BULL. AUSTRAL. MATH. SOC. 37D40, 37F30, 37F35
VOL. 65 (2002) [199–209]

HAUSDORFF DIMENSION OF THE LIMIT SET ON A VISIBILITY MANIFOLD

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In this paper, for a given Fuchsian group Γ, we prove an upper estimate for the Hausdorff dimension of the radial limit set in the visibility manifold. Further, if Γ is a convex cocompact group, we find the exact Hausdorff dimension of the limit set.

1. INTRODUCTION

Imagine an infinite array of points in hyperbolic space. We consider the distribution of these points at large distances from an observation point x. We define the density at the ideal boundary for the array of points viewed from x. That is a class of measures on the ideal boundary, which is called the conformal density or Patterson-Sullivan measure. Suppose H is a n-dimensional complete simply connected Riemannian manifold without conjugate points and Γ is a discrete group of isometries on H, which acts on H freely and properly discontinuously. As an infinite array of points, we consider the orbit Γx of Γ for a point $x \in H$. The conformal density for Γx was constructed by Patterson in the case where $\dim H = 2$ and the sectional curvature of H is constant -1 ([**7**]). His construction was generalised by Sullivan to the case where the sectional curvature of H is constant -1 in all dimensions ([**8**]). In [**9**], Yue performed the same construction when H has a variable negative curvature, and the author proved the existence and some properties of the Patterson-Sullivan measure on a visibility manifold ([**6**]).

We define the visibility manifold, following the notations in [**2**] and [**3**].

DEFINITION 1.1: H satisfies the visibility axiom if for every point $p \in H$ and every number $\varepsilon > 0$, there is $R = R(p, \varepsilon) > 0$ such that for any geodesic $\gamma : \mathbb{R} \to H$ with $d(p, \gamma) \geqslant R$, $\angle_p(\gamma) = \sup\Big\{\angle_p\big(\gamma(t), \gamma(s)\big) \mid t, s \in \mathbb{R}\Big\} \leqslant \varepsilon$.

In particular, when we can get the constant R independently of the choice of p, we say that H satisfies the uniform visibility axiom.

Received 14th June, 2001
This work was supported by a postdoctoral fellowship from KOSEF.

In Definition 1.1 the notation $\angle_p(q_1q_2)$ for q_1 and q_2 in H which means the canonical measurement for the angle consisting of two geodesic rays from p to q_1 and q_2.

In [**1**] there are many properties that are equivalent to the visibility axiom in Definition 1.1, and in this paper we use the property below frequently.

For any distinct two points η and ξ in ∂H, there exists a geodesic line between η and ξ.

We note that there may be more than one geodesics between η and ξ. If we have two geodesic between η and ξ, then the two geodesics bound a flat strip, and in the flat strip, the sectional curvature is 0. In this paper, we assume the uniform visibility axiom on H.

DEFINITION 1.2: Suppose M is a manifold without any conjugate points. If the universal cover H of M satisfies the uniform visibility axiom, we call M a *visibility manifold.*

Suppose H satisfies the uniform visibility axiom. Let ∂H be the ideal boundary, which is the set of points at infinity for H with the cone topology. Then H is diffeomorphic to an open disc D^n and the ideal boundary ∂H of H at infinity is homeomorphic to a sphere S^n. Let Γ be a subgroup of isometries of H, that is torsion free, discrete acting on H freely and properly discontinuously in H. For any point x in H, consider the orbit Γx and its closure $\overline{\Gamma x}$. The limit set of Γ is defined by $L(\Gamma) = \overline{\Gamma x} \cap \partial H$. According to Eberlein [**2**], $L(\Gamma)$ has one point, two points or infinitely many points. From now on, we deal with the case that $L(\Gamma)$ consists of infinitely many points, and call Γ the *Fuchsian* group. Generally, in a 2-dimensional manifold, Γ has been called a Fuchsian group, and in higher dimensional manifolds, Γ has been called a Kleinian group. Here we call Γ Fuchsian group in any dimensional manifold.

We introduce the construction and some properties of the Patterson–Sullivan measure, which were proved in [**6**] for the visibility manifold.

For positive real number s and two fixed points x, y in H, we consider the following Poincaré series

$$g_s(x,y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)},$$

where $d(x,\gamma y)$ is the hyperbolic distance in H. Then there is a positive number $\delta(\Gamma)$ such that $g_s(x,y)$ diverges for $s < \delta(\Gamma)$ and $g_s(x,y)$ converges for $s > \delta(\Gamma)$, that is independent of points $x,\ y \in H$.

Define a family of measures

$$\begin{aligned}\mu_x &= \lim_{s\to\delta(\Gamma)^+} \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}\delta_{\gamma y}, \quad s>\delta(\Gamma),\\ &= \lim_{s\to\delta(\Gamma)^+} \mu_x^s\end{aligned}$$

where $\delta_{\gamma y}$ is the Dirac mass at γy. When at $s=\delta(\Gamma)$ μ_x^s diverges, Γ is of divergence type. Otherwise, Γ is of convergence type. When Γ is of divergence type, μ_x is concentrated on $L(\Gamma)$. In [**6**], we proved that for any other point $x, x' \in H$, $\mu_{x'}$ and μ_x were absolutely continuous and moreover, the Radon–Nikodym derivative at $\xi \in L(\Gamma)$ was

$$\frac{d\mu_{x'}}{d\mu_x}(\xi) = e^{-\delta(\Gamma)\rho_{x.\xi}(x')}, \tag{1.1}$$

where $\rho_{x,\xi}(x')$ is a Busemann function.

For $\gamma\in\Gamma$ we find

$$\gamma^*\mu_x = \mu_{\gamma(x)}. \tag{1.2}$$

Generally, we call the family $\{\mu_x\}$ of measure on $L(\Gamma)$ satisfying (1.1) and (1.2) a $\delta(\Gamma)$-conformal density or an Patterson–Sullivan measure.

In this paper, we estimate the hyperbolic dimension of the limit set of $L(\Gamma)$. For a hyperbolic manifold of constant curvature -1, there is a canonical metric on the ideal boundary in the Poincaré model and the Hausdorff dimension is exactly $\delta(\Gamma)$ in the case where Γ is a convex cocompact group. In a hyperbolic manifold with variable negative curvature, there are many possible equivalent metrics on ∂H (see [**4, 5, 9**]). So the Hausdorff dimension is well defined. Our problem is whether there is natural class of metrics on the visibility manifold. In Section 2, we consider a metric on ∂H which was introduced by Kaimanovich and Hamenstädt for strictly negatively curved manifolds. We show that the metric on ∂H is still well defined on the visibility manifold and we estimate the Hausdorff measure of the radial limit set with respect to this metric. Finally, if Γ is a convex cocompact group, we find the exact Hausdorff dimension of limit set $L(\delta)$.

2. Hausdorff Dimension for Convex Cocompact Group

Let (X,d) be any metric space and $D\geqslant 0$ be a nonnegative constant. Let A be a subset of X. For each $\varepsilon>0$, consider

$$\mathcal{H}_\varepsilon^D(A) \equiv \inf\left\{\sum_{j=1}^\infty \delta_j^D \mid A\subset \bigcup_j B_{x_j}(\delta_j),\ \delta_j\leqslant\varepsilon \text{ and } x_j\in A\right\},$$

https://doi.org/10.1017/S0004972700020244 Published online by Cambridge University Press

where the infimum is taken among all coverings of A by balls of radius less than or equal to ε. The limit measure

$$\mathcal{H}_d^D(A) = \lim_{\varepsilon \to 0} \mathcal{H}_\varepsilon^D(A)$$

is called the D-dimensional Hausdorff measure of A. The Hausdorff dimension $HD(A)$ is defined to be

$$HD(A) \equiv \inf\{D \mid \mathcal{H}_d^D(A) = 0\} \equiv \sup\{D \mid \mathcal{H}_d^D(A) = \infty\}.$$

An easy consequence of this definition is that if $0 < \mathcal{H}_d^D(A) < \infty$ then $HD(A) = D$.

Fix a point $x_0 \in H$. For any x in H and $d > 0$, consider the shadow of the ball $B(x,d)$ from x_0 to ∂H defined by $O_{x_0}(x,d) = \{\eta \in \partial H \mid c_{x_0,\eta} \cap B(x,d) \neq \emptyset\}$, where $c_{x_0,\eta}$ is the geodesic ray from x_0 to η.

Definition 2.1: $\zeta \in \partial H$ is a *radial limit point* if for some $c > 0$ and $x \in H$, ζ belongs to infinitely many shadows $O_x(\gamma x, c)$, for $\gamma \in \Gamma$. We denote the radial limit set by $L^r(\Gamma)$.

In order to estimate the Hausdorff dimension of the radial limit set, first of all we have to define a metric on ∂H. We define a metric on ∂H, which was introduced by Kaimanovich [5] and Hamenstädt [4]. Fix a point $x \in H$.

Definition 2.2: For any two points $\eta, \xi \in \partial H$ let $D_x(\eta,\xi)$ be the minimum distance from x to a geodesic c from ξ to η. The geodesic metric is defined for all $\varepsilon > 0$ to be

$$d_x^\varepsilon(\eta,\xi) := e^{-\varepsilon D_x(\eta,\xi)}.$$

In this definition $D_x(.,.)$ is well defined because the set of the geodesics between two points in H, that is the flat part in H, has a finite width and the range of the distance from x to c is compact interval. We have to show that the metric d_x^ε is well defined.

Theorem 2.3. *There exists $\varepsilon_0 > 0$ such that d_x^ε is a metric for all $0 < \varepsilon < \varepsilon_0$ and $x \in M$.*

Proof: By the definition of d_x^ε, we need to prove only the triangle inequality. Let us choose the three points $\xi_1, \xi_2, \xi_3 \in \partial H$ and let c_1, c_2, c_3 be any geodesics between ξ_1, ξ_2 and ξ_2, ξ_3 and ξ_3, ξ_1, respectively. We can get a point $p \in c_3$ so that $d(x, c_3) = d(x,p)$ and let $R_0 = d(c_3, x) = d(p, x)$, where $d(.,.)$ is the distance induced from the given Riemannian metric. Let γ be the geodesic ray from x to p We prove the triangle inequality by the two steps below.

First, we consider the special case in which the geodesic ray from x to ξ_2 goes through p in the geodesic c_3. Then γ and c_3 orthogonally meet at p, that is,

$\angle_p(\xi_1, \xi_2) = \pi/2$ and $\angle_p(\xi_2, \xi_3) = \pi/2$. Since H satisfies the uniformly visibility axiom, we can get a constant $R = R(\pi/2)$ such that $d(c_1, p) \leqslant R$ and $d(c_2, p) \leqslant R$. So we have $d(x, c_1) \leqslant R_0 + R$ and $d(x, c_2) \leqslant R_0 + R$. Choose $\varepsilon_0 = (\ln 2)/R$. Then for all $\varepsilon < \varepsilon_0$,

$$\begin{aligned}\exp\bigl(-\varepsilon d(x, c_3)\bigr) &= \exp\left(-\varepsilon R_0\right)\\ &\leqslant 2\exp\bigl(-\varepsilon(R_0 + R)\bigr)\\ &\leqslant \exp\bigl(-\varepsilon d(x, c_1)\bigr) + \exp\bigl(-\varepsilon d(x, c_2)\bigr)\\ &\leqslant d_x^\varepsilon(\xi_1, \xi_2) + d_x^\varepsilon(\xi_2, \xi_3).\end{aligned}$$

Next, we consider the general case. For a real number $\theta \in [0, \pi]$, define $f(\theta)$ as the minimum distance $d(p, c)$ from p to a geodesic c between ξ_2 and any point in ∂H with $\angle_p(c) \leqslant \theta$. Then f is a decreasing function in θ and $f(\pi/2) \leqslant R = R(\pi/2)$. We suppose that $\angle_p(\xi_1, \xi_2) = \theta_0$. Then $\angle_p(\xi_2, \xi_3) \geqslant \pi - \theta_0$ because $\angle_p(\xi_1, \xi_3) = \pi$. Let $\varepsilon_0 = (\ln 2)/R$. Since f is a decreasing function in θ, it is allowed that $\exp\bigl(-\varepsilon f(\theta)\bigr) + \exp\bigl(-\varepsilon f(\pi - \theta)\bigr)$ has the minimum value $2\exp\bigl(-\varepsilon f(\pi/2)\bigr)$. Then for all $\varepsilon < \varepsilon_0$, we have

$$\begin{aligned}\exp\bigl(-\varepsilon d(x, c_3)\bigr) &= \exp\left(-\varepsilon R_0\right) \leqslant 2\exp\bigl(-\varepsilon(R_0 + R)\bigr)\\ &\leqslant 2\exp\Bigl(-\varepsilon + f\Bigl(\frac{\pi}{2}\Bigr)\Bigr)\\ &\leqslant \exp\Bigl(-\varepsilon\bigl(R_0 + f(\theta)\bigr)\Bigr) + \exp\Bigl(-\varepsilon\bigl(R_0 + f(\pi - \theta)\bigr)\Bigr)\\ &\leqslant \exp\bigl(-\varepsilon d(x, c_1)\bigr) + \exp\bigl(-\varepsilon d(x, c_2)\bigr)\\ &\leqslant d_x^\varepsilon(\xi_1, \xi_2) + d_x^\varepsilon(\xi_2, \xi_3).\end{aligned}$$

Since the above inequality is true for any geodesic between ξ_1 and ξ_3, we have the triangle inequality

$$d_x(\xi_1, \xi_3) \leqslant d_x(\xi_1, \xi_2) + d_x(\xi_2, \xi_3). \qquad \square$$

From now on, we suppose that H is a visibility manifold with nonpositive sectional curvature. For convenience, we make the followings definitions.

For fixed $x \in H$ and for $\xi, \eta \in \partial H$,

$$\begin{aligned}l_x(\xi, \eta) &= \sup\bigl\{t \geqslant 0 \mid d(\gamma_{x,\xi}, \gamma_{x,\eta}) \leqslant d_1\bigr\}\\ \rho_x^\varepsilon(\xi, \eta) &= \exp\bigl(-\varepsilon l_x(\xi, \eta)\bigr).\end{aligned}$$

This was suggested by Kaimanovich [5] and Hamenstädt [4], who showed that it is a metric for sufficiently small ε in a negatively curved cocompact manifold. The Lemma below says that this is still a metric in ∂H on the visibility manifold.

LEMMA 2.4. *There exists a constant $C > 0$ such that for $\xi, \eta \in \partial H$,*

$$l_x(\xi,\eta) \leqslant D_x(\xi,\eta) + C.$$

PROOF: Let q be the point on a geodesic c between ξ and η that is nearest to x. Let γ_1 and γ_2 be geodesic rays from x to ξ and η, respectively. Then in the triangles (x,q,ξ) and (x,q,η) the angles at the vertex q equal $\pi/2$, the distances from the point q to the ray γ_1 and γ_2 is bounded by $R_1 > 0$ that is independent of the choice η and ξ. Let p_1, p_2 be the points on the geodesic rays γ_1, γ_2 respectively such that $d(p_i, q) = d(q, \gamma_i)$ for $i = 1, 2$. Choose the points q_i on γ_i $(i = 1, 2)$ such that $d(x, q_i) = l_x(\xi, \eta)$, so that $d(q_1, q_2) = d_1$. Let $[q_1, q_2]$ be a geodesic segment between q_1 and q_2. Then there are the three cases:

(1) $d(x,q_1) = d(x,p_1)$,
(2) $d(x,q_1) < d(x,p_1)$,
(3) $d(x,q_1) > d(x,p_1)$.

First, we consider $d(x,q_1) = d(x,p_1)$ and $d(x,q_1) < d(x,p_1)$. Then we can easily see $l_x(\xi,\eta) \leqslant D_x(\xi,\eta)$.

Second, we consider $d(x,q_1) > d(x,p_1)$. Then we can easily check $\angle_{q_1}(q,\xi) > \pi/2$, because $\angle_{q_1}(q,\xi)$ is a exterior angle of the right triangle (q_1,p_1,q). By the visibility axiom, the distance from q_1 to the geodesic ray c from q to ξ is bounded above by $R_1 > 0$. Let p be the point on c nearest to q_1. Since $\angle_p(q,q_1) = \angle_q(p,x) = \pi/2$, we can get $d(q,p) \leqslant d_1$, so that $d(x,q_1) \leqslant d(x,q) + R_2$ for some positive constant R_2.

Summing the above three cases, we get a constant $C > 0$ so that

$$l_x(\xi,\eta) \leqslant d_x(\xi,\eta) + C.$$

∎

For convenience, we use the notation δ instead of $\delta(\Gamma)$ as a critical exponent of Γ. Before the estimate of the Hausdorff measure, we prove a Lemma below, called Sullivan's Shadow Lemma in [**6**], which will play an important role.

LEMMA 2.5. *Let $\{\mu_x\}_{x\in M}$ be a δ-conformal density of Γ. Suppose μ_x does not consist of a single atom. Then there are constants $C \geqslant 1$ and $b_0 \geqslant 0$ such that for all $b \geqslant b_0$*

$$C^{-1}e^{-\delta d(x,\gamma^{-1}x)} \leqslant \mu_x\big(O_x(\gamma^{-1}x, b)\big) \leqslant Ce^{-\delta d(x,\gamma^{-1}x)+2b\delta}.$$

THEOREM 2.6. *Let $\{\mu_x\}$ be δ-conformal density of Γ. There exists a constant $C > 0$ such that if A is a Borel subset of $L^r(\Gamma)$ with $\mu_x(A) > 0$, then $\mathcal{H}_d^{\delta/\varepsilon}(A) \leqslant C\mu_x(A)$, where $d = d_x^\varepsilon$.*

PROOF: Suppose $A \subset L^r(\Gamma)$ is a Borel subset. Let $x \in H$. Since $\mu_x(A) > 0$, almost every point of A is a density point of the measure in the sense that for almost every $a \in A$

$$\lim_{t\to 0} \frac{\mu_x\big[B(a,t)\cap A\big]}{\mu_x\big[B(a,t)\big]} = 1,$$

where $B(a,t) = \{\eta \in \partial H \mid d_x^\varepsilon(a,\eta) \leqslant t\}$ for sufficiently small $\varepsilon > 0$. Let $\Gamma = \{\gamma_i\}_{i=1}^\infty$. Let $\alpha > 0$ be any constant. We can take a set $A' \subset A$ and $t_0 > 0$, such that $\mu_x(A - A') < \alpha$ and

$$\frac{\mu_x\big(B(a,t)\cap A\big)}{\mu_x\big(B(a,t)\big)} > 1 - \alpha \tag{2.1}$$

for all $0 < t < t_0$ and all $a \in A'$.

Let $0 < b < b_0$, where b_0 is the constant in Lemma 2.5. Since every point in $A' \subset L^r(\Gamma)$ lies in infinitely many balls $O_x\big(\gamma_n^{-1}x, b\big)$, we can construct a covering $\Big\{O_x\Big(\gamma_{n_k}^{-1}x, b\Big)\Big\}$ of A' such that $O_x\Big(\gamma_{n_k}^{-1}x, b\Big)$ is a ball whose d_x^ε-radius r_k satisfies $r_k < t < t_0/2$, whose centre is outside the union $\bigcup_{i=1}^{k-1} O_x\big(\gamma_{n_i}^{-1}x, b\big)$, and so that $r_k \geqslant r_{k+1}$ for all k. Then the ball with the radii $r_k/2$ and the same centre are disjoint. Denote the union of these disjoint balls by Ω. By Lemma 2.5, there exists $C_1 > 0$ such that

$$\sum_{k\geqslant 1} e^{-\delta d\left(x,\gamma_{n_k}^{-1}x\right)} \leqslant C_1 \mu_x(\Omega).$$

By the definition of l_x and Lemma 2.4, there exists a constant $C_2 > 0$ such that for all k, $r_k/2 \leqslant C_2 e^{-\varepsilon d\left(x,\gamma_{n_k}^{-1}x\right)}$. Consequently, we have

$$\begin{aligned}
\mathcal{H}_t^{\delta/\varepsilon}(A') &\leqslant \sum_k r_k^{\delta/\varepsilon} = 2^{\delta/\varepsilon}\sum_k \left(\frac{r_k}{2}\right)^{\delta/\varepsilon} \\
&\leqslant C_3 2^{\delta/\varepsilon} \sum e^{-\delta d\left(\gamma_{n_k}^{-1}x,x\right)} \\
&\leqslant C_3 \mu_x(\Omega),
\end{aligned}$$

for some constants $C_3 > 0$, and by (2.1)

$$\mu_x(\Omega) \leqslant \frac{1}{1-\alpha}\mu_x(A).$$

Summarising these results, we have $\mathcal{H}_t^{\delta/\varepsilon}(A') \leqslant C\mu_x(A)$. Letting $t \to 0$, α goes to 0. So we obtain

$$\mathcal{H}^{\delta/\varepsilon}(A) \leqslant C\mu_x(A).$$

∎

THEOREM 2.7. *If there exists a δ-conformal density μ_x of Γ, then the Hausdorff dimension of the radial limit set with respect to d_x^ε satisfies*

$$HD\big(L^r(\Gamma)\big) \leqslant \frac{\delta}{\varepsilon}$$

for sufficiently small ε and for all $x \in H$.

PROOF: If $\mu_x\big(L^r(\Gamma)\big) > 0$, then Theorem 2.6 implies this result. Assume $\mu_x\big(L^r(\Gamma)\big) = 0$. We can get a cover of $L^r(\Gamma)$ as in the proof of Theorem 2.6. Then we have $\mathcal{H}_t^{\delta/\varepsilon}\big(L^r(\Gamma)\big) \leqslant C\mu_x(\Omega) \leqslant C\mu_x(\partial H) < \infty$ for some constant $C > 0$. ∎

We have an upper estimate of the Hausdorff dimension when Γ is a just Fuchsian group. Now we prove that if a Fuchsian group Γ is convex cocompact, $HD\big(L^r(\Gamma)\big)$ is exactly δ/ε with respect to the metric $d_x^\varepsilon = d$ on ∂H.

DEFINITION 2.8: Let Γ be a Fuchsian group of a simply connected visibility manifold.

(i) The geodesic hull of Γ is defined to be $G(\Gamma) = \big(L(\Gamma) \times L(\Gamma) - \text{ diag}\big) \times \mathbb{R}$, that is, the union of all geodesics γ in H with $\gamma(-\infty), \gamma(\infty) \in L(\Gamma)$. Γ is said to be geodesic cocompact if $G(\Gamma)/\Gamma$ is compact.

(ii) The convex hull of Γ is defined to be $H(\Gamma)$, that is, the smallest convex set in $\overline{H}$ containing $L(\Gamma)$. Γ is said to be convex cocompact if $H(\Gamma)/\Gamma$ is compact.

In general $G(\Gamma)$ is not a convex set, but $G(\Gamma)$ is a subset of $H(\Gamma)$.

THEOREM 2.9. *The followings are equivalent.*

(1) *Γ is convex cocompact.*

(2) *Γ is geodesic cocompact.*

(3) *For any point $\xi \in \partial H$, there is a constant $C > 0$ such that the geodesic ray γ from x to ξ is in the C-neighbourhood of a orbit of Γ.*

PROOF: To show (1) implies (2), we first suppose that $g(\Gamma)/\Gamma$ is compact. Since $G(\Gamma)$ is a subset of $H(\Gamma)$, $H(\Gamma)/\Gamma$ is compact.

To show (2) implies (1), we first have by the visibility axiom, that for any $\xi_1, \xi_2, \xi_3 \in \overline{H}$, every point of the interior part of a triangle (ξ_1, ξ_2, ξ_3) is in the $R(\pi/2)$-neighbourhood of the triangle, where $R(\pi/2)$ is a constant corresponding to $\pi/2$ in the uniform visibility axiom. Further, we get that $H(\Gamma)$ is in the $R(\pi/2)$-neighbourhood of the $G(\Gamma)$. So we get that the geodesic cocompactness of Γ implies the convex cocompact of Γ.

To show (3) implies (2), we first choose any two distinct points ξ and η in $L(\Gamma)$. Let $\gamma_{x\xi}$ and $\gamma_{x\eta}$ be the geodesic rays from x to ξ and η, respectively and let $\gamma_{\xi\eta}$ be

https://doi.org/10.1017/S0004972700020244 Published online by Cambridge University Press

the geodesic line from ξ to η. Then by the uniform visibility axiom, $\gamma_{x\xi}$ and $\gamma_{x\eta}$ are in the $R(\pi/2)$-neighbourhood of $\gamma_{\xi\eta}$. Since $\gamma_{x\xi}$ and $\gamma_{x\eta}$ are in the C-neighbourhood of a orbit of Γ, $\gamma_{\xi\eta}$ is in the $\big(C + R(\pi/2)\big)$-neighbourhood of a orbit of Γ, and Γ is geodesic cocompact.

To show (2) implies (3), we suppose Γ is geodesic cocompact. Then there is a constant $C_1 > 0$ such that any point of $G(\Gamma)$ has a distance less than C_1 from an orbit of Γ. Let $\xi \in \partial H$ be any point. Let $x \in H$ be a fixed point. Choose a distinct point $\eta \in \partial H$ from ξ. We can get a unique point p in a geodesic $\gamma_{\xi\eta}$ from ξ to η such that $d(x,p) = d(x,\gamma_{\xi\eta})$. Let $d(x,p) = C_2$. It is easy to see that $\gamma_{x\xi}$ is in the $R(\pi/2)$-neighbourhood of $\gamma_{p\xi}$ and γ_{xp}. By the compactness of $G(\Gamma)/\Gamma$, $\gamma_{p\xi}$ is in the C_1-neighbourhood of a orbit of Γ. And there exists a constant $C_3 > 0$ such that the ball with centre x and the radius $R(\pi/2)$ is in the C_3 neighbourhood of a orbit of Γ. Let $C = \min\{C_1 + C_2, C_3\}$. Therefore the geodesic ray $\gamma_{x\xi}$ is in the C-neighbourhood of a orbit of Γ. ∎

Theorem 2.9 says that every limit point in the convex cocompact set is a radial limit point and $L(\Gamma) = L^r(\Gamma)$.

LEMMA 2.10. *Suppose Γ is a convex cocompact group. Let μ_x be a δ-conformal density. Then there exists $C > 0$ and $r_0 > 0$ such that*

$$\mu_x\big(B(\xi,r)\big) \leqslant C r^{\delta/\varepsilon},$$

where $B(\xi,r) = \big\{\eta \in \partial H \mid d_x^{\varepsilon}(\xi,\eta) < r\big\}$ is a ball with centre $\xi \in L(\Gamma)$ and radius $r_0 > r > 0$.

PROOF: By Theorem 2.8, there is a $C_1 > 0$ such that any point on a geodesic ray from $x \in H$ has a point of the orbit of Γ in a distance $C_1 > 0$. Consider $\xi \in L(\Gamma)$ and $e^{-\varepsilon R} > r > 0$ where $R = R(\pi/2)$ is the constant corresponding to the angle $\pi/2$ in the definition of the uniform visibility axiom. Choose a point u on the geodesic ray from x to ξ such that $d(x,u) = -\ln r/\varepsilon$. We can get a point $\alpha^{-1}(x)$ of orbit $\Gamma(x)$ such that $d\big(u,\alpha^{-1}(x)\big) \leqslant C_1$ for some $\alpha \in \Gamma$. Let η be a point in $B(\xi,r)$. Let $p \in H$ be the point on the geodesic line $\gamma_{\xi\eta}$ between ξ and η such that $d(x,p) = d(x,\gamma_{\xi\eta})$. Then we have that both of $\angle_p(\xi,x)$ and $\angle_p(\eta,x)$ are $\pi/2$, By the definition of uniform visibility, we get points q_1 and q_2 on the geodesic rays $\gamma_{x\xi}$ and $\gamma_{x\eta}$, respectively, such that both of $d(p,q_1)$ and $d(p,q_2)$ are less than and equal to $R(\pi/2)$. By the triangle inequality, we have $d(q_1,q_2) \leqslant 2R$, and by the angle comparison of triangle, we have $d(u,q_1) \leqslant R$. Since

$$d\big(\alpha^{-1}(x),\gamma_{x\eta}\big) \leqslant d\big(\alpha^{-1}(x),u\big) + d(u,q_1) + d(q_1,q_2) \leqslant C_1 + 3R,$$

we have $\eta \in O_x\big(\alpha^{-1}(x), C_1 + 3R\big)$. So it means that $B(\xi,r) \subset O_x\big(\alpha^{-1}(x), C_1 + 3R\big)$.

By Lemma 2.5, we have

$$\mu_x\big(B(\xi,r)\big) \leqslant \mu_x\Big(O_x\big(\alpha^{-1}(x), C_1+3R+d_1\big)\Big) \leqslant C_2 e^{-\delta d\left(x,\alpha^{-1}(x)\right)}.$$

where $C_2 > 0$ is a constant independent of x and r. Since $d\big(x,\alpha^{-1}(x)\big) > -\ln r/\varepsilon - C_1 > 0$, we have a constant $C > 0$ such that

$$\mu_x\big(B(\xi,r)\big) \leqslant C r^{\delta/\varepsilon}.$$

$\square$

THEOREM 2.11. *Let Γ be a convex cocompact group with critical exponent $\delta = \delta(\Gamma)$. Then there exists a constant $C > 0$ such that if A is a Borel subset of the limit set then we have*

$$\mu_x(A) \leqslant C\mathcal{H}_d^{\delta/\varepsilon}(A).$$

PROOF: Let $\{B(\xi_i,r_i)\}$ be any cover of A where the balls $B(\xi_i,r_i)$ in ∂H have centres on the limit set. Let $\varepsilon > 0$. Then by Theorem 2.10, we get for $\varepsilon > r_i > 0$,

$$\mu_x(A) \leqslant \sum_i \mu_x\big(B(\xi_i,r_i)\big) \leqslant C\sum_i r_i^{\delta/\varepsilon}.$$

Letting $\varepsilon \to 0$, we have a constant $C > 0$ such that

$$\mu_x(A) \leqslant C\mathcal{H}_d^{\delta/\varepsilon}(A).$$

$\square$

THEOREM 2.12. *If Γ is a convex cocompact group with critical exponent δ then the Hausdorff dimension of the limit set is δ/ε with respect to the metric d_x^ε.*

PROOF: By Theorem 2.9, we have $L^r(\Gamma) = L(\Gamma)$. Since $\mu_x\big(L^r(\Gamma)\big) = \mu_x\big(L(\Gamma)\big)$ has the positive full measure, Theorem 2.6 implies $HD\big(L(\Gamma)\big) \leqslant \delta/\varepsilon$. By Theorem 2.11, we have $HD\big(L(\Gamma)\big) \geqslant \delta/\varepsilon$, which means that the Hausdorff dimension of the limit set is δ/ε exactly. $\square$

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