London Mathematical Society ISSN 1461–1570

SIMPLE LIE ALGEBRAS OF LOW DIMENSION OVER GF(2)

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Abstract

We classify the simple Lie algebras of dimension at most 9 over GF(2). There is one of dimension 3 and one of dimension 6, there are two of dimension 7 and two of dimension 8, and there is one of dimension 9. The two simple Lie algebras of dimension 8 are restricted Lie algebras. If we extend the ground field to GF(4), then the six-dimensional algebra is no longer simple, and if we extend the ground field to GF(8) then the nine-dimensional algebra is no longer simple. But the other algebras are all central simple.

1. *Introduction*

Helmut Strade gave a talk to the Oxford Algebra Seminar in June 1998, about the classification of simple modular Lie algebras. Towards the end of his talk, he mentioned the possibility of computing low-dimensional simple Lie algebras over GF(2), and the original motivation for this research came from his remarks. I completed the classification of simple Lie algebras of dimension at most 8 over GF(2) later that summer, but did not publish the results at that time, as I gained the impression that all the algebras that I had found were known. A year or so ago, Bettina Eick expressed an interest in my results, and indicated her belief that some of the algebras I had found were new. She also encouraged me to have a look at dimension 9. The results I have obtained are as follows.

Theorem 1. *Let L be a simple Lie algebra over* GF(2)*, and let* $\dim(L) \leqslant 9$*. Then* $\dim(L)$ *is equal to* 3, 6, 7, 8 *or* 9*. There is one simple Lie algebra over* GF(2) *of dimension* 3 *and one of dimension* 6*, there are two of dimension* 7 *and two of dimension* 8*, and there is one simple Lie algebra of dimension* 9*. The two Lie algebras of dimension* 8 *are restricted Lie algebras. The simple Lie algebra of dimension* 6 *is no longer simple if we extend the ground field to* GF(4)*, and the simple Lie algebra of dimension* 9 *is no longer simple if we extend the ground field to* GF(8)*. The other simple Lie algebras are central simple: they remain simple under extension of the ground field.*

Descriptions of these algebras are given in Section 3, at the end of this paper.

It seems that some of these algebras are 'new'. Hogeweij [2] and Hiss [1] have independently investigated the simple Lie algebras of characteristic 2 of classical type. The classical Lie algebras are A_n ($n \geqslant 1$), B_n ($n \geqslant 2$), C_n ($n \geqslant 2$), D_n ($n \geqslant 4$), E_6, E_7, E_8, F_4 and G_2. These are not always simple in characteristic 2, but most of them have simple sections. It seems that the only simple Lie algebra of classical type in characteristic 2 which has dimension at most 9 is A_2, which is of dimension 8 and is isomorphic to sl(3, 2). Eick and Feldvoss (in as yet unpublished work) have investigated the simple Lie algebras of characteristic 2 of classical type and Cartan type. They give strong experimental evidence

Received 20 April 2006, revised 30 May 2006; *published* 3 July 2006.
2000 Mathematics Subject Classification 17B20, 17B50

that the only Lie algebras of Cartan type that give rise to simple Lie algebras of characteristic 2 of dimension at most 9 are the Witt algebras $W(1; (2))$, $W(1; (3))$ and $W(2; (1, 1))$. (See Strade and Farnsteiner [4, Definitions 4.2.1 and 4.2.4] for a description of these algebras.) The algebra $W(1; (2))$ has dimension 4 in characteristic 2, and its derived algebra is a simple Lie algebra of dimension 3. The algebra $W(1; (3))$ has dimension 8 in characteristic 2, and its derived algebra is a simple Lie algebra of dimension 7. Finally, the algebra $W(2; (1, 1))$ is a simple Lie algebra of dimension 8 in characteristic 2. It is isomorphic to sl(3, 2). I am grateful to an anonymous referee who pointed out that the simple Lie algebras that I found of dimension 6 and dimension 9 are $W(1; (2))' \otimes_{\mathrm{GF}(2)} \mathrm{GF}(4)$ and $W(1; (2))' \otimes_{\mathrm{GF}(2)} \mathrm{GF}(8)$ (see Table1).

Strade has recently classified the unsolvable Lie algebras of dimension at most 6 over arbitrary fields, so there is some overlap between his work (not yet published) and this paper. Kaplansky [3] described four infinite families of simple Lie algebras of characteristic 2, but the only 'overlap' between his algebras and the algebras described here is the simple Lie algebra of dimension 3. The smallest algebra in his first family has dimension 14; the smallest in his second family has dimension 3, and the next smallest has dimension 15; the smallest Lie algebra in his third family has dimension 10; and the smallest algebra in his fourth family has dimension 28.

So three of the seven simple Lie algebras of dimension at most 9 over GF(2) are of classical or Cartan type. The six-dimensional simple Lie algebra should occur in Strade's list of non-solvable Lie algebras, and both the six-dimensional and the nine-dimensional simple Lie algebras are derived from $W(1; (2))'$. But two of the simple Lie algebras I have found are 'new', and since they are central simple they also define simple Lie algebras over algebraically closed fields of characteristic 2.

2. *The method*

Consider an n-dimensional Lie algebra over GF(2) with basis $x_1, x_2, \ldots, x_n$. For each i, j with $1 \leqslant i, j \leqslant n$, the Lie product $[x_i, x_j]$ can be expressed as a linear combination

$$[x_i, x_j] = \sum_{k=1}^{n} \lambda_{ijk} x_k,$$

with $\lambda_{ijk} \in \mathrm{GF}(2)$. The coefficients λ_{ijk} are the structure constants of L with respect to the basis $x_1, x_2, \ldots, x_n$.

Table 1: The simple Lie algebras

Dimension	Isomorphism type	Restricted	Central simple
3	$W(1; (2))'$	No	Yes
6	$W(1; (2))' \otimes_{\mathrm{GF}(2)} \mathrm{GF}(4)$	No	No
7	$W(1; (3))'$	No	Yes
7		No	Yes
8	$(3, 2) \cong A_2 \cong W(2; (1, 1))$	Yes	Yes
8		Yes	Yes
9	$W(1; (2))' \otimes_{\mathrm{GF}(2)} \mathrm{GF}(8)$	No	No

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

Since $[x_i, x_i] = 0$ we have $\lambda_{iik} = 0$ for all i, k, and since $[x_i, x_j] + [x_j, x_i] = 0$ we have $\lambda_{ijk} + \lambda_{jik} = 0$ for all i, j, k. In addition, the Jacobi identity

$$[[x_i, x_j], x_k] + [[x_j, x_k], x_i] + [[x_k, x_i], x_j] = 0$$

($i < j < k$) gives additional relations on the structure constants. For $n = 8$ there are 2^{228} sets $\{\lambda_{ijk}\}$ satisfying the conditions $\lambda_{iik} = 0$, $\lambda_{ijk} + \lambda_{jik} = 0$, with 56 additional relations arising from the Jacobi identity. So clearly it is quite impractical to enumerate Lie algebras of dimension 8 by enumerating sets of structure constants.

However, using the following relatively elementary approach, we are able to classify the simple Lie algebras of dimension at most 9 over GF(2). We illustrate our approach by considering the case of eight-dimensional simple Lie algebras.

Let L be a simple Lie algebra of dimension 8 over GF(2). Since L is simple, L is isomorphic to $\mathrm{ad}(L)$. The elements of $\mathrm{ad}(L)$ can be represented as 8×8 matrices with respect to some basis $x_1, x_2, \ldots, x_8$ for L, and L is determined by the matrices for $\mathrm{ad}(x_i)$ ($i = 1, 2, \ldots, 8$). The entries in these matrices are the structure constants for L, so enumerating matrices is equivalent to enumerating sets of structure constants. (Since $L = [L, L]$ the matrices all have zero trace, but this is not much of a reduction.) The major advantage of working with matrices is that if we have the matrices for two or more elements of $\mathrm{ad}(L)$ then we can consider the subalgebra of $\mathrm{sl}(8, 2)$ which they generate. In addition, we can make use of the rational canonical form for matrices.

Suppose that $a \in L$, and let $A = \mathrm{ad}(a)$. Then A can be represented by an 8×8 matrix in $\mathrm{sl}(8, 2)$. By Engel's theorem we may assume that A is not nilpotent. Note that $aA = 0$, so that A is singular. We can write the characteristic polynomial of A in the form $x^k f(x)$, where $0 < k < 8$ and where $f(0) \neq 0$. Since A has zero trace, the coefficient of x^7 in $x^k f(x)$ is zero. This gives 63 possibilities for the characteristic polynomial of A, but most of these possibilities can be ruled out very easily. We can write

$$L = L_0 \oplus L_1,$$

where $L_0 = \ker(A^k)$ and $L_1 = \ker(f(A))$. It is easy to see that $[L_0, L_0] \leqslant L_0$ and that $[L_0, L_1] \leqslant L_1$. We will show that if $f(x)$ has no repeated factors, then $[L_1, L_1] \leqslant L_1$, so that L_1 is an ideal of L, and L is not simple.

So suppose that $f(x)$ has no repeated factors, and let K be the splitting field for $f(x)$ over GF(2). We can think of L as a Lie algebra over K, and then L_1 has a basis consisting of eigenvectors for A, where different basis vectors have different (non-zero) eigenvalues. Let b, c be two such basis vectors, with distinct eigenvalues λ, μ. Then, using the fact that A is a derivation of L, we see that $[b, c]A = (\lambda + \mu)[b, c]$. Note that $\lambda + \mu \neq 0$. If $\lambda + \mu$ is not a root of $f(x)$, then $[b, c] = 0$, and if $\lambda + \mu$ is a root of $f(x)$, then $[b, c] \in L_1$. In either case, $[b, c] \in L_1$, so that $[L_1, L_1] \leqslant L_1$, as claimed.

Eliminating characteristic polynomials $x^k f(x)$ where $f(x)$ has no repeated roots, we are left with 21 possibilities:

$$x^2(x+1)^2(x^2+x+1)^2,$$
$$x(x+1)^3(x^4+x^3+1),$$
$$x^4(x+1)^4,$$
$$x^2(x^3+x+1)^2,$$
$$x(x+1)^2(x^5+x^3+1),$$

$$
\begin{aligned}
&x(x+1)^2(x^5+x^3+x^2+x+1),\\
&x^2(x+1)^3(x^3+x^2+1),\\
&x(x^2+x+1)^2(x^3+x+1),\\
&x(x+1)^5(x^2+x+1),\\
&x^3(x+1)^2(x^3+x+1),\\
&x^6(x+1)^2,\\
&x^2(x^3+x^2+1)^2,\\
&x(x+1)^3(x^4+x^3+x^2+x+1),\\
&x^4(x^2+x+1)^2,\\
&x^2(x+1)^6,\\
&x(x+1)^2(x^2+x+1)(x^3+x^2+1),\\
&x(x+1)^2(x^5+x^2+1),\\
&x^3(x+1)^3(x^2+x+1),\\
&x(x+1)^4(x^3+x+1),\\
&x(x+1)(x^2+x+1)^3,\\
&x^2(x+1)^2(x^4+x+1).
\end{aligned}
$$

However, as we will see, most of these possibilities can easily be eliminated.

Consider the second polynomial $x(x+1)^3(x^4+x^3+1)$ in the list above, for example. Suppose that L is an eight-dimensional Lie algebra over GF(2), and suppose that $a \in L$ and that $A = \mathrm{ad}(a)$ has characteristic polynomial $x(x+1)^3(x^4+x^3+1)$. Write

$$L = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = \ker(A)$, $L_1 = \ker((A+I)^3)$ and $L_2 = \ker(A^4+A^3+I)$. Then it is easy to see that $[L_0, L_i] \leqslant L_i$ for $i = 0, 1, 2$. If we let A act as a derivation on the tensor product $L_1 \otimes L_2$ by setting $(b \otimes c)A = bA \otimes c + b \otimes cA$, then it is straightforward to check that

$$(L_1 \otimes L_2)(A^4+A^3+A^2+A+I)^3 = \{0\}.$$

This implies that

$$[L_1, L_2] \leqslant \ker((A^4+A^3+A^2+A+I)^3) = \{0\}.$$

Similarly, if we let A act as a derivation on the exterior square $L_2 \wedge L_2$ by setting $(b \wedge c)A = bA \wedge c + b \wedge cA$, then we see that

$$(L_2 \wedge L_2)(A^2+A+I)(A^4+A+I) = \{0\},$$

so that

$$[L_2, L_2] \leqslant \ker((A^2+A+I)(A^4+A+I)) = \{0\}.$$

So L_2 is an ideal of L, and L is not simple.

Now let $A = \mathrm{ad}(a)$ have characteristic polynomial $x(x+1)^2(x^5+x^3+1)$. Then

$$L = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = \ker(A)$, $L_1 = \ker((A+I)^2)$ and $L_2 = \ker(A^5+A^3+I)$.

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

The same argument as above shows that

$$\begin{aligned}
&[L_0, L_2] \leqslant L_2,\\
&[L_1, L_2] \leqslant \ker((A^5 + A^4 + A^3 + A^2 + I)^2) = \{0\},\\
&[L_2, L_2] \leqslant \ker((A^5 + A^2 + I)(A^5 + A^3 + A^2 + A + I)) = \{0\}.
\end{aligned}$$

So L_2 is an ideal of L.

Similarly, if A has characteristic polynomial $x(x+1)^2(x^5+x^3+x^2+x+1)$, then $\ker(A^5 + A^3 + A^2 + A + I)$ is a non-trivial proper ideal of L, and if the characteristic polynomial of A is $x^2(x+1)^3(x^3+x^2+1)$, then $\ker(A^3 + A^2 + I)$ is a non-trivial proper ideal of L.

The case when A has characteristic polynomial $x(x^2+x+1)^2(x^3+x+1)$ is very slightly different. Once again we write

$$L = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = \ker(A)$, $L_1 = \ker((A^2 + A + I)^2)$ and $L_2 = \ker(A^3 + A + I)$. Then, using the same arguments as before, we have

$$\begin{aligned}
&[L_0, L_2] \leqslant L_2,\\
&[L_1, L_2] = \{0\},\\
&[L_2, L_2] \leqslant \ker(A^3 + A + I) = L_2.
\end{aligned}$$

So, once again, L_2 is an ideal of L.

We need a different argument in the case when the characteristic polynomial of A is $x(x+1)^5(x^2+x+1)$. We write

$$L = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = \ker(A)$, $L_1 = \ker((A+I)^5)$ and $L_2 = \ker(A^2 + A + I)$. Then

$$\begin{aligned}
&[L_0, L_2] \leqslant L_2,\\
&[L_1, L_2] \leqslant \ker((A^2 + A + I)^5) = L_2,\\
&[L_2, L_2] \leqslant \ker(A + I) \leqslant L_1,
\end{aligned}$$

and so $L_2 + [L_2, L_2]$ is a non-trivial proper ideal of L.

Using similar arguments to these, we can show the following.

- If A has characteristic polynomial $x^3(x+1)^2(x^3+x+1)$, then $\ker(A^3 + A + I)$ is a non-trivial proper ideal of L.
- If we let A have characteristic polynomial $x^6(x+1)^2$, and if we let $L_1 = \ker((A+I)^2)$, then $L_1 + [L_1, L_1]$ is a non-trivial proper ideal of L.
- If A has characteristic polynomial $x(x+1)^2(x^2+x+1)(x^3+x^2+1)$, then it follows that $\ker(A^3 + A^2 + I)$ is a non-trivial proper ideal of L.
- If we let A have characteristic polynomial $x^3(x+1)^2(x^5+x^2+1)$, then $\ker(A^5 + A^2 + I)$ is a non-trivial proper ideal of L.
- If we let A have characteristic polynomial $x^3(x+1)^3(x^2+x+1)$, and if we let $L_2 = \ker(A^2 + A + I)$, then $L_2 + [L_2, L_2]$ is a non-trivial proper ideal of L.
- If A has characteristic polynomial $x(x+1)^4(x^3+x+1)$, then $\ker(A^3 + A + I)$ is a non-trivial proper ideal of L.
- If we let A have characteristic polynomial $x^2(x+1)^2(x^4+x+1)$, and if we let $L_2 = \ker(A^4 + A + I)$, then $L_2 + [L_2, L_2]$ is a non-trivial proper ideal of L.

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This leaves the following possibilities for the characteristic polynomial of A in an eight-dimensional simple Lie algebra over GF(2):

$$\begin{aligned}&x^2(x+1)^2(x^2+x+1)^2,\\&x^4(x+1)^4,\\&x^2(x^3+x+1)^2,\\&x^2(x^3+x^2+1)^2,\\&x^4(x^2+x+1)^2,\\&x^2(x+1)^6,\\&x(x+1)(x^2+x+1)^3.\end{aligned}$$

The first three of these polynomials arise as characteristic polynomials of $\mathrm{ad}(a)$ for $a \in \mathrm{sl}(3, 2)$, but using slightly more complicated arguments than those used above we are able to show that the last four polynomials above cannot arise as characteristic polynomials in a simple Lie algebra.

2.1. $x(x+1)(x^2+x+1)^3$

Let $a \in L$, and let $A = \mathrm{ad}(a)$ have characteristic polynomial $x(x+1)(x^2+x+1)^3$. Write

$$L = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = \ker(A)$, $L_1 = \ker(A+I)$ and $L_2 = \ker((A^2+A+I)^3)$. Then L_0 and L_1 both have dimension 1, and L_2 has dimension 6. Using the same arguments as above, we see that

$$\begin{aligned}&[L_0, L_1] \leqslant L_1,\\&[L_0, L_2] \leqslant L_2,\\&[L_1, L_2] \leqslant L_2,\\&[L_2, L_2] \leqslant L_0 + L_1.\end{aligned}$$

It follows that $L_2 + [L_2, L_2]$ is a non-trivial ideal of L. Furthermore, this ideal is proper unless $[L_2, L_2] = L_0 + L_1$.

First note that $L_0 + L_1$ has dimension 2, so that if $b \in L_2$ then $\dim([b, L_2]) \leqslant 2$. If $\dim([b, L_2]) \leqslant 1$ for all $b \in L_2$ then $\dim([L_2, L_2]) \leqslant 1$, and $L_2 + [L_2, L_2]$ is a non-trivial, proper ideal of L.

So suppose that $\dim([b, L_2]) = 2$ for some $b \in L_2$. Then $[b, L_2] = L_0 + L_1$, and so (since $a \in L_0$) $[b, c] = a$ for some $c \in L_2$. But then, writing $C(a)$, $C(b)$, $C(c)$ for the centralizers in L_2 of a, b, c, we have

$$C(a) \geqslant C(b) \cap C(c),$$

which has codimension at most 4 in L_2. But this is impossible since L_2 has dimension 6 and $C(a) = \{0\}$.

2.2. $x^2(x+1)^6$

Let $a \in L$, and let $A = \mathrm{ad}(a)$ have characteristic polynomial $x^2(x+1)^6$. Write

$$L = L_0 \oplus L_1,$$

where $L_0 = \ker(A^2)$ and $L_1 = \ker((A+I)^6)$.

We have $[L_1, L_1] \leqslant L_0$, which has dimension 2. As in the subsection above, we see that $L_1 + [L_1, L_1]$ is a non-trivial proper ideal of L unless $[L_1, L_1] = L_0$. And, as above, we see that if $[L_1, L_1] = L_0$ then there are elements $b, c \in L_1$ such that $a = [b, c]$. But, as above, we see that this implies that the centralizer of a in L_1 has codimension at most 4, contradicting the fact that L_1 has dimension 6.

2.3. $x^2(x^3 + x^2 + 1)^2$

Let $a \in L$, and let $A = \mathrm{ad}(a)$ have characteristic polynomial $x^2(x^3 + x^2 + 1)^2$. Write

$$L = L_0 \oplus L_1,$$

where $L_0 = \ker(A^2)$ and $L_1 = \ker((A^3 + A^2 + I)^2)$. We have $\dim(L_0) = 2$, $\dim(L_1) = 6$, and $[L_1, L_1] \leqslant L_0$. The proof that $L_1 + [L_1, L_1]$ is a proper, non-trivial ideal of L is the same as in the two subsections above.

2.4. $x^4(x^2 + x + 1)^2$

Let $a \in L$, and let $A = \mathrm{ad}(a)$ have characteristic polynomial $x^4(x^2 + x + 1)^2$. Write

$$L = L_0 \oplus L_1,$$

where $L_0 = \ker(A^4)$ and $L_1 = \ker((A^2 + A + I)^2)$. We have $\dim(L_0) = \dim(L_1) = 4$, and

$$[L_1, L_1] \leqslant \ker(A(A + I)^2) = \ker(A) \leqslant L_0.$$

So $L_1 + [L_1, L_1]$ is an ideal of L, and L can only be simple if $[L_1, L_1] = L_0$. Note that this implies that $L_0 = \ker(A)$, and so the rational canonical form of A is

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&1&0 \end{pmatrix}.$$

If the rational canonical form of A is the first matrix above, then L_1 has a basis b, $[b, a]$, c, $[c, a]$ for some $b, c \in L_1$ where $[b, a, a] = b + [b, a]$ and $[c, a, a] = c + [c, a]$. Using the fact that $[L_1, L_1] \leqslant \ker(A)$, we have

$$0 = [b, [b, a], a] = [b, [b, a, a]] = [b, [b, a]].$$

Similarly, we have $[c, [c, a]] = 0$. So $[L_1, L_1]$ is spanned by $[b, c]$, $[b, [c, a]]$, $[[b, a], c]$, $[[b, a], [c, a]]$. But

$$0 = [b, c, a] = [[b, a], c] + [b, [c, a]]$$

and

$$0 = [b, [c, a], a] = [[b, a], [c, a]] + [b, c] + [b, [c, a]].$$

So $[L_1, L_1]$ has dimension at most 2, and $L_1 + [L_1, L_1]$ is a proper, non-trivial ideal of L.

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

Next, consider the case when the rational canonical form of A is the second matrix above. Then L_1 has a basis b, $[b, a]$, $[b, a, a]$, $[b, a, a, a]$ for some $b \in L_1$, with

$$[b, a, a, a, a] = b + [b, a, a].$$

Since $[L_1, L_1] \leqslant \ker(A)$, we have

$$0 = [b, [b, a, a], a, a] = [b, [b, a, a]].$$

We also have

$$0 = [b, [b, a, a], a] = [[b, a], [b, a, a]] + [b, [b, a, a, a]]$$

and

$$\begin{aligned} 0 &= [[b, [b, a, a, a], a] \\ &= [[b, a], [b, a, a, a]] + [b, [b, a, a, a, a]] \\ &= [[b, a], [b, a, a, a]] + [b, [b, a, a]]. \end{aligned}$$

It follows that

$$[[b, a], L_1] = [b, L_1],$$

and the same argument gives

$$\begin{aligned} [[b, a, a], L_1] &= [[b, a], L_1], \\ [[b, a, a, a], L_1] &= [[b, a, a], L_1]. \end{aligned}$$

So $[L_1, L_1]$ is spanned by $[b, [b, a]]$ and $[b, [b, a, a, a]]$, and $L_1 + [L_1, L_1]$ is a proper, non-trivial ideal of L.

2.5. $x^2(x^3 + x + 1)^2$

Let $a \in L$, and let $A = \mathrm{ad}(a)$ have characteristic polynomial $x^2(x^3 + x + 1)^2$. As we mentioned above, this situation arises when $L = \mathrm{sl}(3, 2)$. We will show that it also arises in the case of one other simple Lie algebra of dimension 8 over GF(2). Write

$$L = L_0 \oplus L_1,$$

where $L_0 = \ker(A^2)$ and $L_1 = \ker((A^3 + A + I)^2)$. We have $\dim(L_0) = 2$, $\dim(L_1) = 6$, and

$$\begin{aligned} &[L_0, L_0] \leqslant L_0, \\ &[L_0, L_1] \leqslant L_1, \\ &[L_1, L_1] \leqslant \ker(A(A^3 + A + I)^2) \leqslant L_0 + L_1. \end{aligned}$$

Clearly, $L_1 + [L_1, L_1]$ is an ideal, and it is a proper ideal unless $\ker(A) = L_0$. So we may assume that the rational canonical form of A is

$$\begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&1&0&1&0&0&0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&1&0 \end{pmatrix}.$$

Consider the case when the rational canonical form of A is the first of these two matrices. Then L_1 has a basis b, $[b, a]$, $[b, a, a]$, $[b, a, a, a]$, $[b, a, a, a, a]$, $[b, a, a, a, a, a]$ for some $b \in L_1$ with

$$[b, a, a, a, a, a, a] = b + [b, a, a].$$

We assume that $L_1 + [L_1, L_1] = L$, so that L is generated by a and b. We assume that L_0 has basis a, a_2 with $[a, a_2] = 0$. Since $[L_0, L_1] \leqslant L_1$, we have

$$[b, a_2] = x_1 b + x_2[b, a] + x_3[b, a, a] + x_4[b, a, a, a] + x_5[b, a, a, a, a] + x_6[b, a, a, a, a, a]$$

for some $x_1, x_2, \ldots, x_6 \in \mathrm{GF}(2)$. Replacing a_2 by $a + a_2$ if necessary, we may assume that $x_2 = 0$. Since $[a, a_2] = 0$,

$$\begin{aligned}
[b, a, a_2] &= [b, a_2, a],\\
[b, a, a, a_2] &= [b, a_2, a, a],\\
&\vdots\\
[b, a, a, a, a, a, a_2] &= [b, a_2, a, a, a, a, a],
\end{aligned}$$

and so $\mathrm{ad}(a_2)$ is determined by the coefficients x_1, x_3, x_4, x_5, x_6. We can assume that the characteristic polynomial of $\mathrm{ad}(a_2)$ is one of

$$x^8, \quad x^2(x+1)^2(x^2+x+1)^2, \quad x^4(x+1)^4, \quad x^2(x^3+x+1)^2,$$

and to ensure that this is so, we require that $x_6 = x_1 + x_4$. With this restriction we show that there are (at most) 2^{22} possibilities for $\mathrm{ad}(b)$. Let B be the matrix representing $\mathrm{ad}(b)$ with respect to the basis a, a_2, b, $[b, a]$, $[b, a, a]$, $[b, a, a, a]$, $[b, a, a, a, a]$, $[b, a, a, a, a, a]$ for L. The first row of B is

$$(0, 0, 0, 1, 0, 0, 0, 0),$$

and we take the second row to be

$$(0, 0, x_1, 0, x_3, x_4, x_5, x_1 + x_4).$$

The third row of B is zero, of course. The fourth and sixth rows of B represent $[b, a, b]$ and $[b, a, a, a, b]$, and we take these to be arbitrary. Since

$$\begin{aligned}
[b, a, a, b] &= [b, a, b, a],\\
[b, a, a, a, a, b] &= [b, a, a, b, a, a],
\end{aligned}$$

the fifth and seventh rows of B are determined by the fourth row. And since

$$\begin{aligned}
&[b, a, a, a, a, a, b, a]\\
&\quad = [b, a, a, a, b, a, a, a] + [b, a, a, a, [b, a, a], a]\\
&\quad = [b, a, a, a, b, a, a, a] + [b, a, a, a, a, [b, a, a]]\\
&\quad = [b, a, a, a, b, a, a, a] + [b, a, a, a, a, b, a, a] + [b, a, a, a, a, a, a, b]\\
&\quad = [b, a, a, a, b, a, a, a] + [b, a, a, a, a, b, a, a] + [b, a, a, b],
\end{aligned}$$

in the last row of B the entries in columns $3, 4, \ldots, 8$ are determined by the entries in the fourth and sixth rows. In summary, B is determined by the coefficients x_1, x_3, x_4 and x_5, by the entries in its fourth and sixth rows, and by the first two entries in its eighth row. This gives 2^{22} possibilities for B. For each of these 2^{22} possibilities we consider the Lie subalgebra of sl(8, 2) generated by A and B. There are 768 possibilities for B where the characteristic polynomials of B and of $A+B$ are one of

$$x^8, \quad x^2(x+1)^2(x^2+x+1)^2, \quad x^4(x+1)^4, \quad x^2(x^3+x+1)^2,$$

and where $\langle A, B\rangle$ has dimension 8. None of these algebras of dimension 8 is simple.

Next, we consider the case when the rational canonical form of A is

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&1&0 \end{pmatrix}.$$

Then L_1 has a basis b, $[b,a]$, $[b,a,a]$, c, $[c,a]$, $[c,a,a]$ for some $b, c \in L_1$, with

$$[b,a,a,a] = b + [b,a],$$
$$[c,a,a,a] = c + [c,a].$$

As above, we assume that L_0 has basis a, a_2 with $[a, a_2] = 0$. Let U be the subspace of L_1 spanned by b, $[b,a]$ and $[b,a,a]$, and let V be the subspace spanned by c, $[c,a]$ and $[c,a,a]$. By considering the derivation action of a on $U \wedge U$, we see that $[U,U] \leqslant L_1$, and similarly $[V,V] \leqslant L_1$. We consider two possibilities: the first possibility is that $[b,a,b] \notin U$, and the second possibility is that $[b,a,b]$ lies in the linear span of b, $[b,a]$ and $[b,a,a]$ for all $b \in L_1$.

So suppose first that $[b,a,b] \notin U$. Then we can assume that $[b,a,b] = c$ and hence that $[b,a,a,b] = [c,a]$. We can also assume that $L_1 + [L_1, L_1] = L$, so this implies that L is generated by a and b. As above, we let B be the matrix representing $\mathrm{ad}(b)$ with respect to the basis a, a_2, b, $[b,a]$, $[b,a,a]$, c, $[c,a]$, $[c,a,a]$ for L. Then, as above, we can take the first five rows of B to be

$$\begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&x_1&0&x_3&x_4&x_5&x_6\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix}.$$

We take the sixth and seventh rows of B to be arbitrary, and we take the first two entries in the eighth row to be arbitrary. The remaining entries in the eighth row are determined by the equation

$$\begin{aligned} [c,a,a,b,a] &= [c,a,a,a,b] + [c,a,a,[b,a]] \\ &= [c,b] + [c,a,b] + [c,[b,a],a,a] + [c,[b,a,a,a]] \\ &= [c,a,b] + [c,[b,a],a,a] + [c,[b,a]] \\ &= [c,b,a] + [c,b,a,a,a] + [c,a,b,a,a]. \end{aligned}$$

This gives 2^{23} possibilities for B. For each of these 2^{23} possibilities we consider the Lie subalgebra of sl(8, 2) generated by A and B. There are 534 possibilities for B where the characteristic polynomials of B and of $A+B$ are one of

$$x^8, \quad x^2(x+1)^2(x^2+x+1)^2, \quad x^4(x+1)^4, \quad x^2(x^3+x+1)^2,$$

and where $\langle A, B\rangle$ has dimension 8. Out of these 534 possibilities, $\langle A, B\rangle$ is simple in 16 cases, with two different isomorphism types.

Finally, consider the case when $[b, a, b]$ lies in the linear span of b, $[b, a]$ and $[b, a, a]$ for all $b \in L_1$. As above, we have a basis $a, a_2, b, [b, a], [b, a, a], c, [c, a], [c, a, a]$ for L. We have $[a_2, b] \in L_1$, and if $[a_2, b]$ is not in the linear span of b, $[b, a]$, $[b, a, a]$ then we may assume that $[a_2, b] = c$. So, replacing a_2 by $a + a_2$ if necessary, we may assume that $[a_2, b] = x_1 b + x_3[b, a, a]$, or that $[a_2, b] = c$. This gives five possibilities for $[a_2, b]$. We let $[b, a, b]$ be an arbitrary element of the linear span of b, $[b, a]$, $[b, a, a]$, and we let $[c, b]$ and $[c, a, b]$ be arbitrary elements of L. If we let B be the matrix representing $\mathrm{ad}(b)$, then this fixes the first four rows of B, and the sixth and seventh rows. The fifth row is determined by the fourth row, since $[b, a, a, b] = [b, a, b, a]$. The first two entries in the eighth row of B are still undetermined, but the remaining entries in the eighth row are determined by the equation

$$\begin{aligned}
[c, a, a, b, a] &= [c, a, a, a, b] + [c, a, a, [b, a]] \\
&= [c, b] + [c, a, b] + [c, [b, a, a, a]] + [c, [b, a], a, a] \\
&= [c, a, b] + [c, [b, a]] + [c, a, b, a, a] + [c, b, a, a, a] \\
&= [c, b, a] + [c, a, b, a, a] + [c, b, a, a, a].
\end{aligned}$$

This gives 5×2^{21} possibilities for B. Note that a and b generate a four-dimensional subalgebra of L, so that we require that $\langle A, B\rangle$ be a four-dimensional subalgebra of $\mathrm{sl}(8, 2)$. There are 7096 possibilities for B which give a four-dimensional subalgebra $M = \langle A, B\rangle$ with the property that each element of M has characteristic polynomial in the set

$$\{x^8,\ x^2(x+1)^2(x^2+x+1)^2,\ x^4(x+1)^4,\ x^2(x^3+x+1)^2\}.$$

For each of these 7096 possibilities for B, we consider the possibilities for C, where C is the matrix representing $\mathrm{ad}(c)$. The first row of C must be

$$(0, 0, 0, 0, 0, 0, 1, 0),$$

and we let the second row represent an arbitrary element of L_1. The third, fourth and fifth rows of C are the same as the sixth rows of B, $[B, A]$ and $[B, A, A]$, respectively, and the sixth row of C is zero. The seventh row of C represents $[c, a, c]$, which we take to be an arbitrary element in the linear span of c, $[c, a]$ and $[c, a, a]$. The eighth row of C represents $[c, a, a, c]$, and is determined by the seventh row, since $[c, a, a, c] = [c, a, c, a]$. This gives 2^9 possibilities for C for each of the 7096 possibilities for B. There are 196 pairs B, C with $\langle A, B, C\rangle$ simple of dimension 8, and the algebras obtained fall into the same two isomorphism types as were found above.

2.6. $x^2(x+1)^2(x^2+x+1)^2$

Next, let $A = \mathrm{ad}(a)$ have characteristic polynomial $x^2(x+1)^2(x^2+x+1)^2$. This case also arises as a possibility in $\mathrm{sl}(3, 2)$. We let

$$L = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = \ker(A^2)$, $L_1 = \ker((A+I)^2)$ and $L_2 = \ker((A^2+A+I)^2)$.

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

We have

$$\begin{aligned}
&[L_0, L_0] \leqslant L_0,\\
&[L_0, L_1] \leqslant L_1,\\
&[L_0, L_2] \leqslant L_2,\\
&[L_1, L_1] \leqslant \ker(A) \leqslant L_0,\\
&[L_1, L_2] \leqslant L_2,\\
&[L_2, L_2] \leqslant \ker(A(A+I)^2) \leqslant L_0 + L_1.
\end{aligned}$$

So $L_2 + [L_2, L_2]$ is an ideal, and if L is simple then we must have $L_0 = \ker(A)$ and $[L_2, L_2] = L_0 + L_1$. So there are four possibilities for the rational canonical form of A. Let us consider the case when

$$A = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix}.$$

Then L_2 has basis b, $[b, a]$, c, $[c, a]$ for some $b, c \in L_2$ with $[b, a, a] = b + [b, a]$ and $[c, a, a] = c + [c, a]$. Also, since $[L_2, L_2] = L_0 + L_1$, we see that L is generated by a, b, c. Since

$$[b, a, b, a] = [b, a, a, b] = [b, a, b],$$

we see that $[b, a, b] \in L_1$. Similarly, $[d, a, d] \in L_1$ for all $d \in L_2$.

First we show that we cannot have $[d, a, d] = 0$ for all $d \in L_2$. For suppose that $[d, a, d] = 0$ for all $d \in L_2$. Then $[b, a, b] = [c, a, c] = 0$. We also have $[b + c, a, b + c] = 0$, and this gives $[b, c, a] = 0$, which implies that $[b, c] \in L_0$. Similarly, we see that $[d, e] \in L_0$ for all $d, e \in L_2$, so that $L_2 + [L_2, L_2]$ is a proper ideal of L. So we can assume that $[b, a, b]$ is a non-trivial element of L_1. We use a similar argument to show that the elements $[d, a, d]$ with $d \in L_2$ cannot all be scalar multiples of $[b, a, b]$. Suppose that they are. Then $[b, a, b]$, $[c, a, c]$ and $[b + c, a, b + c]$ are all scalar multiples of $[b, a, b]$, and this implies that $[b, c, a]$ is a scalar multiple of $[b, a, b]$, and hence that $[b, c] \in L_0 + \mathrm{Sp}\langle [b, a, b]\rangle$. Similarly, we see that $[d, e] \in L_0 + \mathrm{Sp}\langle [b, a, b]\rangle$ for all $d, e \in L_2$, and hence that $L_2 + [L_2, L_2]$ is a proper ideal of L.

So we can assume that $[b, a, b]$ and $[c, a, c]$ form a basis for L_1. We choose a_2 so that a and a_2 form a basis for L_0. Then L has basis $a, a_2, [b, a, b], [c, a, c], b, [b, a], c$ and $[c, a]$. We let A, B and C be the matrices of $\mathrm{ad}(a)$, $\mathrm{ad}(b)$, $\mathrm{ad}(c)$ with respect to this basis. The matrix A is as above. The first row of B is

$$(0, 0, 0, 0, 0, 1, 0, 0),$$

and we let the next three rows represent arbitrary elements of L_2. The fifth row is zero, and the sixth row is

$$(0, 0, 1, 0, 0, 0, 0, 0),$$

representing $[b, a, b]$. We let the last two rows represent arbitrary elements of $L_0 + L_1$. This gives 2^{20} possibilities for B. Once B has been determined, then so have $[B, A]$ and

$[B, A, B]$, and so all the rows of C are determined except for the second and fourth row. Since these two rows represent elements in L_2, this gives 2^8 possibilities for C. Note that $[b, a, b, b] \in L_2$. If $[b, a, b, b]$ lies in the linear span of b and $[b, a]$, then a and b generate a four-dimensional subalgebra of L with basis a, $[b, a, b]$, b, $[b, a]$, and if $[b, a, b, b]$ does not lie in the linear span of b and $[b, a]$ then a and b must generate L. For each of the 2^{18} possibilities for B corresponding to the situation when $[b, a, b, b] \in \mathrm{Sp}\langle b, [b, a]\rangle$, we investigate the subalgebra of $\mathrm{sl}(8, 2)$ generated by A and B. There are 8416 cases when $\langle A, B\rangle$ has dimension 4. However, we can also assume that the characteristic polynomials of the elements in $\langle A, B\rangle$ lie in the set

$$\{x^8,\ x^2(x+1)^2(x^2+x+1)^2,\ x^4(x+1)^4\},$$

and this reduces the number of cases to be considered, to 2848. (We dealt with the possibility of characteristic polynomial $x^2(x^3+x+1)^2$ in Section 2.5.) For each of these 2848 cases we consider the subalgebra $\langle A, B, C\rangle$ for each of the 2^8 possibilities for C mentioned above, though we restrict ourselves to those C such that C, $A+C$, $[C, A]$, $A+[C, A]$, $C+[C, A]$ and $A+C+[C, A]$ have characteristic polynomials in

$$\{x^8,\ x^2(x+1)^2(x^2+x+1)^2,\ x^4(x+1)^4\}.$$

Altogether there are six pairs B, C giving a simple Lie algebra $\langle A, B, C\rangle$ of dimension 8. These simple algebras lie in one of the two isomorphism classes already found. We also investigate $\langle A, B\rangle$ for each of the $2^{20}-2^{18}$ possibilities for B corresponding to the situation when $[b, a, b, b] \notin \mathrm{Sp}\langle b, [b, a]\rangle$. We restrict ourselves to those B such that all the matrices in the linear span of A, B, $[B, A]$ and $[B, A, B]$ have characteristic polynomials in

$$\{x^8,\ x^2(x+1)^2(x^2+x+1)^2,\ x^4(x+1)^4\}.$$

None of these cases gives a Lie algebra of dimension 8.

The other three possibilities for the rational canonical form of A are:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&1&0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&1&0 \end{pmatrix}.$$

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

The first of these three possibilities cannot arise, since we would then have

$$[L_2, L_2] \leqslant \ker(A(A+I)) \leqslant L_0 + L_1,$$

so that for a simple algebra we would need $L_0 + L_1 = \ker(A(A+I))$. In the remaining two cases L_2 has a basis $b, [b,a], [b,a,a], [b,a,a,a]$ for some $b \in L_2$. For L to be simple we need $[L_2, L_2] = L_0 + L_1$, and this then implies that L is generated by a and b. We can find all the simple algebras of this form by considering the different possibilities for $\mathrm{ad}(b)$.

2.7. $x^4(x+1)^4$

The final case to consider is when A has characteristic polynomial $x^4(x+1)^4$. We let $L = L_0 \oplus L_1$, where $L_0 = \ker(A^4)$ and $L_1 = \ker((A+I)^4)$. Then L_0 and L_1 both have dimension 4, and

$$\begin{aligned}
[L_0, L_0] &\leqslant L_0,\\
[L_0, L_1] &\leqslant L_1,\\
[L_1, L_1] &\leqslant \ker(A^3) \leqslant L_0.
\end{aligned}$$

So $L_1 + [L_1, L_1]$ is an ideal of L, and L is simple unless $[L_1, L_1] = L_0$. Note that this implies that $L_0 = \ker(A^3)$. There are a number of possibilities for the rational canonical form for A, and we can deal with these possibilities using similar arguments to those above. Note that we can assume that $\mathrm{ad}(b)$ has characteristic polynomial x^8 or $x^4(x+1)^4$ for all $b \in L$.

3. *The algebras*

The simple Lie algebras of dimension at most 9 over GF(2) are as follows.

There is one simple Lie algebra of dimension 3. It has basis a_1, a_2, a_3 with $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$, $[a_3, a_1] = a_2$. This is isomorphic to $W(1;(2))'$.

There is one simple Lie algebra of dimension 6. It has a vector space basis $x_1, x_2, \ldots, x_6$ such that the matrices for $\mathrm{ad}(x_i)$ $(1 \leqslant i \leqslant 6)$ with respect to this basis are:

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&1&0&1&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&1&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&1&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0 \end{pmatrix}.$$

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

The structure constants for the algebra can be read off from the matrices. It is generated by x_1 and x_2. This algebra is isomorphic to $W(1;\,(2))' \otimes_{\mathrm{GF}(2)} \mathrm{GF}(4)$, and is not simple if we extend the ground field to GF(4).

There are two simple Lie algebras of dimension 7 over GF(2). If we let $x_1, x_2, \ldots, x_7$ be a basis for the algebras then the matrices for $\mathrm{ad}(x_i)$ $(1 \leqslant i \leqslant 7)$ with respect to this basis are:

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}.$$

and

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 1&1&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&1&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&1&0\\ 1&1&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}.$$

In both cases the algebra is generated by x_1 and x_2. The first algebra is isomorphic to $W(1;\,(3))'$. One way to see that these two algebras are not isomorphic is to note that in the second algebra the element $\mathrm{ad}(x_2)$ has minimal polynomial x^4, whereas $W(1;\,(3))'$ has no elements a such that $\mathrm{ad}(a)$ has minimal polynomial x^4.

There are two simple Lie algebras of dimension 8 over GF(2). The first is

$$\mathrm{sl}(3,2) \cong A_2 \cong W(2;(1,1)).$$

If we let $x_1, x_2, \ldots, x_8$ be a basis for the second algebra, then the matrices for $\mathrm{ad}(x_i)$ $(1 \leqslant i \leqslant 8)$ with respect to this basis are:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&1&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&1&1&1&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&1&1&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&1\\ 1&1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&1&0&1&0&0&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&1&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&1\\ 1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&1&0&0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&1&1\\ 1&1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&1&1&0&1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&1&0\\ 0&0&1&0&1&1&0&1\\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

This algebra is generated by x_1 and x_3. Both this algebra and $\mathrm{sl}(3,2)$ are restricted Lie algebras. One way to see that this algebra is not isomorphic to $\mathrm{sl}(3,2)$ is to note that this algebra has 54 elements a such that $\mathrm{ad}(a)$ has minimal polynomial x^4, whereas $\mathrm{sl}(3,2)$ has 42.

There is one simple Lie algebra of dimension 9 over GF(2). If we let $x_1, x_2, \ldots, x_9$ be a basis for this algebra, then the matrices for $\mathrm{ad}(x_i)$ $(1 \leqslant i \leqslant 9)$ with respect to this basis are:

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&1&0
\end{pmatrix},
\quad
\begin{pmatrix}
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1\\
0&0&0&1&0&1&0&1&0\\
0&0&0&0&1&0&1&0&1\\
0&0&0&1&0&1&0&0&0
\end{pmatrix},
$$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1\\
0&0&0&1&0&1&0&1&0
\end{pmatrix},
\quad
\begin{pmatrix}
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0
\end{pmatrix},
$$

$$
\begin{pmatrix}
0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{pmatrix},
\quad
\begin{pmatrix}
0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0
\end{pmatrix},
$$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&1&0&1&0\\
0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{pmatrix},
\quad
\begin{pmatrix}
0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&1&0&1\\
0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0
\end{pmatrix},
$$

$$
\begin{pmatrix}
0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&1&0&0&0\\
0&0&0&1&0&1&0&1&0\\
1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{pmatrix}.
$$

https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press

This algebra is generated by x_1 and x_4. The algebra is isomorphic to

$$W(1;(2))' \otimes_{\mathrm{GF}(2)} \mathrm{GF}(8),$$

and is not simple if we extend the ground field to GF(8).

Appendix A. Magma *program*

A Magma program to construct these algebras can be found at

http://www.lms.ac.uk/jcm/9/lms2006-013/appendix-a.

The program is also available at http://users.ox.ac.uk/˜vlee/simplelie.

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https://doi.org/10.1112/S1461157000001248 Published online by Cambridge University Press