a ring R is called *essential* (in R) if for every $0 \neq J \triangleleft R$, $I \cap J \neq 0$.

A ring R is called *prime essential* [5] if it is semiprime and all its prime ideals are essential.

Recall that a radical is an N-radical [6] if it is left strong (that is, semisimple rings contain no non-zero radical left ideals), left hereditary (that is, left ideals of radical rings are radical) and contains the prime radical β. Throughout the paper instead of "radical class" we write briefly "radical".

All undefined terms and facts used can be found in [7].

We start with some lemmas.

LEMMA 1. [2]
  (i) *Every ideal of a prime essential ring is a prime essential ring.*
  (ii) *If I is an essential ideal of R and I is a prime esssential ring then R is a prime essential ring.*

LEMMA 2. [6, Corollary p. 338]. *If $L < R$ and R is a semiprime ring then $\beta(L) = \{l \in L \mid Ll = 0\}$.*

LEMMA 3. *Let $I \lhd L < R$ and let L/I be a semiprime ring. If M is the largest ideal of R such that $ML \subseteq I$ then the ring R/M is semiprime and $LI \subseteq M$.*

PROOF: If $M \subsetneqq J \lhd R$ then by the choice of M, $JL \nsubseteq I$. Obviously JL is an ideal of L, so since L/I is semiprime, $(JL)^2 \nsubseteq I$. But $(JL)^2 \subseteq J^2$, so $J^2 \neq 0$. This shows that the ring R/M is semiprime.

Notice that LIR is an ideal of R and $LIRL \subseteq I$. Consequently $LIR \subseteq M$. Since R/M is semiprime, $LI \subseteq M$. ∎

The following lemma can be easily deduced from results in [6]. We include its direct proof for completeness.

LEMMA 4. *Let $L < R$.*

    (i) *If $I \lhd L$ then $\overline{I} = \{r \in R \mid RrL \subseteq I\} \lhd R$ and if $I \lhd' L$ then $\overline{I} \lhd' R$.*

    (ii) *If $I \lhd R$ then $I^* = \{l \in L \mid LlR \subseteq I\} \lhd L$ and if $I \lhd' R$ then $I^* \lhd' L$.*

PROOF: (i) The first part of the statement is clear. Suppose now that J, K are ideals of R and $JK \subseteq \overline{I}$. Then $RJKL \subseteq I$. Obviously $JLRK \subseteq JK$, so $RJLRKL \subseteq I$. However RJL and RKL are ideals of L. Hence, since I is a prime ideal of L, $RJL \subseteq I$ or $RKL \subseteq I$. Consequently $J \subseteq \overline{I}$ or $K \subseteq \overline{I}$ and the result follows.

(ii) It is clear that I^* is an ideal of L. Suppose now that $I \lhd' R$ and J, K are ideals of L. If $JK \subseteq I^*$ then $LJKR \subseteq I$. Obviously $LJRLKR \subseteq LJKR \subseteq I$ and LJR, LKR are ideals of R. Since I is a prime ideal of R, $LJR \subseteq I$ or $LKR \subseteq I$. Consequently $J \subseteq I^*$ or $K \subseteq I^*$ and the result follows. ∎

Recall that a class T of rings is called *regular* if every non-zero ideal of each ring in T can be homomorphically mapped onto a non-zero ring in T. The upper radical determined by T is equal the class of all rings which cannot be homomorphically mapped onto non-zero rings in T.

LEMMA 5. [1] *Given a regular class T of rings, the upper radical U_T determined by T is left strong if and only if every non-zero left ideal of each ring in T can be homomorphically mapped onto a non-zero ring in T.*

Now we are ready to prove

THEOREM 6. *The upper radical $U_{\mathcal{E}}$ determined by the class $\mathcal{E}$ of prime essential rings is an N-radical.*

PROOF: Lemma 1 implies that the class $\mathcal{E}$ is regular. Thus to prove that $U_{\mathcal{E}}$ is left strong we can apply Lemma 5. We shall show that if $0 \neq L < R$ and R is prime essential then $L/\beta(L)$ is prime essential. For, let M be an ideal of R maximal with respect to $M \cap \langle L \rangle = 0$, where $\langle L \rangle = L + LR$. Observe that $\langle L \rangle + M/M$ is an essential ideal of

R/M. Therefore by Lemma 1, R/M is prime essential and without loss of generality, we may assume that $M = 0$. Now take ideals P and I of L such that $\beta(L) \subseteq P \lhd' L$ and $\beta(L) \subsetneq I \lhd L$. By Lemma 2, $LI \neq 0$. Hence, since R is semiprime, $LIR \neq 0$. Observe that $LIR \subseteq \overline{I}$, so $\overline{I} \neq 0$. By Lemma 4, $\overline{P} \lhd' R$. Thus $\overline{I} \cap \overline{P} \neq 0$. Since $\langle L \rangle$ is an essential ideal of R and R is semiprime $\left((\overline{I} \cap \overline{P}) \cap \langle L \rangle \right)^2 \neq 0$. This in particular implies that $LR(\overline{I} \cap \overline{P})L \neq 0$, which together with Lemma 2 gives $R(\overline{I} \cap \overline{P})L \nsubseteq \beta(L)$. Hence since $R(\overline{I} \cap \overline{P})L \subseteq I \cap P$, we get $I \cap P \nsubseteq \beta(L)$. This proves that $L/\beta(L)$ is prime essential.

Now we shall show that $U_{\mathcal{E}}$ is left hereditary. If not then there exist a ring $R \in U_{\mathcal{E}}$ and $I \trianglelefteq L < R$ such that $0 \neq L/I \in \mathcal{E}$. Let M be the largest ideal of R satisfying $ML \subseteq I$. Notice that since I is a semiprime ideal of L, $M \cap L \subseteq I$. Consequently by passing to the factor ring R/M we can assume without loss of generality that $M = 0$. By Lemma 3, R is a semiprime ring and $LI = 0$. This obviously gives $\beta(L) = I$. Let $P \lhd' R$ and $0 \neq J \lhd R$. By Lemma 4, $P^* \lhd' L$ and $J^* \lhd L$. Moreover $JL \subseteq J^*$ and by the foregoing, $JL \nsubseteq I$. Hence, since L/I is prime essential, $P^* \cap J^* \nsubseteq I$. Consequently by Lemma 2, $L(P^* \cap L^*) \neq 0$. Further, since R is semiprime, $L(P^* \cap J^*)R \neq 0$. However $L(P^* \cap J^*)R \subseteq P \cap J$, so $P \cap J \neq 0$. This proves that R is prime essential, a contradiction.   □

REMARK. From the proof it also follows that the class of prime essential rings is normal [4].

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