Ergod. Th. & Dynam. Sys. (1988), **8**, 421–424
Printed in Great Britain

Smooth chaotic maps with zero topological entropy

M. MISIUREWICZ AND J. SMÍTAL

Institute of Mathematics, Warsaw University, PKiN IX p., 00-901 *Warsaw, Poland,*
Institute of Mathematics, Komensky University, 842 15 *Bratislava, Czechoslovakia*

(*Received* 13 *March* 1987)

Abstract. We find a class of C^∞ maps of an interval with zero topological entropy and chaotic in the sense of Li and Yorke.

1. *Introduction*

Throughout this paper $f: I \to I$ will be a continuous map of a compact real interval I into itself.

The notion of chaos has been introduced by Li and Yorke [**5**]. The following equivalent definition is given in [**9**] (cf. also [**3**]):

f is *chaotic* if there is some $\varepsilon > 0$ and a non-empty perfect set $S \subset I$ such that for any x, $y \in S$, $x \neq y$, and any periodic point p of f,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \geq \varepsilon, \tag{1}$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0, \tag{2}$$

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| \geq \varepsilon, \tag{3}$$

where f^n denotes the nth iterate of f. The set S is called an ε-scrambled set for f.

It turns out that the chaos in the sense of Li and Yorke is a weaker property than positive topological entropy (and is equivalent to the property that the map has a trajectory which is not approximable by cycles, cf. [**3, 9**]). An example showing this is given in [**9**].

The main aim of the present paper is to give simple examples of chaotic maps with zero entropy.

2. *Preliminary constructions*

We recall that the ω-limit set of $x \in I$, denoted by $\omega_f(x)$, is the set of limit points of the sequence $(f^n(x))_{n=0}^\infty$.

The following theorem will be useful in the sequel:

THEOREM 1. *If there is a point $x \in I$ such that the set $\omega_f(x)$ is infinite and f is not injective on $\omega_f(x)$ then f is chaotic in the sense of Li and Yorke.*

Proof. Choose a, $b \in \omega_f(x)$ such that $a \neq b$, $f(a) = f(b)$. Then any periodic neighbourhoods J_a, J_b of a, b, respectively, must have a point in common. By [**3**], this implies that f is chaotic. □

Remark 1. The condition of Theorem 1 is not necessary, an example is given in [**9**].

Recall that a continuous map $f: I \to I$, where $I = [a, b]$, is called *unimodal* (cf. [**2**]) if there exists $c \in (a, b)$ such that f is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$. We shall call f *weakly unimodal* if there exists $c \in (a, b)$ such that f is non-decreasing on $[a, c]$ and non-increasing on $[c, b]$.

Let f be weakly unimodal. We shall say that $x, y \in I$ are *equivalent* (denoted by $x \sim y$) if there exists $n \geq 1$ such that f^n is constant on $[x, y]$. Clearly, $\sim$ is an equivalence relation. Let $\tilde{I} = I/\sim$ be the factor space obtained by identifying to a point each equivalence class. These classes are closed intervals (possibly degenerated to a point) and thus $\tilde{I}$ is a closed interval (also possibly degenerated). The natural projection $\pi: I \to \tilde{I}$ is continuous and non-decreasing. Since f is continuous, $x \sim y$ implies $f(x) \sim f(y)$. Therefore there exists a unique map $\tilde{f}: \tilde{I} \to \tilde{I}$ such that

$$\text{the diagram} \quad \begin{array}{ccc} I & \xrightarrow{f} & I \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \tilde{I} & \xrightarrow{\tilde{f}} & \tilde{I} \end{array} \quad \text{commutes.} \tag{4}$$

This $\tilde{f}$ is continuous and either monotone or unimodal.

By the period of a periodic point (orbit) we shall understand its smallest period.

LEMMA 1. (a) *If $x \in I$ is a periodic point of f of period k then $\pi(x)$ is a periodic point of $\tilde{f}$ of period k.*

(b) *If $y \in \tilde{I}$ is a periodic point of $\tilde{f}$ of period k then there exists a unique periodic point $x \in I$ of f for which $\pi(x) = y$. The period of x is k.*

Proof. (b) Let $\pi^{-1}(y) = [a, b]$. Then (4) and the equality $\tilde{f}^k(y) = y$ imply that $f^k([a, b]) \subset [a, b]$. Since f is continuous, there is some $x \in [a, b]$ with $f^k(x) = x$. Clearly $\pi(x) = y$. The point x cannot have a period $i < k$ because then by (4), y would have period i.

Let $z \in [a, b]$. Since $z \sim x$, there is some r such that $f^{rk}(z) = f^{rk}(x) = x$. This shows that x is the unique periodic point of f in [a, b].

(a) From (4) it follows that $\tilde{f}^k(\pi(x)) = \pi(x)$. The period of $\pi(x)$ cannot be $i < k$ because then, by (b), the period of x would be also i. □

Let $\mathcal{F}$ be the class of all weakly unimodal maps f, for which

the set $J_f = \{x \in I; f(x) \geq f(y) \quad \text{for all } y \in I\}$ consists of more than one point, (5)

for each $n \geq 0$, $\quad f$ has a periodic orbit of period 2^n, (6)

f has no periodic orbits of other periods. (7)

LEMMA 2. *If $f \in \mathcal{F}$ then $\tilde{f}$ is unimodal and satisfies* (6) *and* (7).

Proof. By Lemma 1, the sets of periods of periodic orbits of f and $\tilde{f}$ are equal. Therefore, if $f \in \mathcal{F}$ then $\tilde{f}$ satisfies (6) and (7). Then, since $\tilde{f}$ has periodic points of period larger than 2, it cannot be monotone. Consequently, it is unimodal. □

3. *Main results*

First we prove the following

THEOREM 2. *Any mapping $f \in \mathscr{F}$ has topological entropy zero and is chaotic in the sense of Li and Yorke.*

Proof. Let $f \in \mathscr{F}$. By (7) and [8], f has topological entropy zero.

By Lemma 2, $\tilde{f}$ is unimodal and satisfies (6) and (7). Therefore it has the same kneading invariant as the Feigenbaum map Φ (see e.g. [**4** Proposition 4.6] for the uniqueness of this kneading invariant, and [**2**] for the description of the Feigenbaum map). Hence the relative positions of the turning point, its images and the periodic points are the same for Φ and $\tilde{f}$. Since π is non-decreasing, by Lemma 1 it is the same also for f. However, for Φ this relative position is well-known. Let c be the critical point of Φ and let a_n be the periodic point of Φ of period 2^n with the largest image under Φ. Then from the known properties of Φ we immediately get

$$\Phi^{2^1}(c) < a_1 < \Phi^{2^3}(c) < a_3 < \cdots < c < \cdots < a_4 < \Phi^{2^4}(c)$$
$$< a_2 < \Phi^{2^2}(c) < a_0 < \Phi^{2^0}(c).$$

Therefore if b_n is the periodic point of f of period 2^n with the largest image under f and $d \in J_f$ (see (5)) then

$$f^{2^1}(d) < b_1 < f^{2^3}(d) < b_3 < \cdots < d < \cdots < b_4 < f^{2^4}(d)$$
$$< b_2 < f^{2^2}(d) < b_0 < f^{2^0}(d). \tag{8}$$

Let $d_1 = \lim_{n\to\infty} b_{2n+1}$, $d_2 = \lim_{n\to\infty} b_{2n}$. Since

$$\Phi(a_0) < \Phi(a_1) < \Phi(a_2) < \Phi(a_3) < \cdots < \Phi(c),$$

we have also

$$f(b_0) < f(b_1) < f(b_2) < f(b_3) < \cdots < f(d)$$

and therefore

$$\lim_{n\to\infty} f(b_i) = f(d_1) = f(d_2) \le f(d). \tag{9}$$

Since

$$\lim_{n\to\infty} \Phi(a_n) = \lim_{n\to\infty} \Phi^{2^n+1}(c) = \Phi(c)$$

and c is not periodic for Φ, the $\tilde{f}$-itineraries of all points

$$y \in [\lim_{n\to\infty} \tilde{f}(\pi(b_n)), \tilde{f}(\pi(d))]$$

are the same (in fact, from the result of [**6**] applied to a slightly modified map f it follows that the above interval is degenerated to a point; however, we do not need to use it). By (9), this means that we can replace in (8) d by d_i, $i = 1, 2$. Consequently

$$d_i \in \omega_f(f(d_1)) \qquad \text{for } i = 1, 2. \tag{10}$$

By the definition of d_1 and by (8), d_1 is not periodic. Hence $\omega_f(f(d_1))$, which by (10) contains the whole trajectory of d_1, is infinite. The point d in (8) is an arbitrary element of J_f and hence $J_f \subset [d_1, d_2]$. Therefore by (5), $d_1 < d_2$. By (9) $f(d_1) = f(d_2)$, and hence by Theorem 1, f is chaotic. □

THEOREM 3. *$\mathcal{F}$ contains a C^∞ map.*

Proof. There exists a C^∞ map g of $[0, 1]$ onto $[0, 1]$ which is weakly unimodal, satisfies (5) and $g(0) = g(1) = 0$. Set $g_\lambda(x) = \lambda g(x)$ for all λ, $x \in [0, 1]$. The map g_λ is of class C^∞ for each λ. By [7], the set $A = \{\lambda;\, g_\lambda$ satisfies (6)$\}$ is closed. Clearly, g_1 satisfies (6) but g_0 does not. Therefore if $\mu = \inf A$ then $\mu > 0$ and g_μ satisfies (6).

Suppose that g_μ does not satisfy (7). Then by [1], if λ is sufficiently close to μ then g_λ satisfies (6), a contradiction. Clearly, g_μ satisfies also (5). □

Remark 3. Let ε be the length of J_g. Then g_μ has a non-empty ε-scrambled set S (cf. [**9**]). Clearly ε can be made arbitrarily close to 1, but less than 1. This result cannot be improved, since if $f : [0, 1] \to [0, 1]$ is a continuous map with zero topological entropy satisfying (1) and (2) for some $x, y \in [0, 1]$ then $\varepsilon < 1$. To see it note that in this case at least one of the sets $\omega_f(x)$, $\omega_f(y)$, say $\omega_f(x)$, must be infinite. Let I_0, I_1 be disjoint closed periodic intervals covering $\omega_f(x)$ and such that $I_0 \cap \omega_f(x) \neq \emptyset \neq I_1 \cap \omega_f(x)$ (cf. e.g. [**9**]). By (2) there exists some $z \in \omega_f(x) \cap \omega_f(y)$ and this z cannot be periodic (cf. [**9**]). Consequently, $\omega_f(y) \subset I_0 \cup I_1$ and since $\operatorname{dist}\{I_0, I_1\} > 0$, (2) implies that for some m, $f^m(x)$, $f^m(y) \in I_0$. Hence for every $n \geq m$, $f^n(x)$, $f^n(y)$ belong to the same interval $I_{i(n)}$, where $i(n) \in \{0, 1\}$. But then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| \leq \max\{\operatorname{diam} I_0, \operatorname{diam} I_1\} < 1.$$

Remark 4. Let h be the tent map ($h(x) = 1 - |2x - 1|$) and let $h_\lambda(x) = \min(\lambda, h(x))$ for $\lambda, x \in [0, 1]$. For each $n \geqq 0$ let $\lambda_n \in [0, 1]$ be the minimal number with the property that $[0, \lambda_n]$ contains a periodic orbit of h of period 2^n. Clearly $\lambda_0 < \lambda_1 < \cdots$. Put $\nu = \lim_{n\to\infty} \lambda_n$ $(=0.8249080\ldots)$. Then $h_\nu \in \mathcal{F}$. In such a way we obtain another simple example of chaotic map with zero topological entropy.

REFERENCES

[1] L. Block. Stability of periodic orbits in the theorem of Šarkovskii. *Proc. Amer. Math. Soc.* **81** (1981), 555–562.

[2] P. Collet & J.-P. Eckmann. *Iterated Maps on the Interval as Dynamical Systems.* Birkhauser: Boston, 1980.

[3] K. Janková & J. Smítal. A characterization of chaos. *Bull. Austral. Math. Soc.* **34** (1986), 283–292.

[4] L. Jonker. Periodic orbits and kneading invariants. *Proc. London Math. Soc.* **39** (3) (1979), 428–450.

[5] T. Y. Li & J. A. Yorke. Period three implies chaos. *Amer. Math. Monthly* **82** (1975), 985–992.

[6] W. de Melo & S. van Strien. A structure theorem in one dimensional dynamics. *Informes de Matématica Série* **A-063/86**, Rio de Janeiro (preprint 1986).

[7] M. Misiurewicz. Structure of mappings of an interval with zero entropy. *Publ. Math. IHES* **53** (1981), 5–16.

[8] M. Misiurewicz & W. Szlenk. Entropy of piecewise monotone maps. *Studia Math.* 67 (1980), 45–63.

[9] J. Smítal. Chaotic functions with zero topological entropy. *Trans. Amer. Math. Soc.* **297** (1986), 269–282.