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EXTENSION FUNCTION AND SUBCATEGORIES OF HAUS

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ABSTRACT. For each Hausdorff space X, let FX be an Hausdorff extension of X. The existence of the largest subcategory of HAUS on which F is a functor and an epi-reflection is investigated.

Let HAUS denote the category of all Hausdorff spaces and continuous functions and HC the full subcategory of H-closed spaces of HAUS. One of the well-known extension functions from obj(HAUS) to obj(HC) is the correspondence that assigns to a Hausdorff space X its Katětov H-closed extension τX (τX is denoted as κX in [PT, PV2, PV3] in honor of Katětov). Herrlich and Strecker [HS1] have shown that κ is not a functor from HAUS to HC. Harris [H1, H2] found, as others have, a subcategory of HAUS on which κ is a functor, but surprisingly, Harris proved that his subcategory is the *largest* on which κ is a functor.

In this note, we have shown in a rather general setting the existence of "largest" subcategories on which an extension function is a functor and, under additional hypothesis, the existence of "largest" subcategories on which an extension function is an epi-reflection. This last result is of some importance in categorical topology since one of the main thrusts in categorical topology is identifying the epi-reflective subcategories of a fixed category whereas this result fixes a category and seeks categories of which the fixed category is an epi-reflective subcategory. Also, the results in this paper partially solve a problem by H. L. Bentley proposed in [Hu]. The author acknowledges and appreciate his useful conversations about this topic with F. Delahan and G. Strecker and thanks the referee for his suggestions.

We use the usual notation of $D \subseteq E$ to denote that a category D is a subcategory of a category E. Let $B \subseteq$ HAUS and A be a full subcategory of HAUS. Let F be an extension function from B to A, i.e., for each object X in B, FX is an object of A and FX is a topological extension of X. Let i denote the function whose domain is obj(B) such that for each $X \in$ obj(B), i_X is the inclusion function from X to FX. We say that A is *F-invariant* if $A \subseteq B$ and for each $X \in$ obj(A), $FX = X$.

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THEOREM. *There is a largest subcategory $C\subseteq B$ for which there exists a functor $G:C\to A$ whose object function is F and which satisfies this condition: for each $f\in M_c(X, Y)$, the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gf} \\ Y & \xrightarrow[i_Y]{} & FY \end{array}$$

commutes. The subcategory C has these properties.

(a) $\mathrm{Obj}(C)=\mathrm{obj}(B)$.

(b) *If A is F-invariant, then $A\subseteq C$ and C is the largest subcategory of B on which i is an A-epi-reflection.*

Proof. Construct C by letting $\mathrm{obj}(C)=\mathrm{obj}(B)$ and for $X, Y\in\mathrm{obj}(C)$, let $M_C(X, Y)=\{f\in M_B(X, Y):f$ has a continuous extension from FX to $FY\}$. Since $A, B\subseteq$ HAUS, then each $f\in M_C(X, Y)$ has a unique extension, denoted as Ff, from FX to FY. First, we prove that C is a category. Since, for $X\in\mathrm{obj}(C)$, $1_X\in M_B(X, X)$ is extended by 1_{FX}, then $1_X\in M_C(X, X)$. If $f\in M_C(X, Y)$ and $g\in M_C(Y, Z)$, then $Fg\circ Ff$ extends $g\circ f$ implying $g\circ f\in M_C(X, Z)$; also, the uniqueness property of the extension yields that $F(g\circ f)=Fg\circ Ff$. This completes the proof that C is a category and also proves that there is a functor from C to A whose object function is F; this functor from C to A is also denoted as F. From our construction of C, it immediately follows that C is the largest subcategory of B on which there is a functor whose object function is F and for which the above diagram commutes. Suppose A is F-invariant. For $X, Y\in\mathrm{obj}(A)$, each $f\in M_A(X, Y)$ is its own extension from $FX=X$ to $FY=Y$; hence, $A\subseteq C$. In particular, for $X, Y\in\mathrm{obj}(A)$, we have $M_A(X, Y)=M_C(X, Y)=M_B(X, Y)=M_{\mathrm{HAUS}}(X, Y)$ by the "fullness" of A is HAUS. Let $X\in\mathrm{obj}(C)$, $Y\in\mathrm{obj}(A)$, and $f\in M_C(X, Y)$. Then $FY=Y$, and since $FX, Y\in\mathrm{obj}(A)$, then $Ff\in M_A(FX, Y)$. This shows that i is an A-epi-reflection of C. Suppose i is an A-epi-reflection of a subcategory D of B. Then by the definition of A-reflection (see [HS2]), $A\subseteq D$. Since $\mathrm{obj}(D)\subseteq\mathrm{obj}(B)=\mathrm{obj}(C)$, then to show that $D\subseteq C$, it suffices to show for $X, Y\in\mathrm{obj}(D)$, $M_D(X, Y)\subseteq M_C(X, Y)$. Let $f\in M_D(X, Y)$. By the definition of A-reflection, $i_Y\in M_D(Y, FY)$. Hence, $i_Y\circ f\in M_D(X, FY)$. Again, by the definition of A-reflection, there is a continuous function $g:FX\to FY$ such that $g\circ i_X=i_Y\circ f$. So, f has a continuous extension from FX to FY implying that $f\in M_C(X, Y)$.

In the case that $A=B=$HAUS, i is a natural transformation from the identity functor from C to HAUS to the functor F. Here are a few examples of extension functions that satisfy the hypothesis of the Theorem.

1. The Theorem applies to any of the H-closed extension functions from HAUS to HC, e.g., the Fomin H-closed extension function σ[Fo, F, PT, PV3], the H-closed extension functions τ' and σ', defined by Katětov [K1, K2], and the Wallman H-closed extension function ω defined by Wenjen [W]. It should be

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noted that the largest subcategory, denoted as T, of HAUS on which σ is a functor is a subcategory of, but not equal to, the largest subcategory, denoted as S, of HAUS on which κ is a functor. To prove this fact, let $f \in M_T(X, Y)$ where $X, Y \in \text{obj}(T) = \text{obj}(\text{HAUS})$ and $g: \sigma X \to \sigma Y$ the continuous extension of f. To show $f \in M_S(X, Y)$, it suffices, by Theorem A in [H2], to show that if $\mathscr{A}$ is a p-cover (an open cover with the property that the union of some finite subfamily is dense) of Y, then $\{f^{-1}(U): U \in \mathscr{A}\}$ is a p-cover of X. By Theorem 7.3 in [PV3], $\mathscr{A}$ extends to an open cover $\mathscr{A}^\sigma$ of σY. Now, $\{g^{-1}(V): V \in \mathscr{A}^\sigma\}$ is an open cover of σX, and there is a finite subfamily $\mathscr{B} \subseteq \mathscr{A}^\sigma$ such that $\cup\{g^{-1}(V): V \in \mathscr{B}\}$ is dense in σX. Since $\mathscr{C} = \{V \cap Y: V \in \mathscr{B}\}$ is a finite subfamily of $\mathscr{A}$ and for $V \in \mathscr{B}$, $X \cap g^{-1}(V) = f^{-1}(V \cap Y)$, then $\cup\{f^{-1}(W): W \in \mathscr{C}\}$ is dense in X implying that $\{f^{-1}(U): U \in \mathscr{A}\}$ is a p-cover. To show that $M_T(X, Y) \neq M_S(X, Y)$, in general, let X be a countable infinite discrete space and Y be Urysohn's well-known example [Ex. 3.14, BPS] of a minimal Hausdorff (and, hence, H-closed) space that is not compact. Since Y has a countable infinite discrete dense subspace, there is a dense embedding map $f: X \to Y$. Since Y is H-closed, then $\kappa Y = Y = \sigma Y$ is an H-closed extension of $f(X)$, and by Theorem 4.4 in [PT], f has a continuous extension $g: \kappa X \to \kappa Y = Y$. Hence, $f \in M_S(X, Y)$. By Theorem 10 in [K2], $\beta X = \sigma X$. So, if there is a continuous extension $h: \sigma X \to \sigma Y = Y$, then Y would be compact; hence, $f \notin M_T(X, Y)$ and $M_S(X, Y) \neq M_T(X, Y)$.

2. Two more extension functions from HAUS that satisfy the hypothesis of the Theorem are the Liu α-closure function [L], from HAUS to the full subcategory of α-spaces and the Liu-Stecker almost real compactification [LS] function from HAUS to its full subcategory of almost real compact spaces. The morphisms of these largest subcategories have recently been characterized by Hunsaker and Naimpally in [HN].

3. The Theorem applies to the Banaschewski minimal Hausdorff extension [B, PT, PV2, H1] function from the category of semi-regular Hausdorff spaces and all continuous functions to its full subcategory of minimal Hausdorff spaces.

4. Another applicable situation is the extension function w, defined by Porter and Votaw in [PV1], from the category REG of regular Hausdorff spaces and all continuous functions to its full subcategory of OCE-regular spaces.

5. The first part of the Theorem applies to the extension function α defined by Alexandroff in [A], from REG to HAUS. A regular Hausdorff space X is known [T] that has the property that αX is not regular.

We conclude this note by formalizing the problem touched upon and motivated by part (b) of the Theorem.

PROBLEM. If A and B are categories (not necessarily subcategories of HAUS) and $A \subseteq B$, then identify those subcategories $C \subseteq B$ such that A is reflective (epi-reflective, coreflective, etc. . . .) in C.

It is interesting to observe that if $A \subseteq D \subseteq C$, A is reflective in C with r as the reflector (resp. A is coreflective in C with c as the coreflector), and $r_X \in M_D(X, A_X)$

(resp. $c_X \in M_D(A_X, X)$) for every $X \in \text{obj}(D)$, then A is reflective (resp. coreflective) in D. In particular, in the setting of part (b) of the Theorem, once the largest subcategory C of B has been identified, then A is reflective in a subcategory D of B if and only if $A \subseteq D \subseteq C$ and for each $X \in \text{obj}(D)$ $i_X \in M_D(X, FX)$.

References

A P. S. Alexandroff, *On bicompact extension of topological spaces* (Russian), Mat. Sbornik N.S. **5** (1939), 420–429.

B B. Banaschewski, *Hausdorffsch-minimale Erweiterungen von Räumen*, Arch. Math. **12** (1961), 355–365.

BPS M. P. Berri, J. R. Porter, and R. M. Stephenson, Jr., *A survey of minimal topological spaces, General Topology and its Relation to Modern Analysis and Algebra III* (Proc. Top. Conf. Kanpur, 1968), Academia, Prague, 1971, pp. 93–114.

F J. Flachsmeyer, *Zur theorie der H-abgeschlossen Erweiterungen*, Math. Z. **94** (1966), 349–381.

Fo S. Fomin, *Extension of topological spaces*, Ann. of Math. **44** (1943), 471–480.

H1 D. Harris, *Structures in topology*, Mem. Amer. Math. Soc. **115** (1971).

H2 ———, *Katětov Extension as a functor*, Math. Ann. **193** (1971), 171–175.

HS1 H. Herrlich and G. Strecker, *H-closed spaces and reflective subcategories*, Math. Ann. **177** (1968), 302–308.

HS2 ———, *Category Theory*, Allyn and Bacon, Boston, 1973.

Hu W. N. Hunsaker, *Problems in categorical topology*, Southern Illinois University Categorical Topology Conference, 1973.

HN W. N. Hunsaker and S. A. Naimpally, *Extensions of continuous functions; reflective functors* (submitted).

K1 M. Katětov, *On H-closed extension of topological space*, Časopis Pěst. Mat. Fys. **72** (1947), 17–32.

K2 ———, *On the equivalence of certain types of extension of topological spaces*, ibid. **72** (1947), 101–106.

L C. T. Liu, *The α-closure αX of a topological space X*, Proc. Amer. Math. Soc. **22** (1969), 620–624.

LS ——— and G. H. Strecker, *Concerning almost real compactifications*, Czech. Math. J. **22** (1972), 181–190.

PT J. R. Porter and J. D. Thomas, *On H-closed and minimal Hausdorff space*, Trans. Amer. Math. Soc. **138** (1969), 159–170.

PV1 ——— and C. Votaw, *S(α) spaces and regular Hausdorff extensions*, Pac. J. Math. **45** (1973), 327–345.

PV2 ———, *H-closed extensions I*, Gen. Top. and its Appl. **3** (1973), 211–224.

PV3 ———, *H-closed extensions II*, Trans. Amer. Math. Soc., to appear.

T H. Tong, *Solutions of problems of P. S. Alexandroff on extensions of topological spaces*, Ann. Mat. Pura Appl. (4) **86** (1970), 47–52.

W C. Wenjen, *On H-closedness and the Wallman H-closed extension* I, II Proc. Japan Acad. **46** (1970), 1102–1106 and **47** (1971), 383–393.

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