Theorems connected with three mutually tangent circles.*

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1. The communication on this subject, as originally made to the society, consisted of a series of theorems, giving (1) expressions for the radii of a great many sets of circles, (2) identities connecting several sets of these radii, and (3) miscellaneous identities closely related thereto. As, however, the paper culminated in a general theorem which may be looked upon as fundamental, and the proof of which makes evident the mode of arriving at the said expressions for radii, and as the relations connecting sets of radii are easily found when attention has been directed to their existence, I have thought it best to print little more than the fundamental theorem and a few auxiliary notes.

2. The theorem is—

If the radii of the two smaller semi-circles of an arbelos† be a *and* b, *and a circle be inscribed in the arbelos, then circles in the three curvilineal triangles cut off by the preceding circle, then circles in the nine curvilineal triangles cut off by the three immediately preceding circles, and so on* ad libitum, *the expression for the radius of any one of these circles is of the form*

$$\frac{ab(a+b)}{\xi a^2 + \eta ab + \zeta b^2}$$

where ξ, η, ζ *are integers.*

We know that if r_1, r_2, r_3 be the radii of three circles in mutual contact, r_4 the radius of one of the circles touching all the said three, and r_5 the radius of the other, then

$$\frac{1}{r_4} + \frac{1}{r_5} = \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3}\ ; \qquad \text{(A)}$$

it being understood that when one of the circles *encloses* the four others its radius is made negative. Now, should it happen that r_1, r_2, r_3, and r_4 are each of the form specified in the theorem, their reciprocals must have the common denominator $ab\ (a+b)$, and hence from (A) r_5 must be of that form likewise. But the radii of the three original circles of the arbelos are of the form specified, for

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* This paper was read at the February meeting.

† See Mr Mackay's paper, pp. 2-11 and fig. 33.

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$$a = \frac{ab(a+b)}{0.a^2 + 1.ab + 1.b^2},$$

$$b = \frac{ab(a+b)}{1.a^2 + 1.ab + 0.b^2},$$

$$\text{and } a+b = \frac{ab(a+b)}{0.a^2 + 1.ab + 0.b^2};$$

and it is known that the radius of the first inscribed circle is so also, being

$$= \frac{ab(a+b)}{a^2 + ab + b^2};$$

hence each of the three next radii has the specified form. The same then follows regarding each of the nine radii succeeding these, and so on. The theorem is thus established.

3. In the proof we have assumed that the radius of the first inscribed circle is known. It is not necessary, however, to do so; for the formula (A) suffices to find it.

The three circles in mutual contact are in this case the three original circles of the arbelos, with the radii a, b, $a+b$; r_4 is unknown, but we see from symmetry that it is the same as r_5; hence from (A) we have

$$\frac{1}{r_5} + \frac{1}{r_5} = \frac{2}{a} + \frac{2}{b} - \frac{2}{a+b},$$

$$\text{and } \therefore \ \frac{1}{r_5} = \frac{a^2 + ab + b^2}{ab(a+b)},$$

as was to be shown.

4. If we denote the reciprocal of the radius

$$\frac{ab(a+b)}{\xi a^2 + \eta ab + \zeta b^2}$$

by (ξ, η, ζ), such calculations of radii can be made with great rapidity. Turning for a moment to the diagram (fig. 33), where, be it observed, the reciprocal of the radius of any circle is given in this notation at the centre, let us calculate the radius of the circle PQR. PQR touches the three mutually tangent circles (1, 1, 0), (1, 1, 1), (0, 1, 0), and the other circle which does so is the circle (0, 1, 1); hence

$$\begin{aligned} (\xi, \eta, \zeta) + (0, 1, 1) &= 2\{(1, 1, 0) + (1, 1, 1) - (0, 1, 0)\}, \\ &= 2\{(2, 2, 1) - (0, 1, 0)\}, \\ &= (4, 2, 2); \\ \therefore \quad (\xi, \eta, \zeta) &= (4, 1, 1), \end{aligned}$$

as appears on the diagram.

5. The radius of the n^{th} circle of any particular set of circles which have properties in common is readily obtainable. The following sets are noteworthy; even the few members of them which appear in the diagram enable one to tell the radius of the n^{th} member.

$$(0, 1, 1), (1, 1, 1), (4, 1, 1), (9, 1, 1), \ldots\ldots\ldots (n^2, 1, 1)$$
$$(0, 1, 1), (1, 1, 4), (4, 1, 9), (9, 1, 16), \ldots\ldots\ldots (n^2, 1, \overline{n+1}^2)$$
$$(0, 1, 1), (1, 1, 9), (4, 1, 25), (9, 1, 49), \ldots\ldots (n^2, 1, \overline{2n+1}^2)$$
$$(0, 1, 1), (1, 1, 16), (4, 1, 49), \ldots\ldots\ldots\ldots\ldots\ldots (n^2, 1, \overline{3n+1}^2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The circles of the first row extend up the horn whose tip is at A, those of the second row up a horn whose tip is at G, and so on.

Similarly we have the series

$$(1, 1, 0), (1, 1, 1), (1, 1, 4), (1, 1, 9), (1, 1, 16), \ldots (1, 1, n^2)$$
$$(1, 1, 0), (4, 1, 1), (9, 1, 4), (16, 1, 9), \ldots\ldots\ldots\ldots (\overline{n+1}^2, 1, n^2)$$
$$(1, 1, 0), (9, 1, 1), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\overline{2n+1}^2, 1, n^2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where every triad of coefficients is got by reversing the order of the corresponding triad in the previous sets, *i.e.*, as we should expect, by interchanging a and b.

Next we have the series which extends up the horn whose tip is C, viz. :—

$$(1, 1, 1), (4, 7, 4), (9, 17, 9), \ldots\ldots\ldots\ldots\ldots\ldots (n^2, 2n^2-1, n^2).$$

Then there are the lateral series

$$(4, 1, 1), (4, 1, 9), (4, 1, 25), \ldots\ldots\ldots\ldots\ldots\ldots (4, 1, \overline{2n-1}^2)$$
$$(4, 7, 4), (4, 7, 12), (4, 7, 28), \ldots\ldots\ldots\ldots\ldots\ldots (4, 7, 4n^2-4n+4).$$

&c., &c.

6. Although in enunciating the theorem of § 2 I have spoken of the arbelos, it should be carefully noted that what we have got to do with is properly not semicircles but circles. We start with one circle (0, 1, 1) touching another (0, 1, 0) internally, then in the space between the two circumferences, and touching them both, we describe a series of circles, each one of which touches the one preceding it and the one following it. Now, there are two such series which can be symmetrically situated with respect to the common diameter AB; first,......the series, (1, 1, 4), (1, 1, 1), (1, 1, 0), (1, 1, 1), (1, 1, 4),.......the

largest of which has AC for diameter; second, the series of which the two largest have AC for a tangent. (See fig. 9.)

If this latter series be described, and circles inscribed in the curvilineal triangles thereby formed, and so on, as before, the radius of every one of these circles likewise can be expressed in the form $ab(a+b)/(\xi a^2+\eta ab+\zeta b^2)$, ζ being now, however, sometimes fractional.

The main series, corresponding to the series (1, 1, 0), (1, 1, 1), ... in the former diagram, is

$$(1,\ 1,\ \tfrac{1}{4}),\ (1,\ 1,\ \tfrac{9}{4}),\ (1,\ 1,\ \tfrac{25}{4}),\ldots\ldots\ldots$$

Each of the latter, it will be seen, falls between two of the former, the two series dovetailing into one with a common law of progression, viz. :—

$$(1,\ 1,\ 0),\ (1,\ 1,\ (\tfrac{1}{2})^2),\ (1,\ 1,\ 1^2),\ (1,\ 1,\ (\tfrac{3}{2})^2),\ (1,\ 1,\ 2^2),\ldots\ldots\ldots$$

Further, the two new lateral series are

$$(4,\ 1,\ 0),\ (4,\ 1,\ 4),\ (4,\ 1,\ 16),\ldots\ldots\ldots$$
$$(4,\ 7,\ 7),\ (4,\ 7,\ 19),\ (4,\ 7,\ 39),\ldots\ldots\ldots$$

Each of these also dovetails with the analogous series in the former diagram, the pair of combined series being

$$(4,\ 1,\ 0),\ (4,\ 1,\ 1),\ (4,\ 1,\ 4),\ (4,\ 1,\ 9),\ldots\ldots(4,\ 1,\ n^2);$$
$$(4,\ 7,\ 4),\ (4,\ 7,\ 7),\ (4,\ 7,\ 12),\ (4,\ 7,\ 19),\ldots(4,\ 7,\ n^2+2n+4).$$

If the one diagram be superposed on the other so that the two can be viewed as one, these and other ties of relationship are well illustrated. For example, it will also be seen that any corresponding pair of circles taken from the two lateral series of the one diagram, touch the two original circles (0, 1, 1), (0, 1, 0) in the points where a circle of the main series of the other diagram touches them.

But the field of such curiosities is unlimited.