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INTERSECTION OF TWO INVARIANT SUBSPACES

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ABSTRACT. It is shown that, if F and G are inner functions, $(H^2 \ominus FH^2)/(H^2 \ominus FH^2) \cap GH^2$ is n-dimensional if and only if G is a Blaschke product of degree n. This is an extension of the well known result for the case $(H^2 \ominus FH^2) \cap GH^2 = \{0\}$.

1. **Introduction**. Let L^2 denote the space of square-integrable functions on the unit circle ∂U with Lebesgue measure $d\theta/2\pi$. Let H^2 denote the usual Hardy class for ∂U, that is, the space of functions in L^2 whose Fourier coefficients with negative indices vanish. H^2 coincides with the space of functions in L^2 whose Poisson extensions into the unit disc U are analytic.

For each $f \in L^2$, put $M_z f = zf$. We call a closed nonzero subspace in L^2 M_z-invariant when it is invariant under an operation of M_z. A M_z-invariant subspace in H^2 is called S-invariant where $S = M_z | H^2$. M_z-invariant subspaces are described completely (cf. [7, Lecture II]). The nonreducing M_z-invariant subspaces of L^2 are precisely the subspaces of the form ψH^2 for some unimodular function ψ on ∂U. This is called Beurling's theorem. So every S-invariant subspace of H^2 has the form GH^2 for some inner function G and hence every S^*-invariant subspace has the form $H^2 \ominus FH^2$ for some inner function F.

It is easy to prove that if $H^2 \ominus FH^2 = (H^2 \ominus FH^2) \cap GH^2$ then G is a constant inner function. In this paper we will show that if F and G are inner functions, $H^2 \ominus FH^2/(H^2 \ominus FH^2) \cap GH^2$ is an n dimensional subspace if and only if G is a Blaschke product of degree n. When $(H^2 \ominus FH^2) \cap GH^2 = \{0\}$, the result is well known [2, p189].

S^*-invariant subspaces were investigated by many people. For example, [1] and [3]. Our result will give information about the structure of them.

Let $z_1, z_2, \cdots$ be distinct points in the open disk and F the Blaschke product with zeros $\{z_k\}$. I_F denotes the linear operator on H^2 defined by

$$I_F(f) = \{(1 - |z_k|^2)^{1/2} f(z_k)\}_{k=1}^{\infty}.$$

Then I_F is a bounded linear operator from H^2 to ℓ^2. If $\{z_k\}$ is uniformly separated then

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$I_F(H^2) = \ell^2$ and so $\mathcal{I}_F$, the restriction of I_F to $H^2 \ominus FH^2$, is a one-one bounded linear operator from $H^2 \ominus FH^2$ onto ℓ^2. This was shown by Shapiro and Shields (cf. [4, Theorem 9.1]). Suppose F and G are Blaschke products with zeros $\{z_k\}$ and $\{s_k\}$, respectively. When $\{z_k\}$ is uniformly separated, $I_G I_F^{-1}$ is a bounded linear operator from ℓ^2 to ℓ^2.

$$\begin{aligned} I_G \mathcal{I}_F^{-1}(\ell^2) &= I_G(H^2 \ominus FH^2) \\ &= [\{(1 - |s_k|^2)^{1/2} f(s_k)\} : f \in H^2 \ominus FH^2] \end{aligned}$$

and $\{f \in H^2 \ominus FH^2 : I_G f = 0\} = (H^2 \ominus FH^2) \cap GH^2$. Thus our result shows that $I_G \mathcal{I}_F^{-1}$ is of finite rank if and only if G is a finite Blaschke product because F is not a finite Blaschke product.

2. **M_z^*-invariant subspace**. We will consider only non-reducing invariant subspaces under M_z. Otherwise problems are easy to solve. The following proposition describes the intersection of a M_z^*-invariant subspace and a M_z-invariant subspace.

PROPOSITION 1. *Let ϕ and ψ be unimodular functions. Then $\phi \bar{H}^2 \cap \psi H^2$ is the L^2-closure of*

$$\psi\{L^2 \cap g(H^2 \ominus zqH^2)\}$$

where q is an inner function and g is a function whose square is a strong outer function in H^1, and $\bar{\phi}\psi = \bar{q}\bar{g}/g$.

Let h be a nonzero function in H^1. Then h is an outer function if and only if k is constant a.e. whenever $kh \in H^1$ and $k \in L^\infty$ with $k \geq 0$ a.e.. We say h is a strong outer function if it has the following property: If $kh \in H^1$ for some Lebesgue measurable k with $k \geq 0$ a.e. then k is constant a.e. [9]. For $u \in L^\infty T_u$ denotes a Toeplitz operator (cf. [2, Chapter 7]). It is easy to see that $\psi \ker T_{\bar{z}\bar{\phi}\psi} = \phi \bar{H}^2 \cap \psi H^2$. Hayashi [6] described completely the kernels of Toeplitz operators, and the formula of the proposition is his result. The author [9] described the finite dimensional kernels of Toeplitz operators independently of [6]. In [9], the author shows that if $\phi \bar{H}^2 \cap \psi H^2$ is an n dimensional subspace and $n \neq 0$ then $\bar{\phi}\psi = \bar{z}^n \bar{g}/g$ for some strong outer function g^2 and $\phi \bar{H}^2 \cap \psi H^2 = \{p\psi g : p$ ranges over all analytic polynomials with degree $\leq n - 1\}$.

PROPOSITION 2. *Let ϕ and ψ be unimodular functions. If $\phi \bar{H}^2 \cap \psi H^2 \neq \{0\}$ then there exists a unimodular function ψ_1 that satisfies the following*:

(1) *$\psi \bar{\psi}_1$ is a simple Blaschke factor*

(2) *$dim\{\phi \bar{H}^2 \cap \psi_1 H^2 / \phi \bar{H}^2 \cap \psi H^2\} = 1$*

PROOF. Let $N = \phi \bar{H}^2 \cap \psi H^2$, and K be the orthogonal complement of N in $\phi \bar{H}^2$, then $N \neq \{0\}$ and $K \neq \{0\}$. Since $N \subset \psi H^2$, there exists $f \in N$ with $\bar{z} f \notin N$. Then $\bar{z} f = k + g$, $k \in K$, $k \neq 0$ and $g \in N$. Hence $f - zk = zg \in \psi H^2$ and so $zk \in \psi H^2$. This implies that $\psi H^2 + [k]$ is a M_z-invariant subspace where $[k]$ is the linear span of k.

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Hence

$$\psi H^2 + [k] = \psi_1 H^2$$

and $\psi = \dfrac{z-\alpha}{1-\bar{\alpha}z}\psi_1$ with $|\alpha| < 1$. Then $\phi\bar{H}^2 \cap \psi_1 H^2 = \phi\bar{H}^2 \cap (\psi H^2 \oplus [k]) = (\phi\bar{H}^2 \cap \psi H^2) \oplus [k]$.

THEOREM 3. *If ϕ and ψ are unimodular functions, $\phi\bar{H}^2/\phi\bar{H}^2 \cap \psi H^2$ is an infinite dimensional space.*

PROOF. If $\phi\bar{H}^2/\phi\bar{H}^2 \cap \psi H^2$ is a finite dimensional subspace, then by Proposition 2 we can show that $\phi\bar{H}^2 = \phi\bar{H}^2 \cap \psi_1 H^2$ for some inner function ψ_1. Then $\phi\bar{H}^2 \subset \psi_1 H^2$ and this contradiction implies the theorem.

3. **S^*-invariant subspace**. Let F and G be inner functions. The intersection of an S^*-invariant subspace $H^2 \ominus FH^2$ and an S-invariant subspace GH^2 has the form $(H^2 \ominus FH^2) \cap GH^2 = \bar{z}(F\bar{H}^2 \cap zGH^2)$. Hence Proposition 1 describes the intersection. Theorem 3 shows that $GH^2/(H^2 \ominus FH^2) \cap GH^2$ is an infinite dimensional subspace. We will study $H^2 \ominus FH^2/(H^2 \ominus FH^2) \cap GH^2$.

LEMMA 1. *If $(H^2 \ominus FH^2) \cap GH^2 \neq \{0\}$ then there exists a Blaschke product B_1 of degree 1 such that*

$$\mathit{dim}\{(H^2 \ominus FH^2) \cap G\bar{B}_1 H^2/(H^2 \ominus FH^2) \cap GH^2\} = 1.$$

The lemma is immediate from Proposition 2.

LEMMA 2. *If $H^2 \ominus FH^2 \subset \psi H^2$ and ψ is a unimodular function then $\bar{\psi}$ is an inner function.*

PROOF. If $f \in H^2 \ominus FH^2$ then f and $(f - f(0))\bar{z}$ are in ψH^2, and hence $f(0)$ belongs to ψH^2. Since there exists $f \in H^2 \ominus FH^2$ with $f(0) \neq 0$, $\bar{\psi} \in H^2$.

LEMMA 3. *If $f \in H^2 \ominus FH^2$ then $\dfrac{f - f(\alpha)}{z - \alpha} \in H^2 \ominus FH^2$ for any α with $|\alpha| < 1$.*

PROOF. Let $g \in FH^2$ and

$$k(\alpha) = \int \frac{f(e^{i\theta}) - f(\alpha)}{e^{i\theta} - \alpha}\overline{g(e^{i\theta})}d\theta, \; |\alpha| < 1.$$

Then k is an analytic function of α and a simple computation yields

$$k^{(n)}(0) = \int e^{-i(n+1)\theta}(f(e^{i\theta}) - \sum_0^n f^{(j)}(0)e^{ij\theta})\overline{g(e^{i\theta})}d\theta.$$

Hence $k^{(n)}(0) = 0$ for any $n \geq 0$ and $k = 0$ (cf. [4]).

PROPOSITION 4. *Let F be an inner function and G an inner function with nontrivial Blaschke part. If $(H^2 \ominus FH^2) \cap GH^2 \neq \{0\}$ then there exists a Blaschke product B_1 of degree 1 such that $G\bar{B}_1 \in H^2$ and*

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$$\dim\{(H^2 \ominus FH^2) \cap G\bar{B}_1 H^2/(H^2 \ominus FH^2) \cap GH^2\} = 1.$$

PROOF. Let $N = (H^2 \ominus FH^2) \cap GH^2$, and K be the orthogonal complement of N in $H^2 \ominus FH^2$, then $N \neq \{0\}$. If $K = \{0\}$ then $H^2 \ominus FH^2 \subset GH^2$. By Lemma 2 G is constant and this contradicts the hypothesis of G. Hence $K \neq \{0\}$. Let $\alpha \in U$ with $G(\alpha) = 0$. There exists $f \in N$ with $f/(z - \alpha) \notin N$. Otherwise $(z - \alpha)^{-1}N \subset N$. Hence

$$N \subset G\left(\frac{z - \alpha}{I - \bar{\alpha}z}\right)^{\ell} H^2 \text{ for any positive integer } \ell. \text{ This contradicts } N \neq \{0\}.$$

Let $f \in N$ with $f/(z - \alpha) \notin N$, then $f/(z - \alpha) \in H^2 \ominus FH^2$ by Lemma 3 because $f(\alpha) = 0$. Hence $f = (z - \alpha)k + (z - \alpha)g$, $k \in K$, $k \neq 0$ and $g \in N$. Hence

$$\frac{f}{1 - \bar{\alpha}z} = \frac{z - \alpha}{1 - \bar{\alpha}z}k + \frac{z - \alpha}{1 - \bar{\alpha}z}g$$

and so $\dfrac{z - \alpha}{1 - \bar{\alpha}z}k$ belongs to GH^2. This implies that $GH^2 + [k]$ is a M_z-invariant subspace and so

$$GH^2 + [k] = G_1 H^2$$

where $G = \dfrac{z - \beta}{1 - \bar{\beta}z}G_1$ with $\beta \in U$. Then

$$(H^2 \ominus FH^2) \cap G_1 H^2 = [k] \oplus (H^2 \ominus FH^2) \cap GH^2.$$

THEOREM 5. *Let n be a nonnegative integer. Let F and G be inner functions. Suppose $(H^2 \ominus FH^2) \cap GH^2 \neq \{0\}$, then the dimension of $H^2 \ominus FH^2/(H^2 \ominus FH^2) \cap GH^2$ is n if and only if G is a Blaschke product of degree n.*

PROOF. Suppose $\dim\{H^2 \ominus FH^2/(H^2 \ominus FH^2) \cap GH^2\} = n$. By Lemma 1, there exists a Blaschke product B of degree n such that

$$H^2 \ominus FH^2 = (H^2 \ominus FH^2) \cap G\bar{B}H^2.$$

Lemma 2 implies that $\bar{G}B \in H^2$ and so G is a Blaschke product of degree $\leq n$.

Conversely suppose G is a Blaschke product of degree n. Proposition 4 implies that $\dim\{H^2 \ominus FH^2/(H^2 \ominus FH^2) \cap GH^2\} = n$.

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