P.-D. Liu
Nagoya Math. J.
Vol. 150 (1998), 197–209

PESIN'S ENTROPY FORMULA FOR ENDOMORPHISMS

PEI-DONG LIU

Abstract. In this paper we prove Pesin's entropy formula for general C^2 (or $C^{1+\alpha}$) (non-invertible) endomorphisms of a compact manifold preserving a smooth measure.

§1. Introduction

Let M be a C^∞ compact Riemannian manifold without boundary, and let $f : M \to M$ be a C^1 map. The Lyapunov exponents of the map f are defined by Oseledec's theorem which states that, for any f-invariant Borel probability measure μ on M, for almost every point $x \in M$ there exists a unique family of numbers

$$-\infty \le \lambda^{(1)}(x) < \lambda^{(2)}(x) < \cdots < \lambda^{(r(x))}(x) < +\infty$$

(the Lyapunov exponents of f at x) and a unique sequence of subspaces of T_xM

$$\{0\} = V^{(0)}(x) \subset V^{(1)}(x) \subset \cdots \subset V^{(r(x))}(x) = T_xM$$

such that

$$\lim_{n\to+\infty} \frac{1}{n} \log |T_xf^n\xi| = \lambda^{(i)}(x)$$

for all $\xi \in V^{(i)}(x) \setminus V^{(i-1)}(x)$, $1 \le i \le r(x)$. For any such a system $f : (M,\mu) \hookleftarrow$, the entropy $h_\mu(f)$ and the Lyapunov exponents can be connected by Ruelle's (or Margulis-Ruelle) inequality ([R]$_1$)

$$h_\mu(f) \le \int_M \sum_i \lambda^{(i)}(x)^+ m_i(x)\, d\mu \tag{1.1}$$

where $a^+ = \max\{a, 0\}$ and $m_i(x) = \dim V^{(i)}(x) - \dim V^{(i-1)}(x)$ (the multiplicity of $\lambda^{(i)}(x)$). Pesin's entropy formula ([Pe]) states that, when f is a

Received August 28, 1996.
This research is supported by the National Natural Science Foundation of China and the Nonlinear Science Project of Climbing Program.

C^2 (or $C^{1+\alpha}$) diffeomorphism and μ is absolutely continuous with respect to the Lebesgue measure on M, then (1.1) is an equality

$$h_\mu(f) = \int_M \sum_i \lambda^{(i)}(x)^+ m_i(x)\, d\mu. \tag{1.2}$$

Such a result has played a fundamental role in smooth ergodic theory and related fields.

Various extensions of Pesin's above result have been made for (essentially) invertible systems (see [LeSt], [KSt], $[\text{LeY}]_1$, $[\text{LeY}]_2$, [LiQ] etc. and see $[\text{M}]_1$ for an alternative proof of Pesin's above result). But in the case of non-invertible systems, extensions have been so far restricted, as far as the author knows, to one-dimensional maps ([Le]), expanding maps ($[\text{M}]_2$ Section IV.5, [H], [B]) and Axiom A endomorphisms ([QZ]). In this paper we aim to extend Pesin's above result to general C^2 (or $C^{1+\alpha}$) non-invertible endomorphisms, that is, to prove the following theorem (the Lebesgue measure on M will be denoted by λ throughout this paper).

THEOREM 1.1. *Let $f : M \to M$ be a C^2 endomorphism and μ an f-invariant Borel probability measure on M. If $\mu \ll \lambda$, then there holds Pesin's formula*

$$h_\mu(f) = \int_M \sum_i \lambda^{(i)}(x)^+ m_i(x)\, d\mu. \tag{1.3}$$

Remark 1.1. Actually Theorem 1.1 can be proved for $C^{1+\alpha}$ ($\alpha > 0$) endomorphisms. But for simplicity of presentation we confine ourselves to the case of C^2 endomorphisms.

Remark 1.2. Under the conditions formulated in Theorem 1.1, it can be verified that $\log|\det T_x f| \in L^1(M,\mu)$ (see Subsection 2.2), and hence $\lambda^{(1)}(x) > -\infty$ for μ-a.e. $x \in M$ by Oseledec's theorem. This was indicated to the author by J. Bahnmüller after he read the first draft of the paper, in which the integrability of $\log|\det T_x f|$ was stated as a condition in the theorem.

Theorem 1.1, among other things, allows us to compute the entropy of an endomorphism via its Lyapunov exponents. For example, the following two results, which have been proved in other ways, can be obtained as natural corollaries of the theorem.

COROLLARY 1.1. ([M]$_2$ Section IV. 5) *Let f be a C^2 expanding endomorphism of M and let μ be the unique f-invariant probability measure which is absolutely continuous with respect to the Lebesgue measure on M. Then one has*

$$h_\mu(f) = \int_M \sum_i \lambda^{(i)}(x) m_i(x)\, d\mu = \int_M \log|\det T_x f| d\mu.$$

COROLLARY 1.2. ([W] Section 8.4) *Suppose that $A : K^p \to K^p$ is a surjective (group) endomorphism of p-dimensional torus. If m is the Haar measure on K^p, then*

$$h_m(A) = \sum_{|\lambda_i|>1} \log|\lambda_i|$$

where $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of the linear transformation $\widetilde{A} : \mathbf{R}^p \to \mathbf{R}^p$ that covers A.

In the rest of this paper we shall address ourselves to the proof of Theorem 1.1. In view of Ruelle's inequality, it remains to prove

$$h_\mu(f) \geq \int_M \sum_i \lambda^{(i)}(x)^+ m_i(x)\, d\mu \tag{1.4}$$

under the conditions presented in the formulation of Theorem 1.1. Since maps considered here are non-invertible and unstable foliations are not available for such maps, we will make use of stable foliations instead. Many ideas used in this paper go back to [Pe], [LeSt] and [LeY]$_2$.

§2. Proof of Theorem 1.1

2.1. Technical preliminaries

In the remaining part of this paper we will always assume that $f : (M, \mu) \hookleftarrow$ satisfies the conditions of Theorem 1.1. Denote by Γ the set of critical points of f. By Sard's Theorem, $\lambda(f^n\Gamma) = 0$ and hence $\mu(f^n\Gamma) = 0$ for all $n \geq 1$ since $\mu \ll \lambda$. Put $g = d\mu/d\lambda$ and $\Gamma' = \{x : g(x) = 0\}$. Write $\widehat{\Gamma} = \bigcup_{k=1}^{+\infty} f^k\Gamma$ and $\widetilde{\Gamma} = \bigcup_{n=0}^{+\infty} f^{-n}(\widehat{\Gamma} \cup \Gamma')$. Clearly $\mu(\widetilde{\Gamma}) = 0$ and $f^{-1}\widetilde{\Gamma} \subset \widetilde{\Gamma}$. Choose a Borel set $\Lambda \subset M \setminus \widetilde{\Gamma}$ such that $\mu(\Lambda) = 1$, $f\Lambda \subset \Lambda$ and every point $x \in \Lambda$ is regular in the sense of Oseledec. It is easy to see that every $x \in \Lambda$ is both a regular point and a regular value of f^n for all $n \geq 1$. Hence for every $x \in \Lambda$ and an arbitrarily fixed natural number n, there is an open ball V centered at x such that $(f^n)^{-1}V$ has a finite number of connected

https://doi.org/10.1017/S0027763000025113 Published online by Cambridge University Press

components $\{U_i\}$ and $f^n|_{U_i} : U_i \to V$ is a C^2 diffeomorphism for each U_i. From this there follows readily the following simple fact.

LEMMA 2.1. *Let n be an arbitrarily fixed natural number. If W is a k-dimensional $(0 < k \leq \dim M)$ $C^{1,\theta}$ $(0 \leq \theta \leq 1)$ embedded submanifold of M, then there exists a k-dimensional $C^{1,\theta}$ embedded submanifold W' of M such that $W' \subset (f^n)^{-1}W$ and*

$$(f^n)^{-1}W \cap \Lambda = W' \cap \Lambda.$$

Set $I = \{x \in \Lambda : \lambda^{(i)}(x) \geq 0 \text{ for all } 1 \leq i \leq r(x)\}$ and $\Delta = \Lambda \setminus I$. Clearly $fI \subset I$ and $f\Delta \subset \Delta$. For $x \in \Delta$, write $E^s(x) = \bigcup_{\lambda^{(i)}(x)<0} V^{(i)}(x)$ and define the stable manifold of f at x as

$$W^s(x) = \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0\}.$$

The arguments of Sections 1–3 of [LiQ, Chapter III] restricted to the deterministic case show that, for μ-a.e. $x \in \Delta$, there exists a sequence of $C^{1,1}$ embedded k-dimensional discs $\{W_n(x)\}_{n=0}^{+\infty}$ (where $k = \dim E^s(x)$) such that $fW_n(x) \subset W_{n+1}(x)$ for all $n \geq 0$ and

$$W^s(x) = \bigcup_{n=0}^{+\infty} (f^n)^{-1} W_n(x)$$

(see also $[\mathrm{R}]_2$ or [RSh]). For $x \in I$, define $W^s(x) = \{x\}$.

Let $\mathcal{B}_\mu(\Lambda)$ denote the completion of the Borel σ-algebra of Λ with respect to μ. Then $(\Lambda, \mathcal{B}_\mu(\Lambda), \mu)$ is a Lebesgue space. Since $f\Lambda \subset \Lambda$ and $\mu(\Lambda) = 1$ we have $h_\mu(f) = h_\mu(f|_\Lambda)$. Hence, in order to prove Theorem 1.1, it is sufficient to prove (1.4) for the map $f|_\Lambda : (\Lambda, \mathcal{B}_\mu(\Lambda), \mu) \hookleftarrow$. Throughout what follows we will consider exclusively this map and we will denote it also by the notation f for simplicity. We now state our main result of this subsection as follows.

LEMMA 2.2. *Let $f : (\Lambda, \mathcal{B}_\mu(\Lambda), \mu) \hookleftarrow$ be as given above. Then there exists a measurable partition η of Λ which has the following properties:*

(1) *$f^{-1}\eta \leq \eta$;*

(2) *For μ-a.e. $x \in \Lambda$, there exists a $\dim E^s(x)$-dimensional $C^{1,1}$ embedded submanifold W_x of M such that $W_x \subset W^s(x)$, $\eta(x) \subset W_x$ and $\eta(x)$ contains an open neighbourhood of x in $W_x \cap \Lambda$ (with respect to the induced topology of $W_x \cap \Lambda$ as a subset of M);*

(3) *For every Borel set $B \subset \Lambda$ the function*

$$P_B(x) = \lambda_x^s(\eta(x) \cap B)$$

is measurable and μ almost everywhere finite, where λ_x^s is the Lebesgue measure on W_x induced by its inherited Riemannian structure as a submanifold of M ($\lambda_x^s = \delta_x$ if $W^s(x) = \{x\}$);

(4) *Let $\{\mu_x^\eta\}_{x\in\Lambda}$ be a canonical system of conditional measures of μ associated with η. Then*

$$\mu_x^\eta \ll \lambda_x^s \qquad \mu - \text{a.e. } x.$$

This lemma is similar to [LiQ] Proposition IV.2.1 (restricted to the deterministic case), which is a variant of [LeSt] Proposition 3.1. A detailed proof of that proposition in [LiQ] is given in [LiQ, Section IV.2] with the needed properties of local stable manifolds being worked out in [LiQ, Chapter III]. The difference between our present situation and that of [LiQ] lies in that we are dealing with a non-invertible endomorphism rather than a diffeomorphism. But one can check that this deficiency can be overcome by the local diffeomorphism property of f (as far as points in Λ are concerned) and Lemma 2.1. That is to say, Lemma 2.2 can be proved by almost the same arguments as the corresponding proof in [LiQ] with some slight modifications caused by applying the local diffeomorphism property and Lemma 2.1 instead of the diffeomorphism property. Here we omit presenting the long arguments and refer the reader to [LiQ] for details.

The conclusion (3) of Lemma 2.2 allows one to define a σ-finite Borel measure λ^* on Λ by

$$\lambda^*(B) = \int \lambda_x^s(\eta(x) \cap B)\, d\mu$$

for each Borel set $B \subset \Lambda$. From Lemma 2.2 (4) it follows that $\mu \ll \lambda^*$. Define

$$h = \frac{d\mu}{d\lambda^*}. \tag{2.1}$$

Then one has the following

LEMMA 2.3. *For μ-a.e. $x \in \Lambda$,*

$$h = \frac{d\mu_x^\eta}{d\lambda_x^s}$$

λ_x^s almost everywhere on $\eta(x)$.

https://doi.org/10.1017/S0027763000025113 Published online by Cambridge University Press

See [LeSt] Proposition 4.1 or [LiQ] Proposition IV.2.2 for a proof.

2.2. Proof of (1.4)

Let η be a measurable partition of Λ as introduced by Lemma 2.2. Let $\{\mu_x^{f^{-1}\eta}\}_{x\in\Lambda}$ be a canonical system of conditional measures of μ associated with the partition $f^{-1}\eta$. Then

$$\begin{aligned} h_\mu(f) \geq h_\mu(f,\eta) &= H_\mu(\eta \mid \bigvee_{n=1}^{+\infty} f^{-n}\eta) \\ &= H_\mu(\eta \mid f^{-1}\eta) \\ &= -\int_\Lambda \log \mu_x^{f^{-1}\eta}(\eta(x))\, d\mu. \end{aligned}$$

So, in order to prove (1.4), it suffices to show

$$-\int_I \log \mu_x^{f^{-1}\eta}(\eta(x))\, d\mu \geq \int_I \sum_i \lambda^{(i)}(x) m_i(x)\, d\mu \tag{2.2}$$

(this is actually an equality) and

$$-\int_\Delta \log \mu_x^{f^{-1}\eta}(\eta(x))\, d\mu \geq \int_\Delta \sum_i \lambda^{(i)}(x)^+ m_i(x)\, d\mu. \tag{2.3}$$

We first prove (2.2). The proof of (2.2) presented below is due to F. Ledrappier and L.-S. Young and was communicated to the author by Bahnmüller [B].

We begin with the Jacobian of $f : (\Lambda, \mu) \hookleftarrow$. Since $T_x f$ is nondegenerate and $g(x) > 0$ ($g = d\mu/d\lambda$) for every $x \in \Lambda$, one can easily choose a countable measurable partition $\alpha = \{A_i\}$ of Λ such that f restricted to each A_i, written f_{A_i}, is injective and $\mu(f_{A_i}(B)) = 0$ if B is a Borel subset of A_i and $\mu(B) = 0$ ($f(B)$ is clearly Borel if B is Borel). By this we can define a measure μ_{A_i} on each A_i by

$$\mu_{A_i}(B) = \mu(f_{A_i}(B)) \qquad \text{for Borel set } B \subset A_i$$

which is clearly equivalent to μ_i ($\mu_i = \mu|_{A_i}$, the restriction of μ to A_i). Define a measurable function $J(f) : \Lambda \to \mathbf{R}^+$ by

$$J(f)(x) = \frac{d\mu_{A_i}}{d\mu_i}(x) \qquad \text{if } x \in A_i.$$

https://doi.org/10.1017/S0027763000025113 Published online by Cambridge University Press

It is easy to see that $J(f)$ is independent of the choice of partition α. We call $J(f)$ the Jacobian of $f : (\Lambda, \mu) \hookleftarrow$. By Radon-Nikodym Theorem, one can easily compute that

$$J(f)(x) = \frac{g(f(x))}{g(x)} |\det T_x f|, \qquad x \in \Lambda. \tag{2.4}$$

We now proceed with the proof of (2.2) and, at the same time, prove that $\log |\det T_x f| \in L^1(M, \mu)$. Suppose that $K \subset \Lambda$ is a measurable set such that $fK \subset K$ and $\mu(K) > 0$. Let $\mathcal{B}_\mu(K)$ be defined analogously to $\mathcal{B}_\mu(\Lambda)$ and let ϵ be the partition of K into single points. Since $(K, \mathcal{B}_\mu(K), \mu)$ is a Lebesgue space (maybe $\mu(K) < 1$), by an interesting result of Parry ([Pa] Lemma 10.5) one has

$$-\log \mu_x^{f^{-1}\epsilon}(\epsilon(x)) = \log J(f)(x), \qquad \mu\text{-a.e. } x \in K.$$

By (2.4), for μ-a.e. $x \in \Lambda$

$$\log J(f)(x) = \log \frac{g(f(x))}{g(x)} + \log |\det T_x f| \geq 0$$

since $J(f) \geq 1$ μ-a.e. on Λ. This yields that

$$\log^- \frac{g(f(x))}{g(x)} \geq -\log^+ |\det T_x f|, \qquad \mu\text{-a.e. } x \in \Lambda$$

which, by [LeSt] Proposition 2.2 and the integrability of $\log^+ |\det T_x f|$, implies that

$$\int_K \log \frac{g \circ f}{g} \, d\mu = 0$$

(note that $fK \subset K$). Hence

$$\begin{aligned} 0 \leq -\int_K \log \mu_x^{f^{-1}\epsilon}(\epsilon(x)) \, d\mu &= \int_K \log |\det T_x f| \, d\mu \\ &= \int_K \sum_i \lambda^{(i)}(x) m_i(x) \, d\mu \end{aligned}$$

by Oseledec's theorem. By taking $K = I$ and $K = \Lambda$ respectively, this proves (2.2) (since $\eta(x) = \{x\}$ for μ-a.e. $x \in I$) and $\log |\det T_x f| \in L^1(M, \mu)$.

Now we proceed to the proof of (2.3). We may assume that $\mu(\Delta) = 1$ and even $\Delta = \Lambda$ without any loss of generality. So in what follows we take this assumption. We first introduce the following measurable functions on Λ:

$$W(z) = \mu_z^{f^{-1}\eta}(\eta(z)),$$
$$X(z) = \frac{g(z)}{g(f(z))} \cdot \frac{h(f(z))}{h(z)},$$
$$Y(z) = \frac{|\det(T_z f|_{E^s(z)})|}{|\det T_z f|},$$

where h is defined by (2.1). We now present several claims, whose proofs will be given a little later.

CLAIM 2.1. *$W = XY$, μ almost everywhere on Λ.*

CLAIM 2.2. *$\log Y \in L^1(\Lambda, \mu)$ and*

$$-\int_\Lambda \log Y \, d\mu = \int_\Lambda \sum_i \lambda^{(i)}(x)^+ m_i(x) \, d\mu.$$

CLAIM 2.3. *$\log X \in L^1(\Lambda, \mu)$ and $\int \log X \, d\mu = 0$.*

Then (2.3) follows immediately from Claims 2.1–2.3. This completes the proof of Theorem 1.1.

In the sequel we give proofs of Claims 2.1–2.3. In order to prove Claim 2.1, we need the following two lemmas.

LEMMA 2.4. *Let $A \subset \Lambda$ be a Borel set such that $\mu(A) > 0$ and $f|_A : A \to fA$ is injective. Then for μ-a.e. $x \in \Lambda$ one has*

$$\int_{(f^{-1}\eta)(x)\cap A} J(f) \, d\mu_x^{f^{-1}\eta} = \mu_{f(x)}^{\eta}(fA). \tag{2.5}$$

Proof. Let $F(x)$ and $G(x)$ denote respectively the function at the left hand and that at the right hand of the equation (2.5). By the uniqueness of canonical systems of conditional measures one has

$$G(x) = \mu_x^{f^{-1}\eta}(f^{-1}(fA)), \qquad \mu\text{-a.e. } x \in \Lambda.$$

Let $\mathcal{B}(f^{-1}\eta)$ denote the σ-algebra generated by $f^{-1}\eta$. Clearly $F(x)$ and $G(x)$ are both measurable with respect to $\mathcal{B}(f^{-1}\eta)$. So, in order to prove (2.5), it suffices to show that for any $C \in \mathcal{B}(f^{-1}\eta)$

$$\int_C F(x) \, d\mu = \int_C G(x) \, d\mu. \tag{2.6}$$

https://doi.org/10.1017/S0027763000025113 Published online by Cambridge University Press

Note that, since $C \in \mathcal{B}(f^{-1}\eta)$,

$$\begin{aligned}\int_C F(x)\,d\mu &= \int_\Lambda \chi_C \int_{(f^{-1}\eta)(x)} \chi_A J(f)\,d\mu_x^{f^{-1}\eta}\,d\mu \\ &= \int_\Lambda \int_{(f^{-1}\eta)(x)} \chi_C\chi_A J(f)\,d\mu_x^{f^{-1}\eta}\,d\mu \\ &= \int_{A\cap C} J(f)\,d\mu \\ &= \mu(f(A\cap C))\end{aligned}$$

and

$$\begin{aligned}\int_C G(x)\,d\mu &= \int_\Lambda \chi_C \int_{(f^{-1}\eta)(x)} \chi_{f^{-1}(fA)}\,d\mu_x^{f^{-1}\eta}\,d\mu \\ &= \int_\Lambda \int_{(f^{-1}\eta)(x)} \chi_C\chi_{f^{-1}(fA)}\,d\mu_x^{f^{-1}\eta}\,d\mu \\ &= \mu(C\cap f^{-1}(fA)) \\ &= \mu(f^{-1}(f(A\cap C))) \\ &= \mu(f(A\cap C)).\end{aligned}$$

This proves (2.6) and completes the proof of Lemma 2.4. □

LEMMA 2.5. *Let A be as given in Lemma 2.4. Then for μ-a.e. $x \in A$ one has*

$$\mu_x^{f^{-1}\eta}(B) = \int_{fB} \frac{1}{J(f)\circ f_A^{-1}}\,d\mu_{f(x)}^\eta$$

for any Borel set $B \subset (f^{-1}\eta)(x) \cap A$, where $f_A = f|_A : A \to fA$.

Proof. Put $\xi = \eta|_{fA}$ and $\zeta = f_A^{-1}\xi = (f^{-1}\eta)|_A$. Write $\nu = \mu|_A$, $\widehat{\nu} = \mu|_{fA}$ and let measure ν_A on A be defined by $d\nu_A/d\nu = J(f)$. It is easy to see that a canonical system of conditional (probability) measures of ν associated with ζ is given by

$$\left\{\nu_x^\zeta : \nu_x^\zeta(\cdot) = \frac{\mu_x^{f^{-1}\eta}(\cdot)}{\mu_x^{f^{-1}\eta}(A)}\right\}_{x\in A}.$$

https://doi.org/10.1017/S0027763000025113 Published online by Cambridge University Press

Then, by [KSt] Proposition II.11.1, a canonical system of conditional measures of ν_A associated with ζ is

$$\left\{ (\nu_A)^\zeta_x : d(\nu_A)^\zeta_x = \frac{J(f)|_{\zeta(x)}}{\int_{\zeta(x)} J(f)\, d\nu^\zeta_x}\, d\nu^\zeta_x \right\}_{x \in A}.$$

Since $f_A : (A, \nu_A) \to (fA, \widehat{\nu})$ is measure-preserving, $f_A\zeta = \xi$ and a canonical system of conditional measures of $\widehat{\nu}$ associated with ξ is given by

$$\left\{ (\widehat{\nu})^\xi_y : (\widehat{\nu})^\xi_y(\cdot) = \frac{\mu^\eta_y(\cdot)}{\mu^\eta_y(fA)} \right\}_{y \in fA},$$

then, by the uniqueness of canonical systems of conditional measures, one has for μ-a.e. $x \in A$

$$(\nu_A)^\zeta_x(B) = (\widehat{\nu})^\xi_{f(x)}(fB) = \frac{\mu^\eta_{f(x)}(fB)}{\mu^\eta_{f(x)}(fA)}$$

if $B \subset \zeta(x)$ is a Borel set. Therefore, for μ-a.e. $x \in A$ and any Borel $B \subset \zeta(x)$,

$$\begin{aligned}
\mu^{f^{-1}\eta}_x(B) &= \mu^{f^{-1}\eta}_x(A) \cdot \nu^\zeta_x(B) \\
&= \mu^{f^{-1}\eta}_x(A) \cdot \int_{\zeta(x)} J(f)\, d\nu^\zeta_x \cdot \int_B \frac{1}{J(f)}\, d(\nu_A)^\zeta_x \\
&= \frac{\int_{(f^{-1}\eta)(x)\cap A} J(f)\, d\mu^{f^{-1}\eta}_x}{\mu^\eta_{f(x)}(fA)} \int_B \frac{1}{J(f)}\, d(\mu^\eta_{f(x)} \circ f_A) \\
&= \int_B \frac{1}{J(f)}\, d(\mu^\eta_{f(x)} \circ f_A) \\
&= \int_{fB} \frac{1}{J(f) \circ f_A^{-1}}\, d\mu^\eta_{f(x)}
\end{aligned}$$

by Lemma 2.4. This completes the proof of the lemma. □

Proof of Claim 2.1. It suffices to show that for μ-a.e. $x \in \Lambda$ one has

$$W(z) = X(z)Y(z), \qquad \mu^\eta_x\text{-a.e. } z.$$

Let $\alpha = \{A_i\}$ be the partition of Λ introduced above. Then for μ-a.e. $x \in \Lambda$ we have for any Borel set $B \subset \eta(x)$

$$\begin{aligned}
\mu_x^\eta(B) &= \frac{1}{W(x)}\mu_x^{f^{-1}\eta}(B) \\
&= \frac{1}{W(x)}\sum_i \mu_x^{f^{-1}\eta}(C_i) \qquad (C_i = B\cap A_i) \\
&= \frac{1}{W(x)}\sum_i \int_{fC_i} \frac{1}{J(f)\circ f_{A_i}^{-1}}\, d\mu_{f(x)}^\eta \qquad \text{(by Lemma 2.5)} \\
&= \frac{1}{W(x)}\sum_i \int_{fC_i} \frac{1}{J(f)\circ f_{A_i}^{-1}(z)} h(z)\, d\lambda_{f(x)}^s \\
&= \frac{1}{W(x)}\sum_i \int_{C_i} \frac{1}{J(f)(z)} h(f(z))|\det(T_zf|_{E^s(z)})|\, d\lambda_x^s \\
&= \frac{1}{W(x)}\int_B \frac{1}{J(f)(z)} h(f(z))|\det(T_zf|_{E^s(z)})|\, d\lambda_x^s
\end{aligned}$$

and, on the other hand,

$$\mu_x^\eta(B) = \int_B h(z)\, d\lambda_x^s.$$

Since Borel set B is arbitrarily chosen, we have

$$h(z) = \frac{1}{W(x)}\cdot\frac{1}{J(f)(z)} h(f(z))|\det(T_zf|_{E^s(z)})|$$

for λ_x^s-a.e. $z\in\eta(x)$. Since $W(z) = W(x)$ for any $z\in\eta(x)$, it follows then that

$$\begin{aligned}
W(z) &= \frac{1}{J(f)(z)}\cdot\frac{h(f(z))}{h(z)}|\det(T_zf|_{E^s(z)})| \\
&= X(z)Y(z), \qquad \mu_x^\eta\text{-a.e. } z\in\eta(x).
\end{aligned}$$

Then there follows the claim. □

Proof of Claim 2.2. Noting that for μ-a.e. $z\in\Lambda$

$$|T_zf|_{E^s(z)}| \le |f|_{C^1},$$

we have $\log^+|T_zf|_{E^s(z)}| \in L^1(\Lambda,\mu)$. By Oseledec multiplicative ergodic theorem we have

$$\int_\Lambda \log|\det T_zf|\, d\mu = \int_\Lambda \sum_i \lambda^{(i)}(z)m_i(z)\, d\mu \tag{2.7}$$

https://doi.org/10.1017/S0027763000025113 Published online by Cambridge University Press

and

$$(2.8) \qquad \int_\Lambda \log|\det(T_z f|_{E^s(z)})|\,d\mu = \int_\Lambda \sum_i \lambda^{(i)}(z)^- m_i(z)\,d\mu.$$

Since $\log|\det T_z f| \in L^1(\Lambda,\mu)$, one has $\sum_i \lambda^{(i)}(z)m_i(z) \in L^1(\Lambda,\mu)$ and hence $\sum_i \lambda^{(i)}(z)^- m_i(z) \in L^1(\Lambda,\mu)$. So, by (2.8), it follows that

$$\log|\det(T_z f|_{E^s(z)})| \in L^1(\Lambda,\mu).$$

Thus, $\log Y$ is integrable and

$$\begin{aligned} -\int \log Y\,d\mu &= \int \sum_i \lambda^{(i)}(z)m_i(z)\,d\mu - \int \sum_i \lambda^{(i)}(z)^- m_i(z)\,d\mu \\ &= \int \sum_i \lambda^{(i)}(z)^+ m_i(z)\,d\mu \end{aligned}$$

which proves Claim 2.2. □

Proof of Claim 2.3. By Claim 2.1,

$$\log W = \log X + \log Y \le 0, \qquad \mu\text{-a.e.}$$

Hence, by Claim 2.2, $\log^+ X \in L^1(\Lambda,\mu)$. Then, by [LeSt] Proposition 2.2, we know that $\log X$ is integrable and $\int \log X\,d\mu = 0$. □

Acknowledgements. The author is very grateful to Dr. J. Bahnmüller for the helpful communication [B] and for his indicating the point stated in Remark 1.2. He also would like to thank Professor Qian Min and Dr. Zhu Shu for helpful discussions.

References

[B] J. Bahnmüller, *Pesin's entropy formula for expanding random dynamical systems*, private communication.

[H] Huyi Hu, *Pesin's entropy formula for expanding maps*, Adv. Math. Beijing, **19** (1990), 338–349.

[KSt] A. Katok and J. M. Strelcyn, Invariant Manifold, Entropy and Billiards; Smooth Maps with Singularities, Lec. Not. Math., 1222, Springer-Verlag, 1986.

[Le] F. Ledrappier, *Some properties of absolutely continuous invariant measures on an interval*, Ergod. Th. Dynam. Sys., **1** (1981), 77.

[LeSt] F. Ledrappier and J. M. Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergod. Th. Dynam. Sys., **2** (1982), 203–219.

[LeY$_1$] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms, Part* I*: Characterization of measures satisfying Pesin's entropy formula, Part* II*: Relations between entropy, exponents and dimension*, Ann. Math., **122** (1985), 509–574.

[LeY$_2$] F. Ledrappier and L.-S. Young, *Entropy formula for random transformations*, Probab. Th. Rel. Fields, **80** (1988), 217–240.

[LiQ] Pei-Dong Liu and Min Qian, Smooth Ergodic Theory of Random Dynamical Systems, Lec. Not. Math., 1606, Springer-Verlag, 1995.

[M$_1$] R. Mañé, *A proof of Pesin's formula*, Ergod. Th. Dynam. Sys., **1** (1981), 95–102.

[M$_2$] R. Mañé, Ergodic Theory and Differentiable Dynamics, Springer-Verlag, 1987.

[Pa] W. Parry, Entropy and Generators in Ergodic Theory, W. A. Benjamin, Inc., New York, 1969.

[Pe] Ya. B. Pesin, *Lyapunov characteristic exponents and smooth ergodic theory*, Russ. Math. Surveys, **32: 4** (1977), 55–114.

[QZ] Min Qian and Zhu-Sheng Zhang, *Ergodic theory of Axiom A endomorphisms*, Ergod. Th. Dynam. Sys., **1** (1995), 161–174.

[R$_1$] D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Bras. Math., **9** (1978), 83–87.

[R$_2$] D. Ruelle, *Characteristic exponents and invariant manifolds in Hilbert space*, Ann. Math., **115** (1982), 243–290.

[RSh] D. Ruelle and M. Shub, Stable manifolds for maps, Lec. Not. Math., 819, Springer-Verlag, 1980, pp. 389–392.

[W] P. Walters, An Introduction to Ergodic Theory, Springer, New York, 1982.

Department of Mathematics and Institute of Mathematics
Peking University, Beijing 100871, P. R. China
lpd@sxx0.math.pku.edu.cn