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A NEW UPPER BOUND FOR THE SUM OF DIVISORS FUNCTION

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Abstract

Robin's criterion states that the Riemann hypothesis is true if and only if $\sigma(n) < e^{\gamma} n \log\log n$ for every positive integer $n \geq 5041$. In this paper we establish a new unconditional upper bound for the sum of divisors function, which improves the current best unconditional estimate given by Robin. For this purpose, we use a precise approximation for Chebyshev's ϑ-function.

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1. Introduction

Let n be a positive integer. The arithmetical function $\sigma : \mathbb{N} \to \mathbb{N}$ is defined by

$$\sigma(n) = \sum_{d \mid n} d$$

and denotes the sum of the divisors of n. The function σ is multiplicative and satisfies $\sigma(p^k) = (p^{k+1} - 1)/(p - 1)$ for every prime number p and every positive integer k. In 1913, Gronwall [9, page 119] found the maximal order of σ by showing that

$$\limsup_{n \to \infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}, \tag{1.1}$$

where $\gamma = 0.5772156\ldots$ denotes the Euler–Mascheroni constant. In the proof of (1.1), Gronwall invoked a result of Mertens [13, page 53], namely that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \sim e^{\gamma} \log x,$$

where p runs over primes not exceeding x. Under the assumption that the Riemann hypothesis is true, Ramanujan [14] showed that the inequality

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n$$

holds for all sufficiently large positive integers n. In 1983, Robin [15, Théorème 1] improved Ramanujan's result by showing that the Riemann hypothesis is true *if and only if*

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n \quad \text{for all } n \geq 5041. \tag{1.2}$$

This criterion for the Riemann hypothesis is called *Robin's criterion* and the inequality (1.2) is called *Robin's inequality*. Robin's inequality holds in many cases (see Choie *et al.* [7, Theorems 1.1–1.2 and Theorems 1.4–1.5], Grytczuk [10, Theorems 1 and 3–4], Banks *et al.* [4, Theorem 2], Solé and Planat [16, Theorem 10] and Broughan and Trudgian [6, Theorem 1]), but remains open in general.

In the other direction, Robin [15, Théorème 2] used a lower bound for Chebyshev's ϑ-function to show that the weaker inequality

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n + \frac{0.6483}{\log\log n} \tag{1.3}$$

holds unconditionally for every positive integer $n \geq 3$, which refines (1.1). If the Riemann hypothesis is false, then, by Robin's criterion, the inequality (1.2) is false when $\sigma(n)/n$ is large. However, by (1.3) the ratio $\sigma(n)/n$ cannot be too large. We note that the constant in (1.3) is an approximation to $(\sigma(12)/12 - e^{\gamma} \log\log 12) \log\log 12$, so a better constant can be achieved if we consider (1.3) for $n \geq n_0 > 12$. The advantage of the inequality (1.3) is that it holds for every positive integer n where $\log\log n$ is positive.

We obtain the following improvement of (1.3).

Theorem 1.1. *Set*

$$\mathcal{A} = \{1, 2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 120, 180, 240, 360, 2520\}.$$

For every positive integer $n \in \mathbb{N} \setminus \mathcal{A}$,

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n + \frac{0.1209}{(\log\log n)^2}. \tag{1.4}$$

2. Preliminaries

A positive integer n is called *colossally abundant* if there is an $\varepsilon > 0$ so that $\sigma(n)/n^{1+\varepsilon} \geq \sigma(k)/k^{1+\varepsilon}$ for every positive integer $k \geq 2$. Suppose M_1 and M_2 are consecutive colossally abundant numbers satisfying the inequality (1.2). Then, Robin [15, Proposition 1, page 192] showed that Robin's inequality (1.2) holds for every positive integer n such that $M_1 \leq n \leq M_2$. In 2006, Briggs [5] verified that Robin's inequality (1.2) holds for every colossally abundant number n with $5041 \leq n \leq 10^{10^{10}}$. Hence, Robin's inequality is fulfilled for every positive integer n so that

$$5041 \leq n \leq 10^{10^{10}}.$$

For further results on colossally abundant numbers, see [2].

Let φ be Euler's totient function. Since φ is multiplicative and $\varphi(p^k) = p^k(1 - 1/p)$ for any prime p and any positive integer k,

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

for every positive integer n. Let $n = q_1^{e_1} \ldots q_k^{e_k}$, where q_i are distinct primes and $e_i \geq 1$. It is easy to show (see, for example, [10, Lemma 2]) that σ and φ are connected by the identity

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{1+e_i}}\right) \frac{n}{\varphi(n)},$$

which implies the inequality

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)}. \tag{2.1}$$

3. Proof of Theorem 1.1

Let k be a positive integer. Throughout this section, we write $N_k = p_1 \ldots p_k$, where p_i is the ith prime number. Chebyshev's ϑ-function is defined by

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

where p runs over primes not exceeding x. Clearly,

$$\log N_k = \vartheta(p_k). \tag{3.1}$$

For $k_0 = \pi(e^{23.85981}) + 1 = 1\,009\,322\,602$, we have $p_{k_0} = 23\,024\,161\,471$. Applying (3.1) and [3, Theorem 1.1],

$$\log\log N_{k_0} = \log\vartheta(p_{k_0}) \leq \log p_{k_0} + \log\left(1 + \frac{0.15}{\log^3 p_{k_0}}\right) \leq 23.85983.$$

Hence, we get the upper bound

$$N_{k_0} \leq e^{e^{23.85983}} \leq 10^{10^{10}}. \tag{3.2}$$

Now we give the proof of Theorem 1.1. For this purpose, we use a lower bound for Chebyshev's ϑ-function, obtained by Dusart [8, Theorem 4.2] in 2016, and an upper bound for the product

$$\prod_{p \leq p_k} \left(1 - \frac{1}{p}\right)^{-1}$$

given in [3, Proposition 6.1].

Proof of Theorem 1.1. Let

$$a_0 = 0.01001 + 1/17.2835 \quad \text{and} \quad k_0 = \pi(e^{23.85981}) + 1 = 1\,009\,322\,602$$

and take $k \geq k_0$. Then $p_k \geq e^{23.85981}$. From (3.1) and the result of Dusart [8, Theorem 4.2],

$$\log N_k > p_k - \frac{0.01 p_k}{\log^2 p_k}. \tag{3.3}$$

Consider the function $f : (1, \infty) \to \mathbb{R}$ given by $t \mapsto \log(1 - 0.01/t^2) + 0.01001/t^2$. Note that $f'(t) \leq 0$ for every $t \geq 3.17$ and $\lim_{t\to\infty} f(t) = 0$. Hence, $f(t) \geq 0$ for every $t \geq 3.17$. Together with (3.3), this implies

$$\log \log N_k > \log p_k - \frac{0.01001}{\log^2 p_k}. \tag{3.4}$$

Note that $x \mapsto x + e^\gamma a_0/x^2$ is a strictly increasing function on $(\sqrt[3]{2e^\gamma a_0}, \infty)$. Combining this remark with (3.4),

$$e^\gamma \log \log N_k + \frac{e^\gamma a_0}{(\log \log N_k)^2} > e^\gamma \log p_k \Big(1 + \frac{a_0 - 0.01001}{\log^3 p_k}\Big). \tag{3.5}$$

Next, by [3, Proposition 6.1],

$$\prod_{p \leq p_k} \Big(1 - \frac{1}{p}\Big)^{-1} < e^\gamma \log p_k \Big(1 + \frac{1}{17.2835 \log^3 p_k}\Big).$$

Together with (3.5), this yields the inequality

$$e^\gamma \log \log N_k + \frac{e^\gamma a_0}{(\log \log N_k)^2} > \prod_{p \leq p_k} \Big(1 - \frac{1}{p}\Big)^{-1} = \frac{N_k}{\varphi(N_k)}. \tag{3.6}$$

Now, let n be a positive integer satisfying $N_k \leq n < N_{k+1}$. We use (2.1), (3.6), the inequality $e^\gamma a_0 \leq 0.1209$ and the fact that $N_k/\varphi(N_k) \geq n/\varphi(n)$ to get

$$e^\gamma \log \log n + \frac{0.1209}{(\log \log n)^2} \geq e^\gamma \log \log N_k + \frac{e^\gamma a_0}{(\log \log N_k)^2} > \frac{N_k}{\varphi(N_k)} \geq \frac{n}{\varphi(n)} > \frac{\sigma(n)}{n}.$$

Hence, the required inequality holds for every positive integer $n \geq N_{k_0}$. From (3.2), we conclude that the inequality (1.4) is correct for every positive integer $n \geq 10^{10^{10}}$. Since Robin's inequality holds for every positive integer n such that $5041 \leq n \leq 10^{10^{10}}$, the required inequality (1.4) also holds for every positive integer n with $5041 \leq n \leq 10^{10^{10}}$. A direct computation for smaller values of n completes the proof.

In a different direction, Ivić [12, Theorem 1] showed that the inequality

$$\frac{\sigma(n)}{n} < 2.59 \log \log n \tag{3.7}$$

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holds for every positive integer $n \geq 7$. After some improvements of the constant in (3.7) (see for example [1], [10] and [15]), the current best such inequality was found by Hertlein [11, Theorem 4] in 2016. Setting $\varepsilon_0 = 5.645 \times 10^{-7}$, he proved that the inequality

$$\frac{\sigma(n)}{n} < (1 + \varepsilon_0)e^{\gamma} \log\log n$$

holds for every positive integer $n \geq 5041$. Now let ε be a real number satisfying $0 < \varepsilon < \varepsilon_0$. We apply Theorem 1.1 to find an upper bound for the smallest positive integer $n_0(\varepsilon)$ so that the inequality

$$\frac{\sigma(n)}{n} < (1 + \varepsilon)e^{\gamma} \log\log n \tag{3.8}$$

holds for every positive integer $n \geq n_0(\varepsilon)$.

Corollary 3.1. *Let $a = 0.1209/e^{\gamma}$ and let ε be a real number with $0 < \varepsilon < \varepsilon_0$. Then the inequality (3.8) holds for every positive integer $n \geq \exp(\exp(\sqrt[3]{a/\varepsilon}))$.*

Proof. This is a direct consequence of Theorem 1.1. □

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