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SPECTRAL MAXIMAL PROJECTIONS AND PROJECTION-RELATIVE DECOMPOSABILITY ON BANACH SPACES

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Abstract. We define and study some properties of spectral maximal projections of a bounded operator on a complex Banach space. Then we apply these results to the new concepts of weakly projection-relative decomposable operators and projection-relative decomposable operators in the spirit of the works of C. Foias [**6**], A. Jafarian [**7**], I. Erdelyi and R. Lange [**5**].

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1. Introduction. Let X be a complex Banach space, $B(X)$ the algebra of all bounded linear operators on X, and $\mathbb{C}$ the field of complex numbers. For an operator $T \in B(X)$, $\sigma(T)$ is the spectrum of T and $\rho(T) = \sigma(T)^c$ its resolvent. For $\lambda \in \rho(T)$ we shall use the notation $R(\lambda, T) = (\lambda - T)^{-1}$. When f is an analytic function defined on an open neighborhood of $\sigma(T)$ we can define the bounded operator $f(T)$ on X by

$$f(T) = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)R(\lambda, T)d\lambda,$$

Γ being an admissible contour surrounding $\sigma(T)$. Let $T \in B(X)$. An *invariant bounded projection under* T is a bounded projection p on X such that pX is invariant for T. Invariant subspaces Y of X or invariant bounded projections p produce the restrictions $T \mid Y$ or T_p as well as the coinduced operators T^Y or T^p on the quotient spaces X/Y or X/pX. We say that Y or p are σ-invariant under T if $\sigma(T \mid Y) \subset \sigma(T)$ or $\sigma(T_p) \subset \sigma(T)$ which implies $\sigma(T) = \sigma(T \mid Y) \cup \sigma(T^Y)$ or $\sigma(T) = \sigma(T_p) \cup \sigma(T^p)$. Moreover Y or p are said to be *hyperinvariant for* T if Y or pX is invariant under each $R \in B(X)$ that commutes with T. $T \in B(X)$ is said to have the *single-valued extension property* if for every function $f : D \to X$ (D open in $\mathbb{C}$) analytic on D, the condition $(\lambda - T)f(\lambda) \equiv 0$ on D implies $f \equiv 0$. For such an operator, the *local resolvent set* $\rho_T(x)$ is defined for every $x \in X$ and there exists a unique X-valued analytic function $\tilde{x}_T$ satisfying the equation $(\lambda - T)\tilde{x}_T(\lambda) = x$ on $\rho_T(x)$. Lastly $X_T(F) = \{x \in X | \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$.

2. Spectral maximal projections.

DEFINITION 2.1. Given $T \in B(X)$, an invariant bounded projection p for T is called *a spectral maximal projection of* T if for any invariant bounded projection q under T, the inclusion $\sigma(T_q) \subset \sigma(T_p)$ implies $qX \subset pX$.

Remark 2.2. If Y is a spectral maximal space of $T \in B(X)$ such that Y is complemented in X, then there exists a spectral maximal projection p such that $Y = pX$. In particular, if X is a Hilbert space, the spectral maximal subspaces are exactly the invariant subspaces $Y = pX$ in which p is a spectral maximal projection of T.

Example 2.3. Let T be a quasispectral operator of class Γ with a spectral measure $E(.)$ of class Γ, then $X_T(F) = E(F)X$ for all closed $F \subset \mathbb{C}$ [**1**, Lemma 1]. Hence $E(F)$ is a spectral maximal projection of T for each closed $F \subset \mathbb{C}$.

Example 2.4. Let $T \in B(X)$ and $\sigma(T)$ be totally disconnected. Let δ be a separate part of $\sigma(T)$ and $A_T(\delta) = \frac{1}{2\pi i}\int_\Gamma R(\lambda, T)d\lambda$ be the spectral projection corresponding to δ, where Γ is a system of curves situated in $\rho(T)$ and surrounding δ. Then $A_T(\delta)$ is a spectral maximal projection of T, $A_T(\delta)X$ being a spectral maximal space of T. See [**3**, Proposition 1.3.10].

Theorem 2.5. *Every spectral maximal projection of* $T \in B(X)$ *is hyperinvariant under* T *and* $\sigma(T) = \sigma(T_p) \cup \sigma(T^p)$.

Proof. Let $R \in B(X)$ commute with T. Then for each $\lambda \in \rho(T)$, $\lambda - R$ is an isomorphism in $B(X)$ commuting with T. We can write $(\lambda - R)pX = qX$ where q is the bounded projection defined by $q = (\lambda - R)p(\lambda - R)^{-1}$. From $T_q = (\lambda - R)T_p(\lambda - R)^{-1}$ it follows that $\sigma(T_q) = \sigma(T_p)$ which implies $qX \subset pX$. Hence $RpX \subset pX$.

Theorem 2.6. *Given* $T \in B(X)$, *let* $f : D \to \mathbb{C}$ *be analytic and injective on an open neighborhood* D *of* $\sigma(T)$. *A projection p in* $B(X)$ *is a spectral maximal projection for* T *if and only if it is a spectral maximal projection for* $f(T)$.

Proof. First we prove the 'if' part of the assertion. Let q be an invariant bounded projection for T that satisfies condition $\sigma(T_q) \subset \sigma(T_p) \subset \sigma(T)$ (the last inclusion is a consequence of the hyperinvariant property of p). Now we can write

$$\begin{aligned} \sigma(f(T)_q) &= \sigma(f(T_q)) \\ &= f(\sigma(T_q)) \\ &\subset f(\sigma(T_p)) = \sigma(f(T_p)) = \sigma(f(T)_p) \end{aligned}$$

and it follows that $qX \subset pX$.

Conversely, let p be a spectral maximal projection of T and let q be an invariant projection under $f(T)$ such that $\sigma(f(T)_q) \subset \sigma(f(T_p)) \subset \sigma(f(T))$. Then

$$\begin{aligned} f(\sigma(T_q)) &= \sigma(f(T_q)) \\ &= \sigma(f(T)_q) \\ &\subset \sigma(f(T)_p) = \sigma(f(T_p)) = f(\sigma(T_p)), \end{aligned}$$

which leads to the desired conclusion.

Definition 2.7. For $T \in B(X)$, we say that an invariant bounded projection p under T is T-*absorbent* if, for any $x \in pX$ and all $\lambda \in \sigma(T_p)$, the equation $(\lambda - T)y = x$ has all solutions y in pX.

Theorem 2.8. *Given* $T \in B(X)$ *and p a spectral maximal projection for* T *then p is* T-*absorbent*.

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Proof. The proof is similar to that of [**5**, Theorem 3.7] and we shall only sketch it. Let $\lambda \in \sigma(T_p)$, $x \in pX$ and let y be a solution of the equation $(\lambda - T)y = x$. If $y \notin pX$, by putting $Y_0 = pX \oplus \mathbb{C}y$ we see that $Y_0 = p_0X$ with p_0 a bounded projection in $B(X)$ invariant under T and from the inclusion $\sigma(T_{p_0}) \subset \sigma(T_p)$ we should have $p_0X \subset pX$ which is preposterous.

COROLLARY 2.9. *Let $T \in B(X)$ have the single-valued extension property. If p is a spectral maximal projection for T, then pX is analytically invariant for T; that is for every function $f : D \to X$ analytic on some open $D \subset \mathbb{C}$, the condition $(\lambda - T)f(\lambda) \in pX$ implies that $f(\lambda) \in pX$.*

Proof. This result is well known when Y is an invariant T-absorbing subspace of X and T has the single-valued extension property [**5**, Theorem 2.26].

3. Weakly projection-relative decomposable operators.

DEFINITION 3.1. $T \in B(X)$ is said to be *weakly projection-relative* (respectively *c-weakly projection-relative*) *decomposable* if for every open cover $\{G_i\}_{1 \le i \le n}$ of $\sigma(T)$, there is a system of spectral maximal projections $\{p_i\}_{1 \le i \le n}$ of T (respectively commuting with T) which performs the following asymptotic spectral decomposition.

1. $\sigma(T_{p_i}) \subset G_i$ for every $1 \le i \le n$.
2. $X = \overline{\sum_{i=1}^{n} p_iX}$.

PROPOSITION 3.2. *Let T be weakly projection-relative (respectively c-weakly projection-relative) decomposable. If $G \subset \mathbb{C}$ is open and $G \cap \sigma(T) \neq \emptyset$, then there exists a non zero spectral maximal projection p (respectively commuting with T) with the property $\sigma(T_p) \subset G$.*

Proof. Let G' be a second open set such that $\{G, G'\}$ is a covering of $\sigma(T)$, $\sigma(T) \not\subset G'$. Then there are p, q spectral maximal projections of T satisfying $\sigma(T_p) \subset G$, $\sigma(T_q) \subset G'$, $X = \overline{pX + qX}$. Now if $p = 0$, we should have $X = qX$ in contradiction with the choice of G'.

LEMMA 3.3. *If p is a spectral maximal projection of an operator T in $B(X)$ and D is a domain such that there is a nonzero analytic X-valued function f satisfying the equation $(\lambda - T)f(\lambda) = 0$ on D, then $D \cap \sigma(T_p) = \emptyset$ or $D \subset \sigma_{point}(T_p)$, where $\sigma_{point}(T_p)$ is the point spectrum of T_p.*

Proof. We shall follow the proof of [**5**, Lemma 6.3], where the key point is the finite dimensional property of the linear manifold $X_n = \bigvee\{f(\lambda_0), f'(\lambda_0), \ldots, f^{(n)}(\lambda_0)\}$ which is complemented in X so that we can associate with X_n a bounded projection p_n invariant under T such that $X_n = p_nX$.

THEOREM 3.4. *Every weakly projection-relative decomposable operator has the single valued extension property.*

Proof. Let T be weakly projection-relative decomposable and $f : D \to X$ be analytic and satisfy the equation $(\lambda - T)f(\lambda) = 0$ on an open set $D \subset \mathbb{C}$. We may assume that $D \cap \sigma(T) \neq \emptyset$ and D is a domain. By Proposition 3.2, there is a nonzero spectral maximal projection p of T such that $\sigma(T_p) \subset D$. If $f \neq 0$ on D then, by Lemma 3.3, $D \subset \sigma(T_p)$, which gives a contradiction, D being open and not void.

THEOREM 3.5. *Given $T \in B(X)$, let $f : \mathbb{D} \to \mathbb{C}$ be analytic and injective on an open neighbourhood D of $\sigma(T)$. Then T is weakly projection-relative (respectively c-projection relative) decomposable if and only if $f(T)$ is.*

Proof. Let $f(T)$ be weakly projection-relative decomposable and $\{G_i\}_{1\le i\le n}$ be an open covering of $\sigma(f(T))$. Since $\sigma(T) \subset D$ the sets $G_i' = G_i \cap D$, $1 \le i \le n$, also form an open covering of $\sigma(T)$. In addition $\{f(G_i')\}_{1\le i\le n}$ is an open covering of $\sigma(f(T))$ and we can find spectral maximal projections p_i of $f(T)$ such that

$$\sigma(f(T)_{p_i}) \subset f(G_i') \qquad (i = 1, 2, \ldots, n), \tag{1}$$

$$X = \overline{\sum_{i=1}^{n} p_i X}. \tag{2}$$

But p_i $(1 \le i \le n)$ are also spectral maximal projections of T by Theorem 2.5 and the inclusion $f(\sigma(T_{p_i})) \subset f(G_i')$ leads to

$$\sigma(T_{p_i}) \subset G_i' \subset G_i \qquad (1 \le i \le n).$$

Thus T is weakly projection-relative decomposable. Now, if p_i commutes with $f(T)$, then p_i commutes with T too. Conversely, the proof is similar.

4. Projection-relative decomposable spectrum.

DEFINITION 4.1. $T \in B(X)$ is said to have *projection-relative* (respectively *c-projection relative*) *decomposable spectrum* if for every open covering $\{G_i\}_{1\le i\le n}$ of $\sigma(T)$, there is an asymptotic projection-relative decomposition induced by a system $\{p_i\}_{1\le i\le n}$ of spectral maximal projections of T (respectively commuting with T) such that

1. $\sigma(T_{p_i}) \subset G_i \quad (1 \le i \le n)$,
2. $X = \overline{\sum_{i=1}^{n} p_i X}$,
3. $\sigma(T) = \bigcup_{i=1}^{n} \sigma(T_{p_i})$.

THEOREM 4.2. *Let T be a weakly projection-relative* (*respectively c-projection-relative*) *decomposable operator. The following statements are equivalent.*

(i) *T has projection-relative* (*respectively c-projection-relative*) *decomposable spectrum.*

(ii) *If $F \subset \sigma(T)$ is closed and $G \supset F$ is open, then there exists a spectral maximal projection p of T* (*respectively commuting with T*) *such that $F \subset \sigma(T_p) \subset G$.*

(iii) *Every system $\{p_i\}_{1\le i\le n}$ of spectral maximal projections* (*respectively commuting with T*) *satisfies $\sigma(T) = \bigcup_{i=1}^{n} \sigma(T_{p_i})$ whenever $X = \overline{\sum_{i=1}^{n} p_i X}$.*

Proof. Obviously (iii) implies (i). We shall prove that (i) $\Rightarrow$ (ii). For this, let $F \subset \sigma(T)$ be closed and $G \supset F$ be open. Then $\{G, F^c\}$ is an open covering of $\sigma(T)$ and so there are spectral maximal projections p, q of T satisfying conditions $\sigma(T_p) \subset G$, $\sigma(T_q) \subset F^c$, $\sigma(T) = \sigma(T_p) \cup \sigma(T_q)$. Consequently $F \subset \sigma(T_p) \subset G$. It remains to prove that (ii) $\Rightarrow$ (iii). Let $\{p_i\}_{1\le i\le n}$ be an arbitrary system of spectral maximal projections of T performing the decomposition $X = \overline{\sum_{i=1}^{n} p_i X}$. If $F = \bigcup_{i=1}^{n} \sigma(T_{p_i}) \ne \sigma(T)$ then there

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exists a spectral maximal projection q of T such that $F \subset \sigma(T_q) \neq \sigma(T)$. Now we have

$$\sigma(T) = \sigma\left(T \Big/ \overline{\sum_{i=1}^{n} p_i X}\right) \subset \bigcup_{i=1}^{n} \sigma(T_{p_i}) \subset \sigma(T_q).$$

It follows that $X = qX$ and $\sigma(T) = \sigma(T_q)$ which is preposterous.

5. Projection-relative quasi decomposable operators.

DEFINITION 5.1. A weakly projection-relative (respectively c-projection relative) decomposable operator is said to be *projection-relative* (respectively *c-projection relative*) *quasi decomposable* if $X_T(F)$ is closed whenever $F \subset \mathbb{C}$ is closed.

THEOREM 5.2. *Every projection-relative (respectively c-projection-relative) quasi decomposable operator has projection-relative (respectively c-projection-relative) decomposable spectrum.*

Proof. Let $\{G_i\}_{1 \leq i \leq n}$ be a finite open covering of $\sigma(T)$ and let $\{p_i\}_{1 \leq i \leq n}$ be a system of spectral maximal projections of T such that $\sigma(T_{p_i}) \subset G_i$ for $1 \leq i \leq n$ and $X = \overline{\sum_{i=1}^{n} p_i X}$. If $F = \bigcup_{i=1}^{n} \sigma(T_{p_i})$ is proper in $\sigma(T)$, then $X_T(F)$ is proper in X, but each $p_i X$ is contained in $X_T(F)$, which is preposterous.

THEOREM 5.3. *If T is a weakly c-projection-relative decomposable operator, then T is in fact a c-projection-relative quasi decomposable operator.*

Proof. Let F be a closed set in $\mathbb{C}$, and G any open set containing F. Since $\{F^c, G\}$ is an open covering of $\sigma(T)$, there exist p_1 and p_2 spectral maximal projections of T such that $\sigma(T_{p_1}) \subset F^c$, $\sigma(T_{p_2}) \subset G$, $X = \overline{p_1 X + p_2 X}$, with p_1, p_2 commuting with T. If $x \in X_T(F)$, there exist $x_{1,n} \in p_1 X$, $x_{2,n} \in p_2 X$ such that $x = \lim_{n\to\infty}(x_{1,n} + x_{2,n})$. Now $p_1 T = T p_1$ and so

$$\begin{aligned}\sigma_T(p_1 x) &\subset \sigma_T(x) \cap \sigma(T_{p_1})\\ &\subset F \cap F^c = \emptyset.\end{aligned}$$

This implies that

$$\begin{aligned}p_1 x = 0 &= \lim_{n\to\infty}(p_1 x_{1,n} + p_1 x_{2,n})\\ &= \lim_{n\to\infty}(x_{1,n} + p_1 x_{2,n})\end{aligned}$$

and so

$$\begin{aligned}x &= \lim_{n\to\infty}(x_{1,n} + x_{2,n}) - \lim_{n\to\infty}(x_{1,n} + p_1 x_{2,n})\\ &= \lim_{n\to\infty}(x_{2,n} - p_1 x_{2,n}).\end{aligned}$$

Since spectral maximal projections are hyperinvariant we have $p_1 x_{2,n} \in p_2 X$ and $x \in p_2 X$. Finally, we obtain

$$\begin{aligned}X_T(F) &\subset p_2 X\\ &\subset X_T(\sigma(T_{p_2}))\\ &\subset X_T(G),\end{aligned}$$

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G being any open set containing F. We have

$$X_T(F) \subset p_2 X \subset \bigcap_{F\subset G} X_T(G) = X_T\left(\bigcap_{F\subset G} G\right) = X_T(F).$$

Thus $X_T(F) = p_2X$ is closed.

6. Projection-relative decomposable operators.

DEFINITION 6.1. $T \in B(X)$ is called *projection-relative* (respectively *c-projection-relative*) *decomposable*, if for every open covering $\{G_i\}_{1\leq i\leq n}$ of $\sigma(T)$, there exists a system $\{p_i\}_{1\leq i\leq n}$ of spectral maximal projections of T (respectively commuting with T) yielding the following spectral decomposition.
1. $\sigma(T_{p_i}) \subset G_i$ *for* $1 \leq i \leq n$.
2. $X = \sum_{i=1}^n p_i X$.

REMARK 6.2. Clearly such an operator is projection-relative quasidecomposable and has projection-relative decomposable spectrum.

EXAMPLE 6.3. If X is a Hilbert space, the concepts of projection-relative decomposable operators and decomposable operators are the same.

EXAMPLE 6.4. Let T be a compact operator on X (or more generally an operator with totally disconnected spectrum). Then T is c-projection-relative decomposable. To see this, let $\{G_i\}_{1\leq i\leq n}$ be a finite open covering of $\sigma(T)$, we can choose open-and-closed subsets δ_i of $\sigma(T)$ such that $\delta_i \subset G_i$ for $1 \leq i \leq n$ and leading to a system $\{A_T(\delta_i)\}_{1\leq i\leq n}$ of spectral maximal projections commuting with T and which yields $\sigma(T_{A_T(\delta_i)}) = \delta_i \subset G_i$ for $1 \leq i \leq n$ and $X = \sum_{i=1}^n A_T(\delta_i)X$.

EXAMPLE 6.5. Quasispectral operators of class Γ on X (in Albrecht's sense [**1**]) with spectral measure $E(.)$ of class Γ are projection-relative decomposable operators. In order to prove this, let us take a finite open covering $\{G_i\}_{1\leq i\leq n}$ of $\sigma(T)$. Then there exists a finite open covering $\{\omega_i\}_{1\leq i\leq n}$ of $\sigma(T)$ with $\overline{\omega_i} \subset G_i$ for every $1 \leq i \leq n$. Put $s_1 = \omega_1$ and $s_i = \omega_i - \bigcup_{j<i} \omega_j$ for $1 \leq i \leq n$. We obtain a finite disjoint covering $\{s_i\}_{1\leq i\leq n}$ of $\sigma(T)$ by Borel sets. The bounded projections $E(\bar{s}_i)$ form a system of spectral maximal projections of T such that

$$\sigma\left(T_{E(\bar{s}_i)}\right) \subset \bar{s}_i \subset \bar{\omega}_i \subset G_i \quad (1 \leq i \leq n) \qquad \text{and}$$

$$X = E(\sigma(T))X = \sum_{i=1}^n E(s_i)X = \sum_{i=1}^n E(\bar{s}_i)X.$$

EXAMPLE 6.6. Prespectral operators of class Γ on X and hence spectral operators are c-projection-relative decomposable operators. This results from the commutativity property of T and E. See [**4**].

THEOREM 6.7. *Let $T \in B(X)$ and $f : D \to X$ be an analytic injective function on an open neighborhood D of $\sigma(T)$. Then $f(T)$ is projection-relative (c-projection-relative) decomposable if and only if T is.*

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Proof. This is similar to that of Theorem 3.5.

PROPOSITION 6.8. *Let T be projection-relative decomposable and p be a spectral maximal projection of T. Then we have $\sigma(T^p) = \overline{\sigma(T) - \sigma(T_p)}$.*

Proof. Suppose that there is $\lambda \in \sigma(T^p) - \overline{\sigma(T) - \sigma(T_p)}$. Then we can find an open covering $\{G_1 \cup G_2\}$ of $\sigma(T)$ such that

$$\lambda \notin G_1 \supset \overline{\sigma(T - \sigma(T_p))},$$
$$G_2 \cap \overline{\sigma(T) - \sigma(T_p)} = \emptyset.$$

Let $\{p_1, p_2\}$ be the spectral maximal projections of T corresponding to this covering of $\sigma(T)$. From the inclusion $\sigma(T_{p_2}) \subset G_2 \cap \sigma(T) \subset \sigma(T_p)$ we have $p_2X \subset pX$. Now let $\dot{x} \in X/pX$ such that $(\lambda - T^p)\dot{x} = 0$. If $x \in \dot{x}$ and $x_1 \in p_1X$, $x_2 \in p_2X$ satisfy $x = x_1 + x_2$, one obtains $(\lambda - T)x_1 = (\lambda - T)x - (\lambda - T)x_2 \in pX \cap p_1X$, a subspace of X invariant under $(\lambda - T_{p_1})^{-1}$ (which exists because $\sigma(T_{p_1}) \subset G_1$ and $\lambda \notin G_1$). It follows that $x_1 = (\lambda - T_{p_1})^{-1}(\lambda - T)x_1 \in p_1X \cap pX$ and $\dot{x} = \dot{x}_1 + \dot{x}_2 = 0$. Hence $\lambda - T^p$ is one to one. Now if we take $\dot{y} \in X|pX$ and $y = y_1 + y_2 \in \dot{y}$, $y_1 \in p_1X$, $y_2 \in p_2X$ we can find $x_1 \in p_1X$ such that $(\lambda - T_{p_1})x_1 = y_1$ (remember that $\lambda \notin G_1$). Consequently one obtains $\dot{y} = \dot{y}_1 = \overline{(\lambda - T_{p_1})x_1} = \overline{(\lambda - T)x_1} = (\lambda - T^p)\dot{x}_1$ which means that $(\lambda - T^p)$ maps X/pX onto X/pX so that $\lambda \notin \sigma(T^p)$ which is preposterous. The opposite inclusion follows from the hyperinvariant property of p.

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