

UNICITY THEOREMS FOR MEROMORPHIC OR ENTIRE FUNCTIONS II

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In 1976, Gross posed the question “can one find two (or possibly even one) finite sets S_j ($j = 1, 2$) such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?”, where $E_f(S_j)$ stands for the inverse image of S_j under f . In this paper, we show that there exists a finite set S with 11 elements such that for any two non-constant meromorphic functions f and g the conditions $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ imply $f \equiv g$. As a special case this also answers the question posed by Gross.

1. INTRODUCTION

By a meromorphic function we shall always mean a meromorphic function in the complex plane. It is assumed that the reader is familiar with the notation of Nevanlinna Theory (see, for example, [3]). We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function f , we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \rightarrow \infty, r \notin E).$$

For any set S and any meromorphic function f let

$$E_f(S) = \bigcup_{a \in S} \{z \mid f(z) - a = 0\},$$

where each zero of $f - a$ with multiplicity m is repeated m times in $E_f(S)$.

In 1976, Gross proved [1] that there exist three finite sets S_j ($j = 1, 2, 3$) such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical. In the same paper Gross posed the following question (see [1, Question 6]): Can one find two (or possibly even one) finite sets S_j ($j = 1, 2$) such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

The present author proved the following result which is partial answer of the above question.

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THEOREM A. (see [7, Theorem 3]). Let $S_1 = \{w \mid w^n - 1 = 0\}$, $S_2 = \{a, b\}$, where $n (> 6)$ is a positive integer, a and b are constants such that $ab \neq 0$, $a^n \neq b^n$, $a^{2n} \neq 1$, $b^{2n} \neq 1$ and $a^n b^n \neq 1$. Suppose that f and g are nonconstant entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$. Then $f \equiv g$.

For the meromorphic case, the present author proved a corresponding theorem, which is the generalisation of Theorem A.

THEOREM B. (see [7, Theorem 2]). Let S_1 and S_2 be defined as in Theorem A, and let $S_3 = \{\infty\}$. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$. Then $f \equiv g$.

The set S such that for any two nonconstant entire functions f and g the condition $E_f(S) = E_g(S)$ implies $f \equiv g$ is called a unique range set (URS in brief) of entire functions (see [2]). In 1982, Gross and Yang proved the following result.

THEOREM C. (see [2, Theorem 3]). The set $S = \{w \mid e^w + w = 0\}$ is a URS of entire functions.

Note that the set $S = \{w \mid e^w + w = 0\}$ contains an infinite number of elements and so Theorem C does not answer the question posed by Gross.

In this paper we give a positive answer to Gross' question. In fact, we prove more generally the following theorem.

THEOREM 1. Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where n and m are two positive integers such that n and m have no common factors and $n \geq 2m + 5$, a and b are two nonzero constants such that the algebraic equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Then the set S is a URS of entire functions.

By Theorem 1, we immediately obtain the following corollary.

COROLLARY 1. Let $S = \{w \mid w^7 + aw^6 + b = 0\}$, where a and b are two non-zero constants such that $b \neq -6^6(a/7)^7$. Then the set S is a URS of entire functions with 7 elements.

For meromorphic functions, we have the following result, which is an extension of Theorem 1.

THEOREM 2. Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where n and m are two positive integers such that $m \geq 2$, $n \geq 2m + 7$ with n and m having no common factors, a and b are two nonzero constants such that the algebraic equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. Then $f \equiv g$.

By Theorem 2, we immediately obtain the following corollary.

COROLLARY 2. Let $S = \{w \mid w^{11} + aw^9 + b = 0\}$, where a and b are two non-zero constants such that $b^2 \neq -2^2 9^9 (a/11)^{11}$. Then for any two nonconstant meromorphic

functions f and g , the conditions $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ imply $f \equiv g$.

2. SOME LEMMAS

The following lemmas will be needed in the proof of our theorems.

LEMMA 1. (see [6]). *Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If*

$$c_1 f + c_2 g = c_3,$$

then

$$T(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, f).$$

LEMMA 2. (see [4]). *Let f_1, f_2, \dots, f_n be linearly independent meromorphic functions satisfying*

$$\sum_{j=1}^n f_j = 1.$$

Then for $k = 1, 2, \dots, n$ we have

$$\begin{aligned} T(r, f_k) < \sum_{j=1}^n N\left(r, \frac{1}{f_j}\right) + N(r, f_k) + N(r, D) - \sum_{j=1}^n N(r, f_j) \\ - N\left(r, \frac{1}{D}\right) + o(T(r)) \quad (r \notin E), \end{aligned}$$

where D denotes the Wronskian of the functions f_1, f_2, \dots, f_n and $T(r)$ denotes the maximum of $T(r, f_j)$, $j = 1, 2, \dots, n$.

LEMMA 3. (see [5]). *Let f be a nonconstant meromorphic function, and let $P(f)$ be a polynomial in f of the form*

$$P(f) = a_0 f^n + a_1 f^{n-1} + \dots + a_{n-1} f + a_n,$$

where $a_0 (\neq 0), a_1, \dots, a_n$ are constants. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

3. PROOF OF THEOREM 2

Let w_1, w_2, \dots, w_n be the roots of the equation $w^n + aw^{n-m} + b = 0$. Since $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, we have from Nevanlinna's second fundamental theorem

$$\begin{aligned} (n - 1)T(r, g) &< \sum_{j=1}^n N\left(r, \frac{1}{g - w_j}\right) + N(r, g) + S(r, g) \\ &= \sum_{j=1}^n N\left(r, \frac{1}{f - w_j}\right) + N(r, f) + S(r, g) \\ &\leq (n + 1)T(r, f) + S(r, g). \end{aligned}$$

Thus

$$(1) \quad T(r, g) < \frac{n + 1}{n - 1}T(r, f) + S(r, g).$$

In the same manner as above, we have

$$(2) \quad T(r, f) < \frac{n + 1}{n - 1}T(r, g) + S(r, f).$$

Again by $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, we obtain

$$(3) \quad \frac{f^n + af^{n-m} + b}{g^n + ag^{n-m} + b} = e^h,$$

where h is an entire function. From Lemma 3, (1) and (3) we have

$$\begin{aligned} (4) \quad T(r, e^h) &\leq T(r, f^n + af^{n-m} + b) + T(r, g^n + ag^{n-m} + b) + O(1) \\ &= nT(r, f) + nT(r, g) + S(r, f) \\ &< \frac{2n^2}{n - 1}T(r, f) + S(r, f). \end{aligned}$$

Let us put

$$(5) \quad f_1 = -\frac{1}{b}f^{n-m}(f^m + a),$$

$$(6) \quad f_2 = e^h,$$

$$(7) \quad f_3 = \frac{1}{b}g^{n-m}(g^m + a)e^h,$$

and let $T(r)$ denote the maximum of $T(r, f_j)$, $j = 1, 2, 3$. From (3), (5), (6) and (7), we obtain

$$(8) \quad f_1 + f_2 + f_3 = 1.$$

From (1), (4), (5), (6) and (7) we have

$$(9) \quad T(r) = O(T(r, f)) \quad (r \notin E).$$

Next, we need the following lemma.

LEMMA 4. f_1, f_2 and f_3 are linearly dependent.

PROOF: Suppose that f_1, f_2 and f_3 are linearly independent. Applying Lemma 2 to the functions f_j ($j = 1, 2, 3$), from (8) and (9) we have

$$(10) \quad T(r, f_1) < \sum_{j=1}^3 N\left(r, \frac{1}{f_j}\right) - N\left(r, \frac{1}{D}\right) + N(r, D) - N(r, f_2) - N(r, f_3) + S(r, f),$$

where

$$(11) \quad D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}.$$

From (5), (6) and (7) we have

$$\begin{aligned} \sum_{j=1}^3 N\left(r, \frac{1}{f_j}\right) &= (n-m)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m+a}\right) \\ &\quad + (n-m)N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^m+a}\right). \end{aligned}$$

By looking at the zeros of f and g , from (5), (6), (7) and (11) we see that

$$(13) \quad N\left(r, \frac{1}{D}\right) \geq (n-m)N\left(r, \frac{1}{f}\right) - 2\bar{N}\left(r, \frac{1}{f}\right) + (n-m)N\left(r, \frac{1}{g}\right) - 2\bar{N}\left(r, \frac{1}{g}\right).$$

From (8) and (11) we get

$$(14) \quad D = \begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}.$$

Since f_2 is entire, from (7) and (14) we have

$$(15) \quad \begin{aligned} N(r, D) - N(r, f_2) - N(r, f_3) &\leq N(r, f_3'') - N(r, f_3) \\ &= 2\bar{N}(r, f_3) = 2\bar{N}(r, g) = 2\bar{N}(r, f). \end{aligned}$$

From Lemma 3, (5), (10), (12), (13) and (15) we deduce

$$\begin{aligned} nT(r, f) &< 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m + a}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + N\left(r, \frac{1}{g^m + a}\right) + 2\overline{N}(r, f) + S(r, f) \\ &< (4 + m)T(r, f) + (2 + m)T(r, g) + S(r, f). \end{aligned}$$

From this and (1) we obtain

$$(16) \quad nT(r, f) < \left(2m + 6 + \frac{2(2 + m)}{n - 1}\right)T(r, f) + S(r, f).$$

Since $n \geq 2m + 7$, (16) is a contradiction, which proves Lemma 4. \square

Next we proceed to prove Theorem 2.

By Lemma 4 we know that f_1 , f_2 and f_3 are linearly dependent. Then there exist three constants c_1 , c_2 and c_3 , at least one of which is not zero, such that

$$(17) \quad c_1 f_1 + c_2 f_2 + c_3 f_3 = 0.$$

If $c_1 = 0$, from (17) we have $c_2 \neq 0$, $c_3 \neq 0$ and

$$f_3 = -\frac{c_2}{c_3} f_2.$$

Hence, from (6) and (7) we obtain

$$g^n + ag^{n-m} = -bc_2/c_3,$$

which is impossible. Thus $c_1 \neq 0$ and

$$(18) \quad f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3.$$

Now combining (8) and (18) we get

$$(19) \quad \left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1.$$

We discuss the following three cases.

CASE I. Assume $c_1 \neq c_2$ and $c_1 \neq c_3$. From (6), (7) and (19) we have

$$(20) \quad -\frac{1}{b} \left(1 - \frac{c_3}{c_1}\right) g^{n-m}(g^m + a) + c^{-h} = 1 - \frac{c_2}{c_1}.$$

From (20) we know that g is an entire function. By Lemma 1, Lemma 3 and (20) we obtain

$$\begin{aligned} nT(r, g) &< \bar{N}\left(r, \frac{1}{g^{n-m}(g^m + a)}\right) + S(r, g) \\ &= \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^m + a}\right) + S(r, g) \\ &< (1 + m)T(r, g) + S(r, g), \end{aligned}$$

which is impossible.

CASE II. Assume $c_1 = c_2$. From (19) we have $c_1 \neq c_3$ and

$$(21) \quad f_3 = \frac{c_1}{c_1 - c_3}.$$

From (7) and (21) we get

$$(22) \quad g^{n-m}(g^m + a) = \frac{bc_1}{c_1 - c_3} e^{-h}.$$

Let a_1, a_2, \dots, a_m be the roots of equation $w^m + a = 0$. From (22) we know that $\infty, 0, a_1, a_2, \dots, a_m$ are Picard exceptional values of g , which is impossible.

CASE III. Assume $c_1 = c_3$. From (19) we have $c_1 \neq c_2$ and

$$f_2 = \frac{c_1}{c_1 - c_2}$$

that is

$$(23) \quad e^h = \frac{c_1}{c_1 - c_2}.$$

From (5), (7), (8) and (23) we get

$$(24) \quad -f^{n-m}(f^m + a) + \frac{c_1}{c_1 - c_2} g^{n-m}(g^m + a) = \frac{bc_2}{c_2 - c_1}.$$

If $c_2 \neq 0$, by Lemma 1 and Lemma 3, we have from (24),

$$\begin{aligned} nT(r, f) &< \overline{N}\left(r, \frac{1}{f^{n-m}(f^m + a)}\right) + \overline{N}\left(r, \frac{1}{g^{n-m}(g^m + a)}\right) + \overline{N}(r, f) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^m + a}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g^m + a}\right) + \overline{N}(r, f) + S(r, f) \\ &< (2 + m)T(r, f) + (1 + m)T(r, g) + S(r, f). \end{aligned}$$

From this and (1) we obtain

$$nT(r, f) < \left(3 + 2m + \frac{2(1 + m)}{n - 1}\right)T(r, f) + S(r, f),$$

which is impossible. Thus $c_2 = 0$. From (24) we deduce

$$(25) \quad f^n - g^n = -a(f^{n-m} - g^{n-m}).$$

If $f \not\equiv g$, from (25) we obtain

$$(26) \quad g^m = -\frac{a(H - v)(H - v^2) \dots (H - v^{n-m-1})}{(H - u)(H - u^2) \dots (H - u^{n-1})},$$

where $H = f/g$, $u = \exp((2\pi i)/n)$ and $v = \exp((2\pi i)/(n - m))$. From (26) we know that H is a nonconstant meromorphic function. Since n and m have no common factors, we have $u^j \neq v^k$ ($j = 1, 2, \dots, n - 1$; $k = 1, 2, \dots, n - m - 1$). Suppose that z_j is a zero of $H - u^j$ of order p_j . From (26) we have $p_j \geq m \geq 2$. Thus

$$(27) \quad \overline{N}\left(r, \frac{1}{H - u^j}\right) \leq \frac{1}{m}N\left(r, \frac{1}{H - u^j}\right) \leq \frac{1}{2}T(r, H) + O(1).$$

By the second fundamental theorem, from (27) we obtain

$$\begin{aligned} (n - 3)T(r, H) &< \sum_{j=1}^{n-1} \overline{N}\left(r, \frac{1}{H - u^j}\right) + S(r, H) \\ &< \frac{n - 1}{2}T(r, H) + S(r, H), \end{aligned}$$

which is impossible. Hence, $f \equiv g$.

This completes the proof of Theorem 2. □

4. SUPPLEMENT TO THEOREM 2

It is reasonable to ask: What can be said if $m = 1$ in Theorem 2? In this section, we prove the following theorem, which is a supplement to Theorem 2.

THEOREM 3. *Let $S = \{w \mid w^n + aw^{n-1} + b = 0\}$, where $n > 8$, and a and b are two nonzero constants such that the algebraic equation $w^n + aw^{n-1} + b = 0$ has no multiple roots. Suppose that f and g are two distinct nonconstant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. Then*

$$f = -\frac{aH(H^{n-1} - 1)}{H^n - 1} \quad \text{and} \quad g = -\frac{a(H^{n-1} - 1)}{H^n - 1},$$

where H is a nonconstant meromorphic function.

PROOF: Proceeding as in the proof of Theorem 2, we have

$$(28) \quad f^n - g^n = -a(f^{n-1} - g^{n-1}).$$

Noting $f \neq g$, from (28) we obtain

$$(29) \quad g = -\frac{a(H^{n-1} - 1)}{H^n - 1},$$

where $H = f/g$. From (29) we know that H is a nonconstant meromorphic function. Thus, from (29) we have

$$f = -\frac{aH(H^{n-1} - 1)}{H^n - 1}.$$

This completes the proof of Theorem 3. □

5. PROOF OF THEOREM 1

If $f \neq g$, noting $N(r, f) = N(r, g) = 0$ and proceeding as in the proof of Theorem 2, we can obtain (26). Since g is a nonconstant entire function, from (26) we know that u^j ($j = 1, 2, \dots, n-1$) are Picard exceptional values of H , which is impossible. Thus $f \equiv g$, which proves Theorem 1. □

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