Notes on the Apollonian problem and the allied theory. Part II.

By John Dougall, M.A., D.Sc.

(*Read* 13*th December* 1907. *Received* 25*th June* 1908.)

In a paper under the above title in Vol. XXIV. of the *Proceedings* it is shown that in a certain system of co-ordinates, the equation of the first degree represents a circle orthogonal to a fixed circle. It follows that any purely graphical theorem regarding right lines in a plane can be extended to orthogonals to a circle. This may be seen otherwise by projecting the figure of right lines on a sphere, the right lines thus becoming circles orthogonal to a circle on the sphere; and then inverting the sphere into the original plane. The geometrical method shows that the extension may also be applied to theorems involving one circle as well as right lines, the circle remaining unchanged, while the lines become orthogonals to a circle; the Pole and Polar Theorem, Pascal's and Brianchon's Theorems are examples. But plane figures involving more than one circle cannot in general be transformed in this way. We cannot, for instance, deduce the construction for a circle touching three great circles on a sphere from the known construction for a circle touching three lines in a plane; nor the Gergonne construction for circles on a sphere from the corresponding method in a plane.

The following pages contain proofs of the extended forms of the Pole and Polar Theorem, Pascal's Theorem, and Gergonne's construction. The theorems in their ordinary forms are not assumed, and the method of proof is as applicable to circles on a sphere as in a plane.

A construction is given for the Apollonian problem, perfectly analogous to the usual construction for a plane or spherical triangle. It may, of course, be got at once by a real or imaginary inversion from the latter case.

I hope to continue the notes in a later communication, and to deduce from them a geometrical theory of conics having double contact with a conic.

2. In the paper cited above, certain conventions were used, which will be adopted here, and it may be convenient for the reader to have a brief account of the chief of them before him :

(a) The *radius* of a circle is supposed to have algebraic sign, so that if the square of the radius is a^2, the circle $-a$ is distinguished from the circle a, although both contain the same points, real and imaginary.

(b) The *angle* Θ between two circles a, b, is defined by the relation

$$d^2 = a^2 + b^2 - 2ab \cos \Theta,$$

where d is the distance between the centres.

(c) Two circles a, b *touch* if $\Theta = 0$, or $d^2 = (a-b)^2$.

(d) The *centre of similitude* of two circles a, b, is that point S in the line of centres A, B, for which SA : SB $= a : b$.

It follows that when a, b touch, the point of contact is the centre of similitude.

(e) The signs of the radii of two inverse circles are so related that the centre of the circle of inversion is their centre of similitude.

(f) Subject to these conventions, it was shown that the angle between two circles is *equal* to the angle between their inverses; that the circle of inversion with radius $-k$ is inverse to the same circle with radius k; and that consequently a circle inverse to itself, *i.e.*, an orthogonal to the circle of inversion, cuts two inverse circles at equal angles; and conversely that a circle cutting two inverse circles at equal angles is orthogonal to the circle of inversion; but that the circle of inversion itself cuts two inverse circles at *supplementary* angles.

(g) It will, however, be convenient, and should hardly be misleading, to refer to the circle of inversion of a, b as their bisecting circle, or the *bisector* of the angle between them. It is the circle coaxal with a, b, and having its centre at the centre of similitude.

3. We take a fixed circle O as the base to which orthogonal circles are drawn. Such circles will be called *orthogonals* simply. Two orthogonals intersect in a pair of *inverse points*.

In general, the centre of O is supposed to be a real point, but the square of its radius is not necessarily positive.

Graphically, a system of orthogonals to a circle is simply a system of circles, the common chord of every pair of which passes through a fixed point.

This holds on a sphere also; in this case the planes of orthogonals to a circle C pass through a fixed point, the pole of the plane of C; thus great circles are orthogonal to the imaginary circle in which the plane at infinity cuts the sphere.

4. The ordinary centre of a circle is the point through which all right lines orthogonal to the circle pass.

By analogy, we define the *centre as to O* of a circle C as the pair of points to which all common orthogonals to C and O pass. These are the limiting points of the coaxal system to which C and O belong, and are inverse points with respect to C and to O alike.

All circles cutting O in the same two points have the same centre as to O.

The orthogonal through the centres as to O of two circles may be called their *line of centres as to O.*

If two circles touch, their line of centres as to O goes through their point of contact, for the two limiting points coincide there.

5. The *common chord*, or *radical axis*, *as to O*, of two circles is the orthogonal through their common points, or coaxal with them.

It is not a determinate circle when the given circles are themselves orthogonals.

The line of centres as to O of the circles being orthogonal to both circles, is orthogonal to their common chord.

The radical axes as to O of three circles are coaxal, for each is orthogonal to two circles, viz., O and the orthogonal circle of the three circles. The common point pair of these radical axes is the *radical centre as to O* of the three circles. It is the centre as to O of their orthogonal circle.

6. The centres as to O of a system of coaxal circles lie on an orthogonal, viz., the orthogonal through their limiting points.

7. Every right line making equal angles with two circles *a*, *b*, being orthogonal to the bisecting circle, § 2 (*f*), goes through the

https://doi.org/10.1017/S0013091500036580 Published online by Cambridge University Press

centre of similitude, and conversely. Similarly every orthogonal making equal angles with two circles a, b, is orthogonal to two circles, O and the bisecting circle, and therefore goes through a fixed point pair, the *centre of similitude as to O* of a, b; and conversely every circle through this point pair cuts a, b, at equal angles.

The centre of similitude as to O of a, b, is the intersection of the two orthogonals which touch them both.

8. The line of centres as to O of two circles, being orthogonal to both, makes both equal and supplementary angles with them, and therefore goes through both centres of similitude as to O.

9. By § 2 (b) every circle which cuts a point circle, *i.e.*, a circle of zero radius, at a finite angle passes through the point.

Also by (d), (g) if two circles a, b, touch, their bisector is the point circle with centre at the point of contact.

Hence every circle cutting two touching circles a, b, at equal angles, goes through their point of contact, and conversely (*cf.* last sentence of § 4).

10. The three centres of similitude as to O of three circles a, b, c, taken in pairs, lie on an orthogonal, the *axis of similitude as to O*. For if these centres are P, Q, R, then the orthogonal PQ, passing through P cuts b, c, at equal angles, and, passing through Q cuts c, a, at equal angles. It therefore cuts a, b, at equal angles, and passes through R (§ 7).

The axis of similitude of a, b, c, as to O, like every circle cutting a, b, c, at equal angles, is orthogonal to the three bisectors, and is coaxal with the orthogonal circle and the ordinary axis of similitude of a, b, c.

11. The three bisectors are coaxal.

Also the bisector of b, $-c$, making supplementary angles with b, c, § 2 (f), makes equal angles with b, c, and is orthogonal to the bisector of b, c.

12. Let a circle x which cuts two circles a, b, at equal angles cut a at P, Q. Since the circle of inversion which inverts a into b inverts x into itself, the points P′, Q′ in which x cuts b are inverse to P, Q. Hence the right lines PP′, QQ′ go through the centre of

similitude of a, b. It follows that any circle through P and P′, being coaxal with the right line PP′ and the circle x, is orthogonal to the circle of inversion and makes equal angles with a, b. Hence orthogonals to O through P, P′ and Q, Q′ pass through the centre of similitude of a, b as to O.

13. Let a circle Σ be orthogonal to two circles A and B, whose radii are a and b, cutting A at P, Q and B at P′, Q′. Since Σ makes equal angles both with a, b and with a, $-b$, it follows from § 12 that the orthogonals PP′, QQ′ intersect at one of the centres of similitude as to O of A, B and the orthogonals PQ′, P′Q at the other.

On this simple theorem the following proofs of two fundamental theorems are based.

14. *The Pole and Polar Theorem.*

If through a fixed point E in the plane of a given circle C any two orthogonals be drawn cutting the circle C in P, Q and P′, Q′ respectively, then the intersections of the orthogonals PP′, QQ′ and those of PQ′, P′Q lie on a fixed orthogonal.

Proof. Draw the circles X,Y orthogonal to C; X through P and Q; Y through P′ and Q′.

Then, by § 13, the orthogonals PP′, QQ′ intersect at one of the centres of similitude as to O of X, Y and the orthogonals PQ′, P′Q at the other. Hence by § 8 the orthogonal through the intersections is the line of centres as to O of X and Y.

Now the circle D orthogonal to C and to two of the orthogonals through E (*i.e.*, the circle orthogonal to C, and with centre as to O at E) is orthogonal to all the orthogonals through E; being orthogonal to PQ and C, it is orthogonal to X, which is coaxal with them.

Since X, and similarly Y, are orthogonal to C and D, they belong to a fixed coaxal system, and, § 6, their line of centres as to O is a fixed orthogonal, which proves the theorem.

15. The proof only requires that X and Y should be orthogonal to a fixed circle besides C. This will be ensured if the circles PQ are orthogonal to a fixed circle H, which is also orthogonal to C; for then X will be orthogonal to H.

In particular, the proof will hold if the circles PQ are coaxal, even if O is not one of their orthogonal circles.

16. The fixed orthogonal of the theorem of § 14 may be called the O - polar of E with respect to C.

All the well known graphical developments follow from the theorem in the ordinary way.

The polar goes through the points of contact of the orthogonals through E touching C. Hence, if an orthogonal cut a circle a in X and X′, the O - pole with respect to a of the orthogonal XX′ is the centre as to O of the orthogonal to a through X, X′.

The case when C is the base O itself is important in certain applications of the theory, *e.g.*, reciprocation. The O - polar of E in that case is the orthogonal whose centre as to O is at E; conversely the O - pole of an orthogonal is its centre as to O. Two orthogonals are therefore conjugate, *i.e.*, the O - pole of either will lie on the other, if they are orthogonal to each other.

17. *Lemma for Pascal's Theorem.*

Let 1, 2, 3, ..., n be points in a right line, not necessarily in order; on 12, 23, ..., $\overline{n-1n}$, $n1$ as diameters describe circles. Let (12), (23), etc., be their radii with signs chosen as follows:—(12) arbitrarily, the rest in turn, so that the circle (23) touches (12), (34) touches (23), ..., $(n1)$ touches $(\overline{n-1\ n})$.

Will $(n1)$ *touch* (12)?

By § 2 (d), we have $21 : 23 = (12) : (23)$ where 21, 23 are signed segments on the right line.

$$\text{Thus} \qquad 12 : 23 = -(12) : (23)$$

$$\text{Similarly} \qquad 23 : 34 = -(23) : (34)$$

$$34 : 45 = -(34) : (45)$$

$$\cdots\cdots\cdot$$

$$\overline{n-1n} : n1 = -(\overline{n-1n}) : (n1).$$

Hence, compounding, we have

$$12 : n1 = (-1)^{n-1}(12) : (n1).$$

For contact of (12), $(n1)$, we need

$$12 : n1 = -(12) : (n1).$$

Therefore, the end members (12), (n1) of the chain will touch *provided n is even.*

By inversion, we find the same result for n points on a circle, the circles 12, 23, etc., being now orthogonals to that circle.

18. *Pascal's Theorem.*

Let 1, 2, 3, 4, 5, 6 be any six points on a circle Σ; then will the intersections of the pairs of orthogonals to O, 12, 45; 23, 56; and 34, 61 lie on an orthogonal.

Draw the orthogonals to Σ through 14, 25, 36.

By § 13, if the orthogonals 12, 45 intersect at P; 23, 56 at Q; 34, 61 at R, and 31, 64 at R′; then P is one of the O - centres of similitude of 14, 25; and so on.

Hence the orthogonal PQ goes through either R or R′; we can determine which it is, by the above lemma.

For that lemma shows that we can choose signs so that each of the circles (12), (23), (34), (45), (56), (61) touches its two neighbours; also that we can take (14) to touch (12) and (34), and therefore to touch (61) and (45) also, (§ 9); similarly with (25) and (36).

Then, since (12), (45) both touch (14) and (25), the point pair P where the orthogonals through 12, 45 intersect is the centre of similitude as to O of (14), (25), (§ 12). Hence PQR is an orthogonal, viz., the axis of similitude as to O of (14), (25), (36).

19. It may happen that PQR′ is also an orthogonal; that is to say, that PQRR′ lie on an orthogonal, This orthogonal, passing through both centres of similitude as to O of 14, 36, viz., R and R′, is their line of centres as to O, and cuts them at right angles; hence PQRR′ cuts all the circles 14, 25, 36 at right angles, and passes through their O - centres. The circles 14, 25, 36 are therefore coaxal.

20. *Brianchon's Theorem* is derived in the usual way by combination of Pascal's Theorem with the Pole and Polar Theorem.

21. *The Apollonian problem*: to describe a circle to touch three given circles a, b, c.

Analysis. Suppose that ρ is such a circle touching a, b, c at X, Y, Z. Since ρ makes equal angles with a, b, c, it is orthogonal to each of the bisecting circles of a, b, c, (§ 10). If then Σ is the circle orthogonal to a, b, c, and therefore orthogonal to the bisectors, the centre of ρ as to Σ is at the common point pair I,I′ of the bisector.

The circle II′X is then orthogonal to ρ, and so to a. Hence X must be one of the points where the circle coaxal with the bisecting circles, and orthogonal to a, cuts a.

Synthesis. Coaxal with the bisecting circles draw the circle orthogonal to a, to meet a at X and X′. The circle through X orthogonal to the bisecting circles will touch a, b, c. For this circle, being orthogonal to the bisecting circles, is orthogonal to the circle through XX′ coaxal with them, and the sign of its radius may therefore be chosen so that it touches a at X; and it makes equal angles with a, b, c.

22. The obvious point of interest in the construction is its perfect analogy with the corresponding construction for a plane rectilineal, or a spherical triangle. To bring out the analogy even more explicitly, let ABC be one of the circular triangles formed by a, b, c; the construction amounts to this:—bisect two of the angles B, C by circles orthogonal to Σ; from I, their common point, draw IX orthogonal to BC and Σ; then the circle through X, with centre as to Σ at I, touches BC, CA, and AB.

23. The *Gergonne construction* is easily deduced, and generalised,

For X, X′ are the common points of two circles orthogonal to Σ, viz., a and IXX′; hence the right line XX′ passes through the centre of Σ. Also the pole of the line XX′ with respect to a, being the centre of the circle through XX′ orthogonal to a, that is of the circle IXX′, which is coaxal with the bisecting circles, lies on the line of centres of these, that is, on the axis of similitude. Thus XX′ is the right line through the radical centre and the pole of the axis of similitude with respect to a.

In the very same way, with orthogonals to O, the orthogonal through X, X′ passes through the centre of Σ as to O. Also the O – pole of the orthogonal XX′ with respect to a, being (§ 16) the O – centre of the circle through XX′ orthogonal to a, *i.e.*, of IXX′, lies on the axis of similitude of a, b, c as to O. Hence X, X′ are given as the intersections of a by the orthogonal through the radical centre as to O and the O – pole with respect to a of the O – axis of similitude.

24. *The Pascal lines of three pairs of points on a circle.*

In the figure of §18 suppose that the radii (14), (25), (36), or a, b, c, say, are assigned to begin with. Suppose also that the points 1, 4 are only given *as a pair*, viz., as the points of intersection of a and Σ; and similarly with 2, 5 and 3, 6. We can affix the marks 1, 4 arbitrarily; then so affix the marks 2, 5 that it is 12, not 15, which cuts (14), (25) at equal angles—by § 13 one of them does so—; then the marks 3, 6, so that it is 23, not 26, which cuts (25), (36) at equal angles. Hence, (14), (25), (36), or a, b, c being given, we can so take the points 1, 2, 3, 4, 5, 6, that each of the nine orthogonals to Σ, a, b, c, (12), ... (61) touches the four which it meets on Σ. And we see easily that *change of the sign* of a is simply equivalent to interchange of the marks 1 and 4 at the points of intersection of a and Σ, a suggestive result from the graphical point of view.

25. *The Gergonne construction in terms of co-orthogonals with the given circles.*

The construction stated in the last sentence of § 23 fails when the base O coincides with Σ, for we have not defined the centre of Σ as to Σ, nor the Σ – pole of one orthogonal with respect to another. But the original construction of § 21 may for this case be stated :— the orthogonal IXX′ joins the Σ – centres (or Σ – poles) of the Σ – axis of similitude and the orthogonal a. The whole construction may therefore be thus stated in terms of orthogonals to Σ :—

Let the given circles cut their orthogonal circle Σ in 14, 25, 36. Join the Σ – pole of the Pascal Σ – line of the Σ – hexagon 123456 by a Σ – line to the Σ – pole of 14. This cuts 14 in the points of contact of a pair of touching circles. The other three pairs of touching circles are found by interchanging 1, 4 or 2, 5 or 3, 6 in this construction.

An identical construction holds for the conics having double contact with a given conic Σ and touching three given lines which cut Σ in 14, 25, 36, Σ – lines being in this case simply right lines.