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THE ABELIAN CASE OF SOLITAR'S CONJECTURE ON INFINITE NIELSEN TRANSFORMATIONS

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ABSTRACT. The paper proves that the group of infinite bounded Nielsen transformations is generated by elementary simultaneous Nielsen transformations modulo the subgroup of those transformations which are equivalent to the identical transformation while acting in a free abelian group. This can be formulated somewhat differently: the group of bounded automorphisms of a free abelian group of countably infinite rank is generated by the elementary simultaneous automorphisms. This proves D. Solitar's conjecture for the abelian case.

1. **Introduction**. Nielsen's method consists of a reduction process which changes any finite set of generators of a subgroup of a free group into a free set of generators (Nielsen-reduced set) ([5], [6]). Since Nielsen's method, applied to a finite set of words, deals with a length function which is minimized, it does not tend to reduce an infinite set of words in a finite number of steps. The Nielsen transformations of rank n ([4], p. 130) build a group which is anti-isomorphic to the group Aut F_n of automorphisms of a free group of rank n and is generated by the elementary Nielsen transformations ([4], Theorem 3.2). In 1970, D. Solitar formulated the problem of generalization of the Nielsen's theory for the case of the free group F_∞ of infinite rank. The notion of the Nielsen transformation can be naturally generalized for F_∞ so that the group of so-called infinite Nielsen transformations is anti-isomorphic with Aut F_∞. It is shown in [3] that any countable set of words in F_∞ can be changed into a Nielsen-reduced set by an infinite Nielsen transformation. A generalization of the elementary Nielsen transformation for F_∞ is given in [1], where the notion of the elementary simultaneous transformation is introduced.

It has been conjectured by D. Solitar that these elementary simultaneous Nielsen transformations generate the subgroup of bounded Nielsen transformations. The conjecture still stands (see [1], p. 100). If by $\bar{F}_\infty$ we denote a free abelian group of countably infinite rank then the automorphism group Aut $\bar{F}_\infty$ is isomorphic with the group of infinite matrices invertible over Z which contain finite numbers of non-zero elements in their rows. The homomorphism Aut $F_\infty \to$ Aut $\bar{F}_\infty$ is the epimorphism [2]

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and hence for the group N of infinite Nielsen transformations there exists an antihomomorphism onto the group of matrices mentioned above. The kernel H of this antihomomorphism consists of those infinite Nielsen transformations which act as the identical transformation if applied to a set of words in abelian group.

2. **Nielsen transformations and matrices**. Let F_∞ be a free group with the base $X = \langle x_i, i \in I\rangle$ where I is the set of natural numbers. Let $W = \langle w_i, i \in I\rangle$ be another base in F_∞. By the same letter W we shall denote the infinite Nielsen transformation which changes the base X into W.

2.1. Definition. *The infinite Nielsen transformation W changes any set of words $V = \langle v_i, i \in I\rangle$ into the set $WV = \langle w_i(V), i \in I\rangle$ where the word $w_i(V)$ is obtained from w_i by substitution of x_j by v_j, $j \in I$, e.g. if $w = x_1x_7$, then $w(V) = v_1v_7$.*

2.2. Definition. *The product of two infinite Nielsen transformations W and U we shall write from the right to the left as UW. The product is defined to carry X into the base $UW = \langle u_i(W), i \in I\rangle$ so that for any set V we get $(UW)V = U(W(V))$.*

According to those definitions the infinite Nielsen transformation X is the identity and the two transformations W and U are inverse to each other if $UW = WU = X$ or, which is the same, $w_i(U) = u_i(W) = x_i$. Referring to the infinite Nielsen transformations we shall omit the word "infinite".

2.3. Definition. *An infinite matrix $W = (\alpha_{ij})$ is called* the exponent matrix *of the set of words $W = \langle w_i, i \in I\rangle$ if $w_i = \Pi x_j^{\alpha}ij$* mod F'_∞.

Obviously the exponent matrix of a Neilsen transformation, i.e. of a corresponding base, is invertible over Z. We shall show that the exponent matrix of the product of two Nielsen transformations W and U is the product of matrices UW. Indeed if $u_k \equiv \Pi x_i^{\beta_{ki}}$, then $u_k(W) \equiv \Pi_i w_i^{\beta_{ki}} \equiv \Pi_i(\Pi_j x_j^{\alpha_{ij}})^{\beta_{ki}} \equiv \Pi_j x_j^{\sigma_{kj}}$, where $\sigma_{kj} = \sum_i \beta_{ki}\alpha_{ij}$, modulo F'_∞, which leads to the result.

In a free abelian group $\bar{F}_\infty$ every set of words $V = \langle v_i, i \in I\rangle$ is unambiguously defined by its exponent matrix V. The Nielsen transformation W of the set V leads to the multiplication of the matrix V by the exponent matrix W from the left side so that the product of two Neilsen transformations W and U acting on a set V multiplies its exponent matrix by the matrix UW, since $U(W(V)) = (UW)V$. We shall also speak of an $n \times n$-exponent matrix for a set of n words in a free group of rank n.

2.4. Definition. *Matrix $W = (\alpha_{ij})$ is called n-*bounded *if W is invertible over Z, $W^{-1} = U = (\beta_{ij})$, and $\sum_j |\alpha_{ij}| \le n$, $\sum_j |\beta_{ij}| \le n$.*

The product of n_1- and n_2-bounded matrices is obviously an n_1n_2-bounded matrix.

2.5. Definition. Nielsen transformation. *W corresponding to the base $W = \langle w_i, i \in I\rangle$ is called n-*bounded *if $L_x(w_i) \le n$, $L_w(x_i) \le n$, where $L_x(w_i)$ is x-length of the word w_i, $L_w(x_i)$ is w-length of the word x_i, $i \in I$.*

The exponent matrix of n-bounded Nielsen transformation is obviously n-bounded.

3. **Elementary simultaneous Nielsen transformations**.

3.1. DEFINITION. *The transformation of an infinite set of words* $\langle w_i, i \in I\rangle$ *of the types 1–3, given below, will be called* elementary simultaneous Nielsen transformations *([1], Definition 4.1.):*

1. *Permutation of words* $w_i, i \in I$;
2. *Change of* w_i *to* w_i^{-1} *for a subset of words* $\langle w_i, i \in M \subseteq I\rangle$;
3. *For* $I = P \cup Q$ *a change of every word* $w_q, q \in Q$ *to* $w_q w_p^{\pm 1}$ *for a* $p, p \in P$, *with no change of the words* $w_p, p \in P$.

3.2. DEFINITION. *The transformations of the type 1, 2, and 3 for* $|P| = 1$ *will be called according to* [1], *Definition 4.2, just* elementary Nielsen transformations.

Let us note here that by means of elementary Nielsen transformations even such a base as $\langle w_i, i \in I)$ where $w_{2k-1} = x_{2k-1}$, $w_{2k} = x_{2k}x_{2k-1}$, $i \in I$ cannot be changed into a Nielsen-reduced set X in a finite number of steps.

3.3. DEFINITION. *The exponent matrix of an elementary simultaneous Nielsen transformation is called* the elementary simultaneous matrix.

3.4 DEFINITION. *An elementary simultaneous Nielsen transformation of a set V induces an* elementary simultaneous transformation of the rows *of exponent matrix V of the next three types:*

1. *Permutation of rows;*
2. *Multiplication of elements of some rows for* -1;
3. *For* $I = P \cup Q$ *a change of every row with number* $q \in Q$ *to a sum or difference of this row and another row with number* $p \in P$, *with no change of the rows with numbers* $p \in P$.

Since a product of n elementary simultaneous matrices is 2^n-bounded matrix, we can say that not every Nielsen transformation is a product of a finite number of elementary simultaneous transformations, but we shall prove here that every n-bounded matrix is a product of a finite number of elementary simultaneous matrices which gives the positive solution for D. Solitar's conjecture in the abelian case.

4. **Lemmas on matrices**. It will be convenient for us to rearrange the set I of natural numbers according to a new ordering φ. Since every exponent matrix A is a function from $I \times I$ to Z, the A can be rewritten as a function φA from $\varphi I \times \varphi I$ to Z.

We shall see that the new ordering does not affect the multiplication of matrices. We say that the ordering φ corresponds to the splitting $I = \cup I_k$ if in every I_k the new ordering coincides with the natural one, and all elements from I_k preceed those from I_{k+1}, that is if $i, j \in I_k$, $i < j$, then $\varphi(i) < \varphi(j)$; if $i \in I_k$, $j \in I_{k+1}$, then $\varphi(i) < \varphi(j)$. So, let $\bar{\alpha}$ be any fixed countable ordinal, then we denote by Φ the set of non-limit ordinals $\Phi = \langle \alpha, 1 \leq \alpha \leq \bar{\alpha}\rangle$. The elements from the set Φ we denote by small Greek letters. Let $\varphi : I \to \Phi$ be a bijection defining the new ordering in I, then any matrix $A = (a_{ij})$ can be rewritten as $\varphi A = (\alpha_{\xi\eta})$ for $\alpha_{\xi\eta} = a_{\varphi^{-1}(\xi)\varphi^{-1}(\eta)}$. We shall check that for the exponent matrices $\varphi(AB) = \varphi A \varphi B$. Indeed, if we denote $AB = C$, where B and C have elements b_{ij} and c_{ij} correspondently and φB, φC have elements $\beta_{\xi\eta}$, $\sigma_{\xi\eta}$, then

$$\sigma_{\xi\eta} = c_{\varphi^{-1}(\xi)\varphi^{-1}(\eta)} = \sum_{\kappa\in I} a_{\varphi^{-1}(\xi)\kappa} b_{\kappa\varphi^{-1}(\eta)}$$

$$= \sum_{\kappa\in I} \alpha_{\xi\varphi(\kappa)}\beta_{\varphi(\kappa)\eta} = \sum_{\mu\in\Phi} \alpha_{\xi\mu}\beta_{\mu\eta},$$

where the last equality holds since the sum is finite.

Such properties of matrices as being n-bounded or elementary simultaneous do not depend on the ordering φ, hence if φA is a product of a finite number of elementary simultaneous matrices, then the same is true for A. We note also that A is elementary simultaneous of the third type if and only if there exists a splitting $I = \cup I_k$, such that for the corresponding ordering φA consists of diagonally placed squares of the form

$$\begin{pmatrix} 1 & & & \epsilon & \\ & \ddots & & \vdots & \\ & & 1 & \epsilon & \\ & & & 1 & \\ & & & \epsilon & 1 \\ & & & \vdots & & \ddots \end{pmatrix},$$

$\epsilon = \pm 1, 0$, corresponding to the elementary Nielsen transformations.

In the next two lemmas we shall consider the ordering φ, corresponding to a splitting $I = P \cup Q$, where $|P|, |Q| = \infty$. A matrix V is supposed n-bounded.

4.1. LEMMA. *If*

$$\varphi V = \begin{pmatrix} E & A \\ O & E \end{pmatrix} \text{ or } \varphi V = \begin{pmatrix} E & O \\ A & E \end{pmatrix}$$

then V is a product of a finite number of elementary simultaneous matrices.

PROOF. We note first that

$$\begin{pmatrix} E & O \\ A & E \end{pmatrix} = \begin{pmatrix} O & E \\ E & O \end{pmatrix}\begin{pmatrix} E & A \\ O & E \end{pmatrix}\begin{pmatrix} O & E \\ E & O \end{pmatrix},$$

where the first and the last matrices are elementary simultaneous, corresponding to the permutation of the rows. Now in the matrix $\begin{pmatrix} E & A \\ O & E \end{pmatrix}$ we use the low rows with numbers $q \in Q$ to act simultaneously on the rows with numbers $p \in P$. We need no more then n simultaneous transformations of the third type to achieve the result.

4.2. LEMMA. *If $\varphi V = \begin{pmatrix} A & B \\ O & E \end{pmatrix}$, then V is a product of a finite number of elementary simultaneous matrices.*

PROOF. Since V is n-bounded we can change φV into $\varphi V' = \begin{pmatrix} A & O \\ O & E \end{pmatrix}$ by a finite number of elementary simultaneous transformations of the rows. Now, since $|Q| = \infty$, we can split $Q = \cup Q_i$, where $|Q_i| = \infty$, $i = 1, 2, \ldots$, then the splitting $I = P \cup (\cup Q_i)$ defines the corresponding new ordering φ' and

$$\varphi' V' = \begin{pmatrix} A & & & & \\ & E & & & \\ & & E & & \\ & & & E & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} A & & & & \\ & A^{-1} & & & \\ & & A & & \\ & & & A^{-1} & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} E & & & & \\ & A & & & \\ & & A^{-1} & & \\ & & & A & \\ & & & & \ddots \end{pmatrix}$$

$$= \varphi' V_1 \varphi' V_2.$$

The matrices $\varphi' V_1$ and $\varphi' V_2$ consist of the squares of the form

$$\begin{pmatrix} A & O \\ O & A^{-1} \end{pmatrix} = \begin{pmatrix} O & E \\ E & O \end{pmatrix} \begin{pmatrix} E & O \\ A & E \end{pmatrix} \begin{pmatrix} E & -A^{-1} \\ O & E \end{pmatrix} \begin{pmatrix} E & O \\ A & E \end{pmatrix} \begin{pmatrix} E & O \\ O & -E \end{pmatrix}.$$

The first and the last matrices here are elementary simultaneous of the first and second types. The three matrices in the middle are by Lemma 4.1 finite products of elementary simultaneous matrices of the third type. This leads to the proof of the lemma. (We used here a well-known trick of J. H. C. Whitehead and S. Eilenberg.)

5. **The main theorem**. We shall show here that in a free abelian group $\bar{F}_\infty$ the action of an n-bounded Nielsen transformation W is equivalent to the action of a finite product of elementary simultaneous Nielsen transformations or, which is the same, that every n-bounded matrix is a product of a finite number of elementary simultaneous matrices.

We shall consider a free abelian group $\bar{F}_\infty$ with the abelian base $X = \langle x_i, i \in I \rangle$, and with another base $W = \langle w_i, i \in I \rangle$ of which the exponent matrix W is n-bounded. By $U = \langle u_i, i \in I \rangle$ we denote the inverse base (such that $w_i(U) = u_i(W) = x_i$, $i \in I$) with the exponent matrix U equal to W^{-1}. For any word $v \in \bar{F}_\infty$ we denote by $I(v)$ the set of generators x_i in the reduced form of v, so that $v = \Pi x_i^{\alpha_i}$, $i \in I(v)$.

5.1. LEMMA. *In the free abelian group $\bar{F}_\infty$ with the bases X, W and U given above, there exists an infinite set of elements $\langle v_q, q \in Q \subseteq I \rangle$ having the following properties:*

1. $v_q = \Pi x_j^{\gamma_j}$ *where*
1*a*. $\Sigma |\gamma_j| \leq n$,
1*b*. $q' < q$ *implies* $\max I(v_{q'}) < \min I(v_q)$,
2. $v_q = u_q \Pi u_j^{\alpha_{qj}}$ *where*
2*a*. $\Sigma_j |\alpha_{qj}| < n^2$,
2*b*. $j \notin Q$.

PROOF. We build a sequence of subgroups in $\bar{F}_\infty$:

$$gp(w_1) \subset gp(x_1, \ldots, x_{t_1}) \subset \ldots \subset gp(w_1, \ldots, w_{T_{k-1}}) \subset gp(x_1, \ldots, x_{t_k})$$
$$\subset gp(w_1, \ldots, w_{T_k}) \subset gp(x_1, \ldots, x_{t_{k+1}}) \subset \ldots$$

such that

(1) $$1 < t_1 < T_1 < \ldots < t_k < T_k < t_{k+1} < \ldots$$

(2) $$x_{t_{k+1}} \notin gp(x_1, \ldots, x_{t_k}), \qquad k > 0.$$

Now we shall define a set of elements $\langle v_r, r \in R \subseteq I\rangle$ which satisfies the required properties except $2b$, and later we shall choose a subset $\langle v_q, q \in Q \subseteq R\rangle$ satisfying all the properties. We define v_{r_1} as being equal to u_1 (hence we take $r_1 = 1$), and assume that $v_{r_1}, v_{r_2}, \ldots, v_{r_n}$ are defined as required. Then there exists a natural number k such that $t_{k-1} > \max I(v_{r_n})$. We then consider the element

$$u_{t_{k+1}} = \prod_{i \leq t_{k-1}} x_i^{\beta_i} \prod_{j > t_{k-1}} x_j^{\gamma_j}.$$

At least one γ_j here is not equal to zero, since otherwise

$$x_{t_{k+1}} = u_{t_{k+1}}(W) = \prod_{i \leq t_{k-1}} w_i^{\beta_i} \in gp(x_1, \ldots, x_{t_k})$$

which contradicts (2). We put now $r_{n+1} = t_{k+1}$ and define

(3) $$v_{r_{n+1}} = u_{t_{k+1}} \prod_{i \leq t_{k-1}} x_i^{-\beta_i} = \prod_{j > t_{k-1}} x_j^{\gamma_j}.$$

In this way the set $\langle v_r, r \in R \subseteq I\rangle$ is consequently defined. Since every r is equal to some t_{k+1}, we have from (1) that $|R| = \infty$, $|I \backslash R| = \infty$. The properties $1a$, $1b$ are satisfied because

$$L_x(u_{t_{k+1}}) \leq n, \quad \text{and} \quad \min I(v_{r_{n+1}}) > t_{k-1} > \max I(v_{r_n}).$$

We have also

$$v_{r_{n+1}} = u_{t_{k+1}} \prod_{i \leq t_{k-1}} x_i^{-\beta_i} = u_{t_{k+1}} \prod_{i \leq t_{k-1}} w_i^{-\beta_i}(U) = u_{t_{k+1}} \prod u_j^{\alpha_j},$$

where because of (1) $j \leq t_k < t_{k+1}$. Since $L_x(w_i) \leq n$ we get $L_u(w_i(U)) \leq n$ and hence the number of factors u_j is not greater than $(n-1)n$. This gives the properties 2, $2a$.

Obviously, every subset of $\langle v_r, r \in R \subseteq I\rangle$ also satisfies the same properties. We shall choose now a subset $\langle v_q, q \in Q \subseteq R\rangle$ to satisfy $2b$. A word v_r, we shall call proper if there exists an infinite subset of words v_r which do not contain $u_{r'}$ in its expression through u_j. We note that in every subset of n^2 words from $\langle v_r, r \in R\rangle$ we can find a proper word. Indeed, if $v_{s_1}, v_{s_2}, \ldots, v_{s_{n^2}}$ belong to $\langle v_r, r \in R\rangle$, and no v_s is proper then for every s only a finite number of words do not contain u_s and hence there exist v_r containing $u_{s_1}, u_{s_2}, \ldots, u_{s_{n^2}}$ which contradicts $2a$. The required set $\langle v_q, q \in Q \subseteq R\rangle$ can be defined now inductively: we take v_{q_1} as a proper word with the minimal index in $\langle v_r, r \in R\rangle$. Let $v_{q_1}, \ldots, v_{q_n}$ be proper words chosen to satisfy $2b$. Then there exists an infinite subset $R' \subset R$ such that for every $r' \in R'$, $v_{r'}$ does not contain $u_{q_1}, \ldots, u_{q_n}$. We now choose $v_{q_{n+1}}$ to be the proper word with the minimal index in the subset $\langle v_{r'}, r' \in R' \subset R\rangle$. So, the required set $\langle v_q, q \in Q\rangle$ is defined which finishes the proof.

5.2. THEOREM. *Every n-bounded matrix is a product of a finite number of elementary simultaneous matrices.*

PROOF. Let W be an n-bounded matrix, then we can treat it as the exponent matrix for the correspondent base W in $\bar{F}_\infty$. By means of Lemma 5.1 we construct the set of words $\langle v_q, q \in Q\rangle$. We note that every v_q is a primitive element in the free abelian group $\bar{F}_\infty$ because we can put it instead of u_q into the base U. For every v_q we shall consider a subgroup $\bar{F}_{(q)} = gp(x_i, i \in I(v_q))$. By the property $1a$ rank $(\bar{F}_{(q)}) \leq n$. Since v_q is primitive, it can be included into a base in $\bar{F}_{(q)}$. In [7] a method is given to build a base V_q in $\bar{F}_{(q)}$ containing v_q as the first element. This base V_q has $n \times n$ (or smaller) exponent matrix defined by its first row $\gamma_1, \gamma_2, \ldots, \gamma_n$, corresponding to the expression of v_q through x_i, $i \in I(v_q)$. Because of the property $1a$ there exists only a finite number of possibilities for the first row and hence we get only a finite number of different exponent matrices V_q, $q \in Q$. It follows that there exists n_0 such that every matrix V_q is n_0-bounded. According to J. H. C. Whitehead's results [8], [9] (or see [4], pp. 166–167; [1], p. 119), the set V_q can be changed into $\langle x_i, i \in I(v_q)\rangle$ by not more than $(n_0 - 1)n$ Whitehead's transformations which coincide with the elementary Nielsen transformations in an abelian group. It means that every matrix V_q is a product of not more than $(n_0 - 1)n$ elementary matrices. Since by the property $1b$, $\bar{F}_{(q)} \cap \bar{F}_{(q')}$ is trivial for $q \neq q'$, we have $\bar{F}_\infty = \Pi_q^X \bar{F}_{(q)} \times \bar{F}$ which gives the splitting

$$I = \bigcup_q \langle x_i, i \in I(v_q)\rangle \cup \langle x_i, i \notin \bigcup_q I(v_q)\rangle,$$

and the corresponding new ordering φ. We shall suppose that elements from V_q are indexed with i, $i \in I(v_q)$ and denote by V the base in $\bar{F}_\infty$ where

$$V = \bigcup_q V_q \cup \langle x_i, i \notin \bigcup_q I(v_q)\rangle.$$

The exponent matrix V in the ordering φ has a form

$$\varphi V = \begin{pmatrix} V_{q_1} & & & \\ & V_{q_2} & & \\ & & \ddots & \\ & & & E \end{pmatrix}.$$

This matrix is a product of not more than $(n_0 - 1)n$ elementary simultaneous matrices and hence the same is true for V and V^{-1}. By Lemma 5.1 the sets Q and $P = I \backslash Q$ are infinite. We denote by φ' the ordering corresponding to the splitting $I = P \cup Q$. Then since by $2b$ $v_q(W) = u_q(W) \Pi u_p^{\alpha_{qp}}(W) = x_q \Pi x_p^{\alpha_{qp}}$ we have

$$\varphi' V \varphi' W = \begin{pmatrix} A & B \\ C & E \end{pmatrix} = \begin{pmatrix} A - BC & B \\ O & E \end{pmatrix} \begin{pmatrix} E & O \\ C & E \end{pmatrix}. \tag{1}$$

The matrix $\begin{pmatrix} A & B \\ C & E \end{pmatrix}$ is $n_0 n$-bounded as the product of bounded matrices. The matrix $\begin{pmatrix} E & O \\ C & E \end{pmatrix}^{-1} = \begin{pmatrix} E & O \\ -C & E \end{pmatrix}$ and hence by the property $2a$, since $C = (\alpha_{qp})$, is a n^2-bounded. It follows now that the left matrix of the last product in (1) is also invertible and bounded. By Lemmas 4.2 and 4.1, VW is a product of a finite number of elementary simultaneous matrices. Since the same is true for V^{-1} the proof is complete.

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