Proceedings of the Edinburgh Mathematical Society (1980), **23**, 145–150 ©

ON SOME FUNCTIONAL EQUATIONS FROM ADDITIVE AND NON-ADDITIVE MEASURES-I

by PL. KANNAPPAN

(Received 20th July 1978)

1. Introduction

It is known that while the Shannon and the Rényi entropies are additive, the measure entropy of degree β proposed by Havrda and Charvat (**7**) is non-additive. Ever since Chaundy and McLeod (**4**) considered the following functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_iy_j) = \sum_{i=1}^{m} f(x_i) + \sum_{j=1}^{n} f(y_j) \tag{1.1}$$

$$(x_i,\ y_j \geqq 0,\quad \Sigma x_i = 1 = \Sigma y_j)$$

which arose in statistical thermodynamics, (1.1) has been extensively studied (**1**, **5**, **6**, **8**). From the algebraic properties of symmetry, expansibility and branching of the entropy (viz. Shannon entropy H_n, etc.) one obtains the sum representation

$$\left(\text{viz. } H_n(P) = \sum_{i=1}^{n} f(P_i),\ P = (p_1, \ldots, p_n) \quad \text{with} \quad \sum_{i=1}^{n} p_i = 1\right) \ (\mathbf{13}),$$

which with the property of additivity yields the functional equation (1.1), (**9**, **10**).

If we write

$$g(p) = \frac{p^\beta - p}{2^{1-\beta}-1}, \quad p \in [0, 1], \quad (\beta \neq 1, 0^\beta = 0, 1^\beta = 1) \tag{1.2}$$

then the entropy of degree β (**7**)

$$H_n^\beta(P) = \frac{\sum_{i=1}^{n} p_i^\beta - 1}{2^{1-\beta}-1}$$

takes the sum form

$$H_n^\beta(P) = \sum_{i=1}^{n} g(p_i)$$

and g satisfies the functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(x_iy_j) = \sum_{i=1}^{m} g(x_i) + \sum_{j=1}^{n} g(y_j) + c \sum_{i=1}^{m} g(x_i) \cdot \sum_{j=1}^{n} g(y_j) \tag{1.3}$$

$$(x_i,\ y_j \geqq 0,\quad \Sigma x_i = 1 = \Sigma y_j)$$

similar to (1.1) with $c = (2^{1-\beta}-1)$, treated in (**3**, **11**).

The equations (1.1) and (1.3) were studied under various hypotheses: such as the equations holding for all positive integers m, n or (1.1) holding for only one pair $m=2$, $n=3$ and under some regularity condition on f or g etc. In this paper we propose to solve the functional equations (1.1) and (1.3) holding for some (arbitrary but) fixed pair (m, n) when the functions involved are Lebesgue measurable using a different technique and to show that the solutions depend upon the pair (m, n) and to obtain in the case of (1.3) more solutions than hitherto known.

2. Auxiliary result

Let $\Gamma_n=\{P=(p_1, \ldots, p_n)\colon p_i \geqq 0, \Sigma p_i=1\}$. The measurable solutions of the functional equations (1.1) and (1.3) are obtained in Section 3 based on the solution of the following functional equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} F(x_i, y_j)=0 \tag{2.1}$$

$(X=(x_i) \in \Gamma_m,\ Y=(y_j) \in \Gamma_n)$ holding for some fixed pair of integers m, $n \geqq 3$. Now, we prove the following lemma:

Lemma 1. *Let* $F\colon [0, 1]\times[0, 1] \to \boldsymbol{R}$ *(reals), be measurable in each variable and satisfy the functional equation* (2.1) *for fixed* m, n $(\geqq 3)$. *Then* F *is given by*

$$F(x, y)=F(0, y)(1-mx)+F(x, 0)(1-ny)-F(0, 0)(1-mx)(1-ny). \tag{2.2}$$

Proof. For fixed $(y_1, y_2, \ldots, y_n)=\boldsymbol{y} \in \Gamma_n$, define

$$h(x)=\sum_{j=1}^{n} F(x, y_j), \quad x \in [0, 1]. \tag{2.3}$$

Then h is measurable and (2.1) can be rewritten as

$$\sum_{i=1}^{m} h(x_i)=0 \quad (x_i \geqq 0, \Sigma x_i=1) \tag{2.4}$$

From (**12**), it follows that (for $m \geqq 3$)

$$h(x)=a(1-mx) \quad (x \in [0, 1]) \tag{2.5}$$

where a is a function of $y_1, \ldots, y_n$.

Using (2.3) and (2.5), we get

$$\sum_{j=1}^{n} F(x, y_j)=a(y_1, \ldots, y_n)(1-mx),$$

$$\sum_{j=1}^{n} F(0, y_j)=a(y_1, \ldots, y_n)$$

so that

$$\sum_{j=1}^{n} l_x(y_j)=0 \tag{2.6}$$

where

$$l_x(y)=F(x,y)-F(0,y)(1-mx), \quad \textit{for} \quad y\in[0,1], \tag{2.7}$$

is measurable. Again, from (**12**), it follows that (for $n\geqq 3$),

$$l_x(y)=b(x)(1-ny), \quad \text{for} \quad y\in[0,1]. \tag{2.8}$$

From (2.7) and (2.8) results, $b(x)=l_x(0)=F(x,0)-F(0,0)(1-mx)$ so that we obtain

$$F(x,y)=F(0,y)(1-mx)+F(x,0)(1-ny)-F(0,0)(1-mx)(1-ny)$$

which is precisely (2.2). This proves Lemma 1.

Remark 1. Suppose the equation (2.1) holds for 3 pairs (m,n), (γ,s) and (p,q). Then

$$\left.\begin{aligned} &F(0,y)(\gamma-m)x+F(x,0)(s-n)y-F(0,0)[(\gamma-m)x+(s-n)y\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-(\gamma s-mn)xy]=0\\ &F(0,y)(p-m)x+F(x,0)(q-n)y-F(0,0)[(p-m)x+(q-n)y\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-(pq-mn)xy]=0 \end{aligned}\right\} \tag{2.9}$$

From (2.9), it is easy to see that, $F(0,y)$ and $F(x,0)$ are given by

$$\left.\begin{aligned} F(0,y)&=F(0,0)\left[1-\frac{pqs-pqn-mns-\gamma sq+\gamma sn+mnq}{(\gamma-m)(q-n)-(p-m)(s-n)}y\right]\\ F(x,0)&=F(0,0)\left[1-\frac{pq\gamma-pqm-mn\gamma-\gamma sp+\gamma sn+mnp}{(s-n)(p-m)-(\gamma-m)(q-n)}x\right], \end{aligned}\right\} \tag{2.10}$$

provided $(\gamma-m)(q-n)-(s-n)(p-m)\neq 0$.

(2.10) gives explicit values for $F(0,y)$ and $F(x,0)$ in terms of $x, y, m, n, \gamma, s, p, q$.

It also follows from (2.9) that,

$$F(x,y)=F(0,y)(1-mx)=f(y)(1-mx)$$

where f is arbitrary, when (2.1) holds for the 2 pairs (m,n) and (m,s);

$$F(x,y)=F(x,0)(1-ny)=g(x)(1-ny)$$

where g is arbitrary, when (2.1) holds for the 2 pairs (m,n) and (γ,n);

$$F(x,y)=(1-mx)(1-ny), \tag{2.11}$$

when (2.1) holds for the 3 pairs (m,n), (γ,n) and (m,s).

From (2.11), it follows that, when (2.1) holds for all pairs (m,n), then

$$F(x,y)=0 \tag{2.12}$$

(a result found in (**2**), under the condition of continuity of F).

3. Solutions of the equations (1.1) and (1.3)

Let f, g: $[0,1]\to R$, be measurable and satisfy the functional equations (1.1) and (1.3), for a fixed pair m, n ($\geqq 3$) where c is a constant. When $c=0$, (1.3) and (1.1) are one and the same. We follow the convention $0\log 0=0$, $0^\beta=0$, $1^\beta=1$.

By letting

$$F(x, y) = g(xy) - yg(x) - xg(y) - cg(x)g(y), \tag{3.1}$$

for $x, y \in [0, 1]$, we see that F is measurable in each variable and that (1.3) takes the form (2.1). Thus by Lemma 1 and (3.1), we obtain

$$\begin{aligned} g(xy) = {} & yg(x) + xg(y) + cg(x)g(y) \\ & + g(0)(1 - y - cg(y))(1 - mx) + g(0)(1 - x - cg(x))(1 - ny) \\ & - g(0)(1 - cg(0))(1 - mx)(1 - ny) \end{aligned} \tag{3.2}$$

for $x, y \in [0, 1]$, where c is a constant.

First, let us consider the case when $c = 0$ (i.e the equation (1.1)). Then (3.2) becomes

$$g(xy) = yg(x) + xg(y) + d[1 - x - y + (m + n - mn)xy]$$

where

$$d = g(0). \tag{3.3}$$

i.e.

$$g(xy) - d - d(m + n - mn)xy = x(g(y) - d) + y(g(x) - d)$$

i.e.

$$h(xy) = xk(y) + yk(x), \quad (x, y \in [0, 1]) \tag{3.4}$$

where

$$h(x) = g(x) - d(1 + m + n - mn)x,$$

and

$$k(x) = g(x) - d$$

are measurable functions.

Now, (3.4) can be rewritten as

$$\frac{h(xy)}{xy} = \frac{k(y)}{y} + \frac{k(x)}{x}, \quad x, y \in]0, 1],$$

a Pexider equation, so that, the measurability of the functions imply,

$$k(x) = Ax \log x + Bx, \quad x \in]0, 1],$$

where A, B are constants, that is,

$$g(x) = Ax \log x + Bx + d, \quad x \in]0, 1]. \tag{3.5}$$

But, obviously (3.5) holds for $x = 0$ in view of (3.3) and the fact that $0 \log 0 = 0$.

Now, g given by (3.5) is a solution of (1.1), provided $B = (mn - m - n)d$. Thus we have proved the following theorem.

Theorem 1. *Let f: $[0, 1] \to \mathbf{R}$, be measurable and satisfy* (1.1) *for a fixed pair m, n* ($\geqq 3$). *Then f is a solution of* (1.1) *if and only if*

$$f(x) = Ax \log x + d(mn - m - n)x + d, \tag{3.6}$$

where A, d are constants.

Now, let us treat the remaining case when $c \neq 0$ in (3.2) (i.e. the equation (1.3)).

With $y = 1$, (3.2) gives

$$c[g(1)+(n-1)d]g(x)=x[dm(n-1)-(1+mcd)(g(1)+(n-1)d)]+dc(g(1)+(n-1)d). \tag{3.7}$$

Hence, when $g(1)+(n-1)d\neq 0$, we obtain

$$g(x)=\frac{1}{c}\left[\frac{dm(n-1)}{\alpha}-(1+mcd)\right]x+d, \quad x\in[0,1] \tag{3.8}$$

where

$$\alpha=g(1)+(n-1)d. \tag{3.9}$$

Now, g given by (3.8) is a solution of (1.3), provided

$$c\alpha^2+\alpha[1-(n-m)cd]-dm(n-1)=0. \tag{3.10}$$

Finally, when $g(1)+(n-1)d=0$, by taking $x_1=1=y_1$, $x_2=0=\ldots=x_m=y_2=\ldots=y_n$ in (1.3), we get $g(1)=0=g(0)$, so that (3.2) becomes

$$g(xy)=yg(x)+xg(y)+cg(x)g(y),$$

which can be reduced to a Cauchy equation

$$h(xy)=h(x)h(y), \quad \text{for} \quad x,y\in\,]0,1], \tag{3.11}$$

where

$$h(x)=1+c\frac{g(x)}{x}, \quad x\in\,]0,1]. \tag{3.12}$$

Now the solutions of (3.11) are given by

$$h(x)=0 \quad (x\in\,]0,1]), \qquad \text{or} \qquad h(x)=x^{\beta} \quad (x\in\,]0,1])$$

with the convention $1^{\beta}=1$.

The solution $h(x)=0$ is not possible, since $h(1)=1$ by (3.12) and $g(1)=0$. Thus (3.12) yields

$$g(x)=\frac{x(x^{\beta}-1)}{c}, \tag{3.13}$$

for $x\in\,]0,1]$. Evidently, since $g(0)=0$, (3.13) holds for $x=0$ also, provided $\beta\neq -1$. But the solution corresponding to $\beta=-1$ is contained in the solution (3.8) for $d=1/c$, $g(1)=0$. (The solution is surprisingly independent of m and n). Thus, we have proved the following theorem.

Theorem 2. *Let g: $[0,1]\rightarrow \boldsymbol{R}$ be measurable. Then g is a solution of the equation* (1.3) *holding for fixed m, $n\geqq 3$ if and only if g has either the form* (3.8) *with the condition* (3.10) *or the form* (3.13).

Remark 2. If $d=g(0)=0$ and $g(1)\neq 0$, then (3.8) yields

$$g(x)=-\frac{x}{c},$$

(a solution independent of m and n (**3**, **11**)). For examples, the solutions of (1.3) (arising from (3.8)) are $g(x)=-10x+2$ or $g(x)=-3x+2$, when $m=3=n$, $c=1$, $d=2$; $g(x)=10x-2$ or $g(x)=3x-2$ when $m=3=n$, $c=-1$, $d=-2$ etc.

But there exists no solution of (1.3) (arising from (3.8)) when $m=3=n$, $c=-1$, $d=2$ or $m=3=n$, $c=1$; $d=-2$ etc for the condition (3.10) is not satisfied in either case; $\alpha^2-\alpha+12=0$ and $\alpha^2+\alpha+12=0$ have no real solutions for $\alpha(=g(1)+2d)$.

Remark 3. If the equation (1.1) holds for all m and n, it follows from Theorem 1, (3.6), that $d=0$ and the measurable solution of (1.1) takes the form $f(x)=Ax\log x$.

If the equation (1.3) holds for all m and n, then the measurable solutions of (1.3) are given by $g(x)=-x/c$ or by (3.12) $g(x)=(x^{\delta+1}-x)/c$. For, the equation (3.10) for $m=n$ gives $c\alpha^2+\alpha-dm(m-1)=0$, which for any two distinct values of m yields $d=0$ so that the solution of (1.3) corresponding to (3.8) becomes $g(x)=-x/c$ and the other solution is given by (3.12).

Whereas the solution (3.13) of (1.3) is known (**3**, **11**), the solution (3.8) of (1.3) involving m and n is new. Further the solution of (1.3) is obtained in Theorem 2 for fixed pair m, n and under the weaker regularity condition on g i.e. under the Lebesgue measurability of g.

REFERENCES

(**1**) J. ACZÉL und Z. DARÓCZY, Charakterisierung der Entropien positiver ordnung und der Shannonschen Entropie, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 95–121.

(**2**) M. BEHARA, *Additive and non-additive measures of entropy with applications* (McMaster University report, 1978).

(**3**) M. BEHARA and P. NATH, *Additive and non-additive entropies of finite measurable partitions, Probability and Information Theory*, Vol. 2, Springer-Verlag, 1973, V296, 102–138.

(**4**) T. W. CHAUNDY and J. B. MCLEOD, On a functional equation, *Edin. Math. Notes* **43** (1960), 7–8.

(**5**) DARÓCZY, Über die charakterisierung der Shannonschen Entropy, *Proc. Colloq. on Information Theory*, I, *Jónos I 3o ly ai Math. Soc., Budapest* (1968), 135–139.

(**6**) Z. DARÓCZY, On the measurable solutions of a functional equation, *Acta Math. Acad. Sci. Hungar.* **22** (1971), 11–14.

(**7**) J. HAVRDA and F. CHARVAT, Quantification method of classification process, Concept of structural a-entropy, *Kybernetika (Prague)*, **3** (1967), 30–35.

(**8**) PL. KANNAPPAN, On Shannon's entropy, directed divergence and inaccuracy, *Z. Wahr. Verw. Geb.* **22** (1972), 95–100.

(**9**) PL. KANNAPPAN, On directed divergence and inaccuracy, *Z. Wahr. Verw. Geb.* **25** (1972), 49–55.

(**10**) PL. KANNAPPAN, On a functional equation connected with generalized directed divergence, *Aeq. Math.* (1974), 81–93.

(**11**) PL. KANNAPPAN, On generalization of some measures in information theory, *Glasnik Mat.* **29** (1974), 81–93.

(**12**) PL. KANNAPPAN, On a generalization of sum form functional equation -1 (submitted).

(**13**) C. T. NG, Representation of measures of information with the branching property, *Information and Control* **25** (1974), 45–56.

UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO