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NORMAL HILBERT COEFFICIENTS AND ELLIPTIC IDEALS IN NORMAL TWO-DIMENSIONAL SINGULARITIES

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Abstract. Let $(A,\mathfrak{m})$ be an excellent two-dimensional normal local domain. In this paper, we study the elliptic and the strongly elliptic ideals of A with the aim to characterize elliptic and strongly elliptic singularities, according to the definitions given by Wagreich and Yau. In analogy with the rational singularities, in the main result, we characterize a strongly elliptic singularity in terms of the normal Hilbert coefficients of the integrally closed $\mathfrak{m}$-primary ideals of A. Unlike p_g-ideals, elliptic ideals and strongly elliptic ideals are not necessarily normal and necessary, and sufficient conditions for being normal are given. In the last section, we discuss the existence (and the effective construction) of strongly elliptic ideals in any two-dimensional normal local ring.

§1. Introduction and notations

Let $(A,\mathfrak{m})$ be an excellent two-dimensional normal local ring, and let I be an $\mathfrak{m}$-primary ideal of A. The integral closure $\bar{I}$ of I is the ideal consisting of all solutions z of some equation with coefficients $c_i \in I^i$: $Z^n + c_1 Z^{n-1} + c_2 Z^{n-2} + \cdots + c_{n-1} Z + c_n = 0$. Then $I \subseteq \bar{I} \subseteq \sqrt{I}$. We say that I is *integrally closed* if $I = \bar{I}$ and I is *normal* if $I^n = \overline{I^n}$ for every positive integer n. By a classical result of Rees [29], under our assumptions, the filtration $\{\overline{I^n}\}_{n\in\mathbb{N}}$ is a good I-filtration of A and it is called the *normal filtration*.

We may define the Hilbert–Samuel function $\bar{H}_I(n) := \ell_A(A/\overline{I^{n+1}})$ for all integers $n \geq 0$, and it becomes a polynomial for large n. (Here, $\ell_A(M)$ is the length of the A-module M.) This polynomial is called the *normal Hilbert polynomial*

$$\bar{P}_I(n) = \bar{e}_0(I)\binom{n+2}{2} - \bar{e}_1(I)\binom{n+1}{1} + \bar{e}_2(I),$$

and the coefficients $\bar{e}_i(I)$, $i = 0,1,2$, are the *normal Hilbert coefficients*.

A rich literature is available on the normal Hilbert coefficients $\bar{e}_i(I)$, and this study is considered an important part of the theory of blowing-up rings (see, e.g., [2], [3], [10], [11], [12], [17]–[19], [33]).

From the geometric side, any integrally closed $\mathfrak{m}$-primary ideal I of A is represented on some resolution (see [16]). Let

$$f\colon X \to \operatorname{Spec} A$$

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be a resolution of singularities with an anti-nef cycle $Z > 0$ on X, so that $I = I_Z = H^0(\mathcal{O}_X(-Z))$ and $I\mathcal{O}_X = \mathcal{O}_X(-Z)$. We say that $I = I_Z$ is *represented by* Z *on* X. The aim of this paper is to join the algebraic and the geometric information on A taking advantage of the theory of the Hilbert functions and of the theory of the resolution of singularities.

For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, we write $h^i(\mathcal{F}) = \ell_A(H^i(X, \mathcal{F}))$. If $I = I_Z$ is an $\mathfrak{m}$-primary integrally closed ideal of A represented by Z on X, one can define for every integer $n \geq 0$ a decreasing chain of integers $q(nI) := q(\overline{I^n}) = h^1(\mathcal{O}_X(-nZ))$ where $q(0I) := p_g(A)$ is the *geometric genus* of A. It is proved that $q(nI)$ stabilizes for every I and $n \geq p_g(A)$. We denote it by $q(\infty I)$.

These integers are independent of the representation, and they are strictly related to the normal Hilbert polynomial. The keys of our approach can be considered Theorem 2.2 and Proposition 2.6, consequences of Kato's Riemann–Roch formula (see [14], [24]). In particular, the following holds:

(1) $\overline{P}_I(n) = \ell_A(A/\overline{I^{n+1}})$ for all $n \geq p_g(A) - 1$.
(2) $\bar{e}_1(I) - e_0(I) + \ell_A(A/I) = p_g(A) - q(I)$.
(3) $\bar{e}_2(I) = p_g(A) - q(nI) = p_g(A) - q(\infty I)$ for all $n \geq p_g(A)$.

Moreover, we have

$$\bar{e}_0(I) = -Z^2, \qquad \bar{e}_1(I) = \frac{-Z^2 + ZK_X}{2}.$$

This makes the bridge between the theory of the normal Hilbert coefficients and the theory of the singularities. This is the line already traced by Lipman [16], Cutkosky [4], and more recently by Okuma, Watanabe, and Yoshida (see [23]–[25]).

Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$. It is known that A is a rational singularity (see [1]) if and only if every integrally closed $\mathfrak{m}$-primary ideal I of A is normal (see [4], [16]), equivalently $\bar{e}_2(I) = 0$, that is, I is a p_g-ideal, as proved in [23], [24]. Inspired by a paper by Okuma [22], we investigate the integrally closed $\mathfrak{m}$-primary ideals of *elliptic singularities* (see [38]) and of *strongly elliptic singularities* (see [41]). All the preliminary results are contained in §2.

In §3, we prove the main results of the paper. We define *the elliptic and the strongly elliptic ideals* aimed by the study of nonrational singularities. We recall that if Q is a minimal reduction of I, then we denote by $\bar{\mathrm{r}}(I) := \min\{r \mid \overline{I^{n+1}} = Q\overline{I^n}, \text{for all } n \geq r\}$, *the normal reduction number* of I and this integer exists and does not depend on the choice of Q. Okuma proved that if A is an elliptic singularity, then $\bar{\mathrm{r}}(I) = 2$ for any integrally closed $\mathfrak{m}$-primary ideal of A (see [22, Theorem 3.3]). According to Okuma's result, we define *elliptic ideals* to be the integrally closed $\mathfrak{m}$-primary ideals satisfying $\bar{\mathrm{r}}(I) = 2$. In Theorem 3.2, we prove that elliptic ideals satisfy $\bar{e}_2(I) = \bar{e}_1(I) - e_0(I) + \ell_A(A/I) > 0$ attaining the minimal value according to the inequality proved by Sally [35] and Itoh [12]. In particular, if I is an elliptic ideal, then $p_g(A) > q(I) = q(\infty I)$. If A is not a rational singularity, then elliptic ideals always exist (see Proposition 3.3). In particular, we prove the following proposition.

Proposition 1.1 (See Proposition 3.3). *If A is not a rational singularity, then for any $\mathfrak{m}$-primary integrally closed ideal I of A, $\overline{I^n}$ is either a p_g-ideal or an elliptic ideal for every $n \geq p_g(A)$.*

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Yau in [41], Laufer in [15], and Wagreich in [38] introduced interesting classes of elliptic singularities. An excellent two-dimensional normal local ring A is a strongly elliptic singularity if $p_g(A) = 1$, that is, p_g is almost minimal.

Among the elliptic ideals, in Theorem 3.9, we define strongly elliptic ideals those for which $\bar{e}_2 = 1$ and equivalent conditions are given. The following result characterizes algebraically the strongly elliptic singularities.

THEOREM 1.2 (See Theorem 3.14). *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$, and assume that $p_g(A) > 0$. The following conditions are equivalent*:

(1) *A is a strongly elliptic singularity.*
(2) *Every integrally closed ideal of A is either a p_g-ideal or a strongly elliptic ideal.*

Notice that p_g-ideals are always normal, but elliptic ideals are not necessary normal (see Proposition 3.16 and Examples 3.15 and 3.25). Moreover, if A is strongly elliptic and I is not a p_g-ideal, then Proposition 3.16 and Theorem 3.23 give necessary and sufficient conditions for being I normal.

THEOREM 1.3. *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$. Assume that A is a strongly elliptic singularity. If $I = I_Z$ is an elliptic ideal (equivalently, I is not a p_g-ideal) and D is the minimally elliptic cycle on X, then I^2 is integrally closed (equivalently, I is normal) if and only if $-ZD \geq 3$ and if $-ZD \leq 2$, then $I^2 = QI$.*

For any normal surface singularity which is not rational, p_g-ideals and elliptic ideals exist plentifully. But this is no longer true for strongly elliptic ideals.

In §4, we show that there exist excellent two-dimensional normal local rings having no strongly elliptic ideals (see Example 4.8). Finally, Corollary 4.7 gives necessary and sufficient conditions for the existence of strongly elliptic ideals in terms of the existence of certain cohomological cycles. When there exist, we present an effective geometric construction (see Example 4.9).

§2. Preliminaries and normal reduction number

Let $(A, \mathfrak{m})$ be an excellent two-dimensional normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$, and let I be an integrally closed $\mathfrak{m}$-primary ideal of A. With the already introduced notation, then there exists a resolution $X \to \operatorname{Spec} A$ and a cycle Z such that I is represented on X by Z. When we write I_Z, we always assume that $\mathcal{O}_X(-Z)$ is generated by global sections, namely $I\mathcal{O}_X = \mathcal{O}_X(-Z)$, and note that $I_Z = H^0(X, \mathcal{O}_X(-Z))$. Recall that the geometric genus $p_g(A) = h^1(\mathcal{O}_X)$ is independent of the choice of the resolution.

Okuma, Watanabe, and Yoshida introduced a natural extension of the integrally closed ideals in a two-dimensional rational singularity, that is, the p_g-ideals. With the previous notation,

$$p_g(A) \geq h^1(\mathcal{O}_X(-Z)),$$

and if the equality holds, then Z is called a p_g-cycle and $I = I_Z$ is called a p_g-ideal. In [23], [24], the authors characterized the p_g-ideals in terms of the normal Hilbert polynomial.

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They proved that A is a rational singularity if and only if every integrally closed $\mathfrak{m}$-primary ideal is a p_g-ideal. Starting by $p_g(A)$, we define the following chain of integers.

Definition 2.1. We define $q(I) := h^1(\mathcal{O}_X(-Z))$ and more in general $q(nI) := q(\overline{I^n}) = h^1(\mathcal{O}_X(-nZ))$ for every integer $n \geq 1$.

We put $q(0I) = h^1(\mathcal{O}_X) = p_g(A)$. Notice that $q(nI)$ is in general very difficult to compute, but it is independent of the representation (see [23, Lemma 3.4]). These invariants are strictly related to the normal Hilbert polynomial, and their interplay is very important in our approach.

The following formula is called a Riemann–Roch formula. The result was proved in [14] in the complex case, but it holds in any characteristic (see [40]).

Theorem 2.2 (Kato's Riemann–Roch formula [40, Theorem 2.2]). *Let $I = I_Z$ be an $\mathfrak{m}$-primary integrally closed ideal represented by an anti-nef cycle Z on X. Then we have*

$$\ell_A(A/I) + q(I) = -\frac{Z^2 + K_X Z}{2} + p_g(A),$$

where K_X denotes the canonical divisor.

We recall here the properties of the sequence $\{q(nI)\}$. Propositions 2.3 and 2.5 on $I = I_Z$ follow from the long exact sequence attached to the short exact sequence

$$(\dagger) \quad 0 \to \mathcal{O}_X(-(n-1)Z) \to \mathcal{O}_X(-nZ)^{\oplus 2} \to \mathcal{O}_X(-(n+1)Z) \to 0$$

(see [24, Lemma 3.1]).

Proposition 2.3. *With the previous notation, the following facts hold*:

(1) $0 \leq q(I) \leq p_g(A)$.
(2) $q(kI) \geq q((k+1)I)$ *for every integer* $k \geq 0$ *and if* $q(nI) = q((n+1)I)$ *for some* $n \geq 0$, *then* $q(nI) = q(mI)$ *for every* $m \geq n$. *Hence,* $q(nI) = q((n+1)I)$ *for every* I *and* $n \geq p_g(A)$. *We denote it by* $q(\infty I)$.

We use the above sequence for computing the following important algebraic numerical invariants of the normal filtration $\{\overline{I^n}\}$. Let $\mathbb{Z}_+$ denote the set of positive integers.

Definition 2.4 (cf. [26]). Let $I \subset A$ be $\mathfrak{m}$-primary integrally closed ideal, and let Q be a minimal reduction of I. Define:

$$\mathrm{nr}(I) := \min\{r \in \mathbb{Z}_+ \mid \overline{I^{r+1}} = Q\overline{I^r}\},$$
$$\bar{\mathrm{r}}(I) := \min\{r \in \mathbb{Z}_+ \mid \overline{I^{n+1}} = Q\overline{I^n} \text{ for all } n \geq r\}.$$

We call $\bar{\mathrm{r}}(I)$ the *normal reduction number* and $\mathrm{nr}(I)$ the *relative normal reduction number*.

The normal reduction number exists (see [21], [29]), and it has been studied by many authors in the context of the Hilbert function and of the Hilbert polynomial (see, e.g., [2], [3], [10], [12], [19]). The main difficulty of the normal filtration with respect to the I-adic filtration is that the Rees algebra of the normal filtration is not generated by the part of degree one because $I\overline{I^n} \neq \overline{I^{n+1}}$. By the definition, we deduce that $\mathrm{nr}(I) \leq \bar{\mathrm{r}}(I)$ and we see that, in general, they do not coincide. Note that the definitions of $\mathrm{nr}(I)$ and of $\bar{\mathrm{r}}(I)$ are independent on the choice of a minimal reduction Q of I (see, e.g., [10, Theorem 4.5]). It is also a consequence of the following result in [26, §2].

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PROPOSITION 2.5. *The following statements hold.*

(1) *For any integer $n \geq 1$, we have*

$$2 \cdot q(nI) + \ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = q((n+1)I) + q((n-1)I).$$

(2) *We have*

$$\mathrm{nr}(I) = \min\{n \in \mathbb{Z}_+ \mid q((n-1)I) - q(nI) = q(nI) - q((n+1)I)\},$$
$$\bar{\mathrm{r}}(I) = \min\{n \in \mathbb{Z}_+ \mid q((n-1)I) = q(nI)\}.$$

From the propositions above, we have that $\bar{\mathrm{r}}(I) \leq p_g(A) + 1$. In [26, Theorem 2.9], the authors showed that $p_g(A) \geq \binom{\mathrm{nr}(I)}{2}$.

Rossi [32, Corollary 1.5] proved the following upper bound on the reduction number $r(I)$ for every $\mathfrak{m}$-primary ideal I (here, $r(I)$ denotes the reduction number for the I-adic filtration) in a two-dimensional Cohen–Macaulay local ring A in terms of the Hilbert coefficients:

$$r(I) \leq e_1(I) - e_0(I) + \ell_A(A/I) + 1.$$

The bound gives, as a consequence, several interesting results, in particular a positive answer to a long-standing conjecture stated by Sally in the case of local Cohen–Macaulay rings of almost minimal multiplicity (see [32], [34]). Later, the inequality was extended by Rossi and Valla (see [33, Theorem 4.3] for special multiplicative I-filtrations). The result does not include the normal filtration. It is natural to ask if the same bound also holds for $\bar{\mathrm{r}}(I)$. The answer is negative as we will show later, but we prove that the analogue upper bound holds true for $\mathrm{nr}(I)$. We need some preliminary results.

From Riemann–Roch formula (Theorem 2.2), we get

$$\ell_A(A/\overline{I^{n+1}}) + q((n+1)I) = -\frac{(n+1)^2Z^2 + (n+1)ZK_X}{2} + p_g(A).$$

Using this, we can express $\bar{e}_0(I), \bar{e}_1(I), \bar{e}_2(I)$ as follows.

PROPOSITION 2.6 [24, Theorem 3.2]. *Assume that $I = I_Z$ is represented by a cycle $Z > 0$ on a resolution X of* $\mathrm{Spec}(A)$. *Let $\bar{P}_I(n)$ be the normal Hilbert polynomial of I. Then*:

(1) $\overline{P}_I(n) = \ell_A(A/\overline{I^{n+1}})$ *for all $n \geq p_g(A) - 1$.*
(2) $\bar{e}_0(I) = e_0(I)$.
(3) $\bar{e}_1(I) - e_0(I) + \ell_A(A/I) = p_g(A) - q(I)$.
(4) $\bar{e}_2(I) = p_g(A) - q(nI) = p_g(A) - q(\infty I)$ *for all $n \geq p_g(A)$.*

Moreover, we have

$$\bar{e}_0(I) = -Z^2, \qquad \bar{e}_1(I) = \frac{-Z^2 + ZK_X}{2}.$$

THEOREM 2.7. *Let $(A, \mathfrak{m})$ be an excellent two-dimensional normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$. Let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal. Then*

$$\mathrm{nr}(I) \leq \bar{e}_1(I) - \bar{e}_0(I) + \ell_A(A/I) + 1.$$

If we put $r = \mathrm{nr}(I)$, *equality holds if and only if the following conditions hold true*:

(1) $\ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = 1$ *for* $n = 1, \dots, r-1$ *if* $r > 1$,
(2) $q((r-1)I) = q(\infty I)$.

When this is the case, $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$, $q(I) = p_g(A) - \bar{\mathrm{r}}(I) + 1$, *and* $\bar{e}_2(I) = p_g(A) - q(\infty I) = r(r-1)/2$.

Proof. By virtue of Proposition 2.6, it is enough to show

$$\mathrm{nr}(I) \le p_g(A) - q(I) + 1.$$

If we put $\Delta q(n) := q(nI) - q((n+1)I)$ for every integer $n \ge 0$, then $\Delta q(n)$ is nonnegative and decreasing since $\ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = \Delta q(n-1) - \Delta q(n)$. We have

$$\mathrm{nr}(I) = \min\{n \in \mathbb{Z}_+ \mid \Delta q(n-1) = \Delta q(n)\}, \quad \bar{\mathrm{r}}(I) = \min\{n \in \mathbb{Z}_+ \mid \Delta q(n-1) = 0\}.$$

Put $a = p_g(A) - q(I)$. Then $\Delta q(0) = a \ge \mathrm{nr}(I) - 1$ and $\mathrm{nr}(I) = a + 1$ if and only if

$$\Delta q(0) = a > \Delta q(1) = a-1 > \cdots > \Delta q(a-1) = 1 > \Delta q(a) = 0 = \Delta q(a+1).$$

Now, assume $\mathrm{nr}(I) = a+1$. Then $a = r-1$ and for every n with $1 \le n \le a = r-1$, we have

$$\ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = \Delta q(n-1) - \Delta q(n) = (a-n+1) - (a-n) = 1.$$

Moreover, for every $n \ge a+1$, we have

$$\ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = \Delta q(n-1) - \Delta q(n) = 0,$$

and thus $\overline{I^{n+1}} = Q\overline{I^n}$. Hence, $\bar{\mathrm{r}}(I) = a+1 = \mathrm{nr}(I)$. Furthermore,

$$\bar{e}_2(I) = p_g(A) - q(\infty I) = q(0I) - q((r-1)I) = \sum_{i=0}^{r-2} \Delta q(i) = \frac{r(r-1)}{2}.$$

One can prove the converse similarly. ∎

Note that, if the equality holds in the previous result, then the normal filtration $\{\overline{I^n}\}$ has almost minimal multiplicity following the definition given in [33, 2.1]. In the following example, we show that Theorem 2.7 does *not* hold if we replace $\mathrm{nr}(I)$ by $\bar{r}(I)$. The example shows that for all $g \ge 2$, there exist an excellent two-dimensional normal local ring A and an integrally closed $\mathfrak{m}$-primary ideal I such that $\mathrm{nr}(I) = 1, \bar{r}(I) = g+1, q(I) = g-1$ and $\ell_A(A/I) = g$.

The following ideal I satisfies $\bar{e}_1(I) = \bar{e}_0(I) - \ell_A(A/I) + 1$, but $\bar{\mathrm{r}}(I) \not\le 2$.

EXAMPLE 2.8 [27, Example 3.10]. Let $g \ge 2$ be an integer, and let K be a field of $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K = p$, where p does not divide $2g+2$. Then $R = K[X,Y,Z]/(X^2 - Y^{2g+2} - Z^{2g+2})$ is a graded normal K-algebra with $\deg X = g+1$, $\deg Y = \deg Z = 1$. Let $A = R^{(g)}$ be the gth Veronese subring of R:

$$A = K[y^g, y^{g-1}z, y^{g-2}z^2, \dots, z^g, xy^{g-1}, xy^{g-2}z, \dots, xz^{g-1}],$$

where x, y, z denote, respectively, the image of X, Y, Z in R. Then A is a graded normal domain with $A_k = R_{kg}$ for every integer $k \ge 0$. Let $I = (y^g, y^{g-1}z) + A_{\ge 2}$ and

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$Q = (y^g - z^{2g}, y^{g-1}z)$. Then the following statements hold:

(1) $p_g(A) = g$.
(2) $\mathrm{nr}(I) = 1$ and $\bar{\mathrm{r}}(I) = g+1$. Indeed,
  (a) $\overline{I} = I$ and $\overline{I^n} = I^n = QI^{n-1}$ for every $n = 2, \dots, g$.
  (b) $\ell_A(\overline{I^{g+1}}/Q\overline{I^g}) = 1$ $(\overline{I^{g+1}} = I^{g+1} + (xy^{g^2-1}))$.
  (c) $\overline{I^{n+1}} = Q\overline{I^n}$ for every $n \geq g+1$.
(3) $\bar{e}_0(I) = 4g-2$, $\bar{e}_1(I) = 3g-1$, $\bar{e}_2(I) = g$, and $\ell_A(A/I) = g$.
(4) $q(nI) = g-n$ for every $n = 0, 1, \dots, g$; $q(gI) = q(\infty I) = 0$.

The first statement follows from that $a(A) = 0$ and $g = g(\mathrm{Proj}(A))$. For the convenience of the readers, we give a sketch of the proof in the case of $g = 2$ (see [27, Proof of Example 3.10]). Let $A = K[y^2, yz, z^2, xy, xz] = R^{(2)}$ with $\deg x = 3$ and $\deg y = \deg z = 1$, and $I = (y^2, yz, z^4, xy, xz) \supset Q = (y^2 - z^4, yz)$. Then one can easily see that $e_0(I) = \ell_A(A/Q) = 4g-2 = 6$, $\ell_A(A/I) = p_g(A) = g = 2$, and $I^2 = QI$, $\overline{I} = I$. In particular, $\mathrm{nr}(I) = 1$.

Claim 1. *$f_0 \in K[y,z]_{2n} \cap \overline{I^n} \Longrightarrow f_0 \in I^n$ for each $n \geq 1$.*

The normality of $I_0 = (y^2, yz, z^4)K[y,z] \subset K[y,z]$ implies the above claim.

Claim 2. *$0 \neq f_1 \in K[y,z]_{2n-3}$, $xf_1 \in \overline{I^n} \Longrightarrow n \geq 3$.*

By assumption and Claim 1, we have $(y^6 + z^6)f_1^2 = (xf_1)^2 \in \overline{I^{2n}} \cap K[y,z]_{2\cdot 2n} \subset I^{2n}$. The degree (in y and z) of any monomial in $I^{2n} = (y^2, yz, z^4, xy, xz)$ is at least $4n = \deg(y^6 + z^6)f_1^2$. Hence, $(y^6 + z^6)f_1^2 \in (y^2, yz)^{2n}$, and the highest power of z appearing in $(y^6 + z^6)f_1^2$ is at most $2n$. Therefore, $n \geq 3$.

Claim 3. *If $n \leq 2$, then $\overline{I^n} \cap A_n \subset I^n \cap A_n$.*

Any $f \in \overline{I^n} \cap A_n$ can be written as $f = f_0 + xf_1$ for some $f_0 \in K[y,z]_{2n}$ and $f_1 \in K[y,z]_{2n-3}$). Let $\sigma \in \mathrm{Aut}_{K[y,z]^{(2)}}(A)$ such that $\sigma(x) = -x$. Then, since $\sigma(I) = I$, we obtain $\sigma(f) = f_0 - xf_1 \in \overline{I^n}$. Hence,

$$f_0 = \frac{f + \sigma(f)}{2} \in \overline{I^n} \quad \text{and} \quad xf_1 = \frac{f - \sigma(f)}{2} \in \overline{I^n}.$$

By Claims 1 and 2, we have $f_0 \in I^n$ and $f_1 = 0$. Therefore, $f = f_0 \in I^n \cap A_n$, as required.

Claim 4. *$xy^3 \in \overline{I^3} \setminus Q\overline{I^2}$.*

Since $(xy^3)^2 = (y^6)^2 + (y^3z^3)^2 \in (I^3)^2$, we get $xy^3 \in \overline{I^3}$. Assume $xy^3 \in Q\overline{I^2} = (a,b)\overline{I^2}$, where $a = y^2 - z^4$ and $b = yz$. Then $axy + bxz^3 = xy^3 = au + bv$ for some $u, v \in \overline{I^2}$. Since a, b form a regular sequence, we can take an element $h \in A_1$, so that $u - xy = bh$ and $xz^3 - v = ah$. So we may assume $u, v \in A_2$, and thus $u, v \in \overline{I^2} \cap A_2 \subset I^2$. However, this yields $xy^3 = au + bv \in QI^2 = I^3$, which is a contradiction.

Claim 5. *$q(I) = 1$, $q(2I) = q(\infty I) = 0$, $\ell_A(\overline{I^3}/Q\overline{I^2}) = 1$, and $\overline{I^{n+1}} = Q\overline{I^n}$ for each $n \geq 3$.*

By Proposition 2.3, we have $2 = p_g(A) = q(0\cdot I) \geq q(I) \geq q(2\cdot I) \geq 0$. If $q(I) = q(2\cdot I)$, then $q(2\cdot I) = q(3\cdot I)$. This implies $\ell_A(\overline{I^3}/Q\overline{I^2}) = 0$ from Proposition 2.5. This contradicts Claim 4. Hence, $q(I) = 1$ and $q(2\cdot I) = 0$. The other assertions follow from Proposition 2.5. In particular, $\bar{\mathrm{r}}(I) = 3$.

Claim 6. *$\bar{e_1}(I) = 3g-2 = 5$ and $\bar{e_2}(I) = g = 2$.*

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By Proposition 2.6, we have

$$\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + p_g(A) - q(I) = 6 - 2 + 2 - 1 = 5,$$
$$\bar{e}_2(I) = p_g(A) - q(\infty I) = \ 2 - 0 = 2.$$

§3. Elliptic and strongly elliptic ideals

We define the Rees algebra $\bar{\mathcal{R}}(I)$ and the associated graded ring $\bar{G}(I)$ associated with the normal filtration as follows:

$$\bar{\mathcal{R}}(I) := \bigoplus_{n\geq 0} \overline{I^n} t^n \subset A[t].$$
$$\bar{G}(I) := \bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}} \cong \bar{\mathcal{R}}(I)/\bar{\mathcal{R}}(I)(1).$$

$\bar{\mathcal{R}}(I)$ (resp. $\bar{G}(I)$) is called the *normal Rees algebra* (resp. *the normal associated graded ring*) of I. We recall that the a-invariant of a graded d-dimensional ring R with maximal homogeneous graded ideal $\mathfrak{M}$ was introduced by Goto and Watanabe [9] and defined as $a(R) := \max\{n \mid [H^d_{\mathfrak{M}}(R)]_n \neq 0\}$, where $[H^d_{\mathfrak{M}}(R)]_n$ denotes the homogeneous component of degree n of the graded R-module $H^d_{\mathfrak{M}}(R)$.

It is known that A is a rational singularity if and only if $\bar{r}(A) = 1$ (see [27, Proposition 1.1]). In [23], [24], the authors introduced the notion of p_g-ideals, characterizing rational singularities.

Theorem 3.1 (cf. [7], [10], [23], [24]). *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal nonregular local domain containing an algebraically closed field $k = A/\mathfrak{m}$. Let $I = I_Z$ be an $\mathfrak{m}$-primary integrally closed ideal of A. Put $\bar{G} = \bar{G}(I)$ and $\bar{\mathcal{R}} = \bar{\mathcal{R}}(I)$. Then the following conditions are equivalent*:

(1) $\bar{\mathrm{r}}(I) = 1$.
(2) $q(I) = p_g(A)$.
(3) $I^2 = QI$ *and* $\overline{I^n} = I^n$ *for every* $n \geq 1$.
(4) $\bar{e}_1(I) = e_0(I) - \ell_A(A/I)$.
(5) $\bar{e}_2(I) = 0$.
(6) $\bar{G}$ *is Cohen–Macaulay with* $a(\bar{G}) < 0$.
(7) $\bar{\mathcal{R}}$ *is Cohen–Macaulay.*

When this is the case, I is said to be a p_g-ideal.

Proof. Since $QI^{n-1} \subset I^n \subset \overline{I^n}$ for every $n \geq 2$, (1) ⇔ (3) is trivial. (1) ⇔ (5) (resp. (6) ⇔ (7)) follows from [7, Part II, Proposition 8.1] (resp. [7, Part II, Corollary 1.2]). Moreover, the equivalence of (4), (5), and (7) follows from [7, Part II, Theorem 8.2]. (2) ⇔ (4) follows from Proposition 2.5. □

It is known that A is a rational singularity if and only if any integrally closed $\mathfrak{m}$-primary ideal is a p_g-ideal (see [23], [24]). We define

$$\bar{r}(A) := \max\{\bar{\mathrm{r}}(I) \mid I \text{ is an integrally closed } \mathfrak{m}\text{-primary ideal}\}.$$

Then A is a rational singularity if and only if $\bar{r}(A) = 1$ (see [27, Proposition 1.1]).

Okuma proved in [22, Theorem 3.3] that if A is an elliptic singularity, then $\bar{r}(A) = 2$. For the definition of elliptic singularity, we refer to [38, p. 428] or [22, Definition 2.1].

We investigate the integrally closed $\mathfrak{m}$-primary ideals such that $\bar{\mathrm{r}}(I)=2$ with the aim to characterize elliptic singularities. Next result extends and completes a result by Itoh [12, Proposition 10], by using a different approach.

THEOREM 3.2. *Let $(A,\mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k=A/\mathfrak{m}$, and let $I\subset A$ be an $\mathfrak{m}$-primary integrally closed ideal. Put $\bar{G}=\bar{G}(I)$ and $\bar{\mathcal{R}}=\bar{\mathcal{R}}(I)$. Then the following conditions are equivalent*:

(1) $\bar{\mathrm{r}}(I)=2$.
(2) $p_g(A)>q(I)=q(\infty I)$.
(3) $\bar{e}_1(I)=e_0(I)-\ell_A(A/I)+\bar{e}_2(I)$ *and* $\bar{e}_2(I)>0$.
(4) $\ell_A(A/\overline{I^{n+1}})=\bar{P}_I(n)$ *for all* $n\ge 0$ *and* $\bar{e}_2(I)>0$.
(5) $\bar{G}$ *is Cohen–Macaulay with* $a(\bar{G})=0$.

When this is the case, I is said to be an elliptic ideal and $\ell_A([H^2_{\mathfrak{M}}(\bar{G})_0)=\ell_A(\overline{I^2}/QI)=\bar{e}_2(I)$.

Proof. (1) $\Longleftrightarrow$ (2) : It follows from Proposition 2.5(2).

(2) $\Longleftrightarrow$ (3) : By Proposition 2.6, we have

$$\bar{e}_1(I)=e_0(I)-\ell_A(A/I)+\bar{e}_2(I)-\{q(I)-q(\infty I)\}.$$
$$\bar{e}_2(I)=p_g(A)-q(\infty I)\ge 0.$$

The assertion follows from here.

(2) $\Longleftrightarrow$ (4) : Assume $I=I_Z=H^0(X,\mathcal{O}_X(-Z))$ for some resolution $X\to\operatorname{Spec}A$. By Kato's Riemann–Roch formula, for every integer $n\ge 0$, we have

$$\ell_A(A/\overline{I^{n+1}})+h^1(\mathcal{O}_X(-(n+1)Z))=-\frac{(n+1)^2Z^2+(n+1)K_XZ}{2}+p_g(A).$$

Hence,

$$\begin{aligned}\ell_A(A/\overline{I^{n+1}})&=\bar{e}_0(I)\binom{n+2}{2}-\bar{e}_1(I)\binom{n+1}{1}+\{p_g(A)-q((n+1)I)\}\\&=\bar{P}_I(n)-\{q((n+1)I)-q(\infty I)\}.\end{aligned}$$

Assume (4). By replacing 0 to n in the above equation, we get $q(I)=q(\infty I)$, hence (2). Conversely, if $q(I)=q(\infty I)$, then since $q((n+1)I)=q(\infty I)$ for all $n\ge 1$, the above equation implies (4).

(1) $\Longrightarrow$ (5) : Put $Q=(a,b)$. Since $\overline{I^{n+1}}\colon a=\overline{I^n}$, a^*, the image of a in $\bar{G}$ is a nonzero divisor of $\bar{G}$.

By assumption, we have $\overline{I^{n+1}}\cap Q=Q\overline{I^n}\cap Q=Q\overline{I^n}$ for every $n\ge 2$. On the other hand, we have $\overline{I^2}\cap Q=QI$ by [10, Theorem in p. 371] or [11, Theorem]. Then it is well known that a^*, b^* form a regular sequence in $\bar{G}$, and thus $\bar{G}$ is Cohen–Macaulay (see also [37]) and $2=\bar{\mathrm{r}}(I)=a(\bar{G})+\dim A=a(\bar{G})+2$. Thus, $a(\bar{G})=0$, as required.

(5) $\Longrightarrow$ (1) : Since $\bar{G}$ is Cohen–Macaulay, we have $\bar{\mathrm{r}}(I)=a(\bar{G})+\dim A=0+2=2$. □

We notice that if A is *not* a rational singularity, then elliptic ideals always exist.

PROPOSITION 3.3. *Let $(A,\mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k=A/\mathfrak{m}$, and let $I\subset A$ be an $\mathfrak{m}$-primary integrally closed ideal which is not a p_g-ideal. Then there exists a positive integer n such that $\overline{I^n}$ is an elliptic ideal. In particular, if A is not a rational singularity, then for any $\mathfrak{m}$-primary*

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integrally closed ideal I of A, then $\overline{I^n}$ is either a p_g-ideal or an elliptic ideal for every $n \geq p_g(A)$.

Proof. Let n be a positive integer such that $\ell_A(A/\overline{I^n}) = \bar{P}_I(n-1)$. Since the integral closure of $(\overline{I^n})^p$ coincides with $\overline{I^{np}}$ for p large, we have

$$\bar{e}_0(I^n) = n^2\bar{e}_0(I); \quad \bar{e}_1(I^n) = n\bar{e}_1(I) + \binom{n}{2}\bar{e}_0(I); \quad \bar{e}_2(I^n) = \bar{e}_2(I).$$

After substituting the $\bar{e}_i(I^n)$'s with the corresponding expressions in terms of the $\bar{e}_i(I)$'s we conclude that

$$\begin{aligned}\bar{e}_2(I^n) - \bar{e}_1(I^n) + \bar{e}_0(I^n) - \ell_A(A/\overline{I^n}) &= \bar{e}_0(I)\binom{n+1}{2} - \bar{e}_1(I)n + \bar{e}_2(I) - \ell_A(A/\overline{I^n}) \\ &= \bar{P}_I(n-1) - \ell_A(A/\overline{I^n}) = 0.\end{aligned}$$

Since I is not a p_g-ideal, then $\bar{e}_2(I^n) = \bar{e}_2(I) > 0$. Hence, by Theorem 3.2, then $\overline{I^n}$ is an elliptic ideal. ∎

We denote by $\mathfrak{M} = \mathfrak{m} + \bar{\mathcal{R}}_+$ the homogeneous maximal ideal of $\bar{\mathcal{R}}$. As usual, we say that $\bar{\mathcal{R}}$ is (FLC) if $\ell_A(H^i_{\mathfrak{M}}(\bar{\mathcal{R}})) < \infty$ for every $i \leq \dim A = 2$.

Proposition 3.4. *Assume I is an elliptic ideal, then $\bar{\mathcal{R}}$ is (FLC) but not Cohen–Macaulay with*

$$H^2_{\mathfrak{M}}(\bar{\mathcal{R}}) = [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_0 \cong [H^2_{\mathfrak{M}}(\bar{G})]_0.$$

Proof. Note that $\bar{\mathcal{R}}_{\mathfrak{M}}$ is a universally catenary domain which is a homomorphic image of a Cohen–Macaulay local ring. Hence, it is an (FLC) because $\bar{\mathcal{R}}$ satisfies Serre condition (S_2). Thus, $H^0_{\mathfrak{M}}(\bar{\mathcal{R}}) = H^1_{\mathfrak{M}}(\bar{\mathcal{R}}) = 0$ and $H^2_{\mathfrak{M}}(\bar{\mathcal{R}})$ has finite length.

Put $\mathcal{N} = \bar{\mathcal{R}}_+$. Then we obtain two exact sequences of graded $\bar{\mathcal{R}}$-modules.

$$0 \to \mathcal{N} \to \bar{\mathcal{R}} \to {}_hA \to 0,$$

$$0 \to \mathcal{N}(1) \to \bar{\mathcal{R}} \to \bar{G} \to 0,$$

where ${}_hA$ can be regarded as $\bar{\mathcal{R}}/\mathcal{N}$ which is concentrated in degree 0. One can easily see that $H^0_{\mathfrak{M}}(\mathcal{N}) = H^1_{\mathfrak{M}}(\mathcal{N}) = 0$, and we get

$$0 \to H^2_{\mathfrak{M}}(\mathcal{N}) \to H^2_{\mathfrak{M}}(\bar{\mathcal{R}}) \to {}_hH^2_{\mathfrak{m}}(A) \to H^3_{\mathfrak{M}}(\mathcal{N}) \to H^3_{\mathfrak{M}}(\bar{\mathcal{R}}) \to 0, \tag{3.1}$$

$$0 \to H^2_{\mathfrak{M}}(\mathcal{N})(1) \to H^2_{\mathfrak{M}}(\bar{\mathcal{R}}) \to H^2_{\mathfrak{M}}(\overline{G}) \to H^3_{\mathfrak{M}}(\mathcal{N})(1) \to H^3_{\mathfrak{M}}(\bar{\mathcal{R}}) \to 0. \tag{3.2}$$

For any integer $n \leq -1$, the first exact sequence (3.1) yields

$$0 \to [H^2_{\mathfrak{M}}(\mathcal{N})]_n \to [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_n \to 0.$$

In addition, the second exact sequence (3.2) yields

$$0 = [H^1_{\mathfrak{M}}(\bar{G})]_n \to [H^2_{\mathfrak{M}}(\mathcal{N})]_{n+1} \to [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_n.$$

Then $[H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_{-1} \subset [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_{-2} \subset \cdots \subset [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_n = 0$ for $n \ll 0$, and thus $[H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_n = 0$ for all $n \leq -1$.

For any integer $n \geq 1$, the first exact sequence (3.1) yields

$$0 \to [H^2_{\mathfrak{M}}(\mathcal{N})]_n \to [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_n \to 0.$$

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Moreover, as $a(\bar{G}) = 0$, we have

$$[H^2_{\mathfrak{M}}(\mathcal{N})]_{n+1} \to [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_n \to [H^2_{\mathfrak{M}}(\bar{G})]_n = 0 \text{ (ex)}.$$

Hence, we get $[H^2_{\mathcal{M}}(\bar{\mathcal{R}})]_n = 0$ for all $n \geq 1$.

Since $a(\bar{\mathcal{R}}) = -1$, we have $[H^3_{\mathfrak{M}}(\mathcal{N})]_1 \cong [H^3_{\mathfrak{M}}(\bar{\mathcal{R}})]_1 = 0$. Hence, we get

$$H^2_{\mathfrak{M}}(\bar{\mathcal{R}}) = [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_0 \cong [H^2_{\mathfrak{M}}(\bar{G})]_0,$$

as required. ∎

Corollary 3.5. *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain, and let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal.*

Then I is an elliptic ideal if and only if $0 \neq H^2_{\mathfrak{M}}(\bar{\mathcal{R}}) = [H^2_{\mathfrak{M}}(\bar{\mathcal{R}})]_0 \hookrightarrow H^2_{\mathfrak{m}}(A)$, where the last map is induced from the natural surjection $\bar{\mathcal{R}} \to {}_h A = \bar{\mathcal{R}}/\bar{\mathcal{R}}_+$.

Proof. Assume I is an elliptic ideal, then from the proof of Proposition 3.4 and Theorem 3.2, we conclude our assertions. Conversely, by our assumption, we can conclude that $\bar{G}(I)$ is Cohen–Macaulay with $a(\bar{G}(I)) = 0$ by a similar argument as in the proof of Proposition 3.4. Hence, I is an elliptic ideal by Theorem 3.2. ∎

For a cycle $C > 0$ on X, we denote by $\chi(C)$ the Euler characteristic of $\mathcal{O}_C$.

Definition 3.6. Let Z_f denote the *fundamental cycle*, namely, the nonzero minimal anti-nef cycle on X. The ring A is called *elliptic* if $\chi(Z_f) = 0$.

The following result follows from Theorem 3.2 and [22, Theorem 3.3].

Corollary 3.7. *If A is an elliptic singularity, then for every integrally closed ideal $I \subset A$, the following facts hold*:

(1) *$\bar{G}(I)$ is Cohen–Macaulay with $a(\bar{G}(I)) \leq 0$.*
(2) *I is elliptic or a p_g-ideal.*

Since there always exists an ideal I with $q(I) = 0$, we have $\bar{r}(A) = 2$.

The result above gives some evidence about a positive answer to the following question:

Question 3.8. *Assume $\bar{r}(A) = 2$, is it true that A is an elliptic singularity?*

We can give a positive answer to Question 3.8 if $\bar{e}_2(I) \leq 1$ for all integrally closed $\mathfrak{m}$-primary ideals. In the following result, we describe the integrally closed $\mathfrak{m}$-primary ideals satisfying this minimal condition.

Theorem 3.9. *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain over an algebraically closed field. Let $I \subset A$ be an $\mathfrak{m}$-primary integrally closed ideal, and let Q be a minimal reduction of I. Put $\bar{G} = \bar{G}(I)$ and $\bar{\mathcal{R}} = \bar{\mathcal{R}}(I)$. Then the following conditions are equivalent*:

(1) *$\bar{r}(I) = 2$ and $\ell_A(\overline{I^2}/QI) = 1$.*
(2) *$q(I) = q(\infty I) = p_g(A) - 1$.*
(3) *$\bar{e}_2(I) = 1$.*
(4) *$\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + 1$ and $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$.*
(5) *$\bar{G}$ is Cohen–Macaulay with $a(\bar{G}) = 0$ and $\ell_A([H^2_{\mathfrak{M}}(\bar{G})]_0) = 1$.*

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When this is the case, I is said to be a strongly elliptic ideal and $\bar{\mathcal{R}}$ is a Buchsbaum ring with $\ell_A(H^2_{\mathfrak{M}}(\bar{\mathcal{R}})) = 1$.

Proof. (1) $\Longrightarrow$ (2) : By Theorem 3.2, we have $p_g(A) > q(I) = q(\infty I)$. In particular, $q(2I) = q(I)$. By Proposition 2.5(1), $p_g(A) - q(I) = \ell_A(\overline{I^2}/QI) = 1$. Conversely, (2) $\Longrightarrow$ (1) again by Proposition 2.5.

(2) $\Longrightarrow$ (3) : By Proposition 2.6(4), we have

$$\bar{e}_2(I) = p_g(A) - q(I) = 1.$$

(3) $\Longrightarrow$ (2) : Since $p_g(A) - q(\infty I) = \bar{e}_2(I) = 1$, by assumption, we have $p_g(A) - 1 = q(\infty I) \leq q(I) \leq p_g(A)$. If $q(I) = p_g(A)$, then I is a p_g-ideal, and thus $\bar{e}_2(I) = 0$. This is a contradiction. Hence, $q(\infty I) = q(I) = p_g(A) - 1$, as required.

(1), (3) $\Longrightarrow$ (4) : It follows from Theorem 3.2 (1) $\Longrightarrow$ (3) and the fact that $1 < \mathrm{nr}(I) \leq \bar{\mathrm{r}}(I) = 2$.

(4) $\Longrightarrow$ (1) : By Proposition 2.5(1), we have

$$\begin{aligned} \ell_A(\overline{I^2}/QI) &= (p_g(A) - q(I)) - (q(I) - q(2I)), \\ \ell_A(\overline{I^3}/Q\overline{I^2}) &= (q(I) - q(2I)) - (q(2I) - q(3I)), \\ \vdots \quad &= \quad \vdots \end{aligned}$$

By a similar argument as in [10] and Proposition 2.6, we get

$$\begin{aligned} \bar{e}_2(I) &= \sum_{n=1}^{\infty} n \cdot \ell_A(\overline{I^{n+1}}/Q\overline{I^n}), \\ \bar{e}_1(I) - \bar{e}_0(I) + \ell_A(A/I) &= \sum_{n=1}^{\infty} \ell_A(\overline{I^{n+1}}/Q\overline{I^n}). \end{aligned}$$

Thus, our assumption implies $\ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = 1$ for some unique integer $n \geq 1$. On the other hand, since $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$, we must have $n = 1$.

(1) $\Longrightarrow$ (5) : Suppose (1). Then Theorem 3.2 (1) $\Longrightarrow$ (5) implies that $\bar{G}$ is Cohen–Macaulay with $a(\bar{G}) = 0$.

We remark that $\sqrt{\mathfrak{M}} = \sqrt{\bar{G}_+}$ in $\bar{G}$; hence, by [19, Proposition 3.1], we have $[H^2_{\mathfrak{M}}(\bar{G})]_0 \cong \overline{I^2}/QI \cong A/\mathfrak{m}$ has length 1. In particular, by Proposition 3.4, $H^2_{\mathfrak{M}}(\bar{\mathcal{R}})$ becomes an $A/\mathfrak{m}$-vector space, and thus $\bar{\mathcal{R}}$ is Buchsbaum.

(5) $\Longrightarrow$ (1) : By Theorem 3.2 (5) $\Longrightarrow$ (1), we have $\bar{\mathrm{r}}(I) = 2$. In addition, $\ell_A(\overline{I^2}/QI) = \ell_A([H^2_{\mathfrak{M}}(\bar{G})]_0) = 1$. ∎

It is clear that if I is a strongly elliptic ideal, then I is an elliptic ideal. In some cases, they are equivalent. Notice that the converse is *not* true in general. For instance, let $A = k[[x^2, y^2, z^2, xy, xz, yz]]/(x^4 + y^4 + z^4)$. Then A is a two-dimensional normal local domain with the maximal ideal $\mathfrak{m} = (x^2, y^2, z^2, xy, xz, yz)$. Then $\mathfrak{m}$ is a normal ideal, and $Q = (x^2, y^2)$ is a minimal reduction of $\mathfrak{m}$ with $\mathfrak{m}^3 = Q\mathfrak{m}^2$. Moreover, $\bar{\mathrm{r}}(\mathfrak{m}) = r(\mathfrak{m}) = 2$ and $\ell_A(\mathfrak{m}^2/Q\mathfrak{m}) = 3$ imply that $\mathfrak{m}$ is an elliptic ideal but not a strongly elliptic ideal.

Notice that (1) is equivalent to (3) follows also from [13].

Proposition 3.10. *Let $(A, \mathfrak{m})$ be a two-dimensional Gorenstein excellent normal local domain. Then $\mathfrak{m}$ is an elliptic ideal if and only if $\mathfrak{m}$ is a strongly elliptic ideal.*

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Proof. Assume $\mathfrak{m}$ is an elliptic ideal and Q be its minimal reduction. Since $\bar{r}(\mathfrak{m}) = 2$, $\mathfrak{m}\overline{\mathfrak{m}^2} \subset Q$, and we have $\overline{\mathfrak{m}^2}/Q\mathfrak{m} \cong (\overline{\mathfrak{m}^2} + Q)/Q \hookrightarrow A/Q$, whose image is contained in $(Q : \mathfrak{m})/Q$. Since the latter has length 1, $\ell_A(\overline{\mathfrak{m}^2}/Q\mathfrak{m}) = 1$ and $\mathfrak{m}$ is strongly elliptic. ∎

EXAMPLE 3.11. Let $A = \mathbb{C}[[x,y,z]]/(x^a + y^b + z^c)$ be a Brieskorn hypersurface, where $2 \le a \le b \le c$. Then:

(1) $\mathfrak{m}$ is a p_g-ideal if and only if $(a,b) = (2,2),(2,3)$.
(2) $\mathfrak{m}$ is an elliptic ideal (equivalently strongly elliptic) if and only if

$$(a,b) = (2,4),(2,5),(3,3),(3,4).$$

In particular, if $p \ge 1$ and $(a,b,c) = (2,4,4p+1)$, then $p_g(A) = p$ and $\mathfrak{m}$ is a (strongly) elliptic ideal. It follows from [26, Theorem 3.1 and Proposition 3.8].

EXAMPLE 3.12. Proposition 3.10 does not hold if $I \neq \mathfrak{m}$. Let A be any two-dimensional excellent normal local domain with $p_g(A) > 1$. Then there exist always integrally closed ideals I with $q(I) = 0$. Since $q(I) = q(2I) = 0$, $\bar{r}(I) = 2$, and $\bar{e}_2(I) = p_g(A)$. Thus, 3.10 does not hold for such I.

We recall that an excellent normal local domain for which every integrally closed $\mathfrak{m}$-primary ideal is a p_g-ideal is a rational singularity ($p_g(A) = 0$). This result suggests to study the next step.

DEFINITION 3.13 (e.g., [41]). An excellent normal local domain A is a *strongly elliptic singularity* if $p_g(A) = 1$.

Note that any strong elliptic singularity is an elliptic singularity. The following result characterizes algebraically the strongly elliptic singularities.

THEOREM 3.14. *Let $(A,\mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$ and assume that $p_g(A) > 0$. The following facts are equivalent*:

(1) *A is a strongly elliptic singularity.*
(2) *Every integrally closed ideal of A is either a p_g-ideal or a strongly elliptic ideal.*

Proof. It depends by the fact that always there exists an integrally closed ideal I of A such that $q(I) = 0$. Thus, $p_g(A) = \bar{e}_2(I)$. ∎

If A is a rational singularity, then every integrally closed $\mathfrak{m}$-primary ideal is normal. This is not true if A is an elliptic singularity, even if we assume A is a strongly elliptic singularity.

EXAMPLE 3.15.

(1) Let $A = k[X,Y,Z]/(X^3 + Y^3 + Z^3)$, then A is Gorenstein, $p_g(A) = 1$, and the maximal ideal $\mathfrak{m}$ is normal. If we consider $I = (x,y,z^2)$, then I^2 is not normal.
(2) Cutkosky showed that if $A = \mathbb{Q}[[X,Y,Z]]/(X^3 + 3Y^3 + 9Z^3)$ ($\mathbb{Q}$ rational numbers), then for every integrally closed ideal $I \subset A$, I^2 is also integrally closed and hence normal. This is because the elliptic curve does not have any $\mathbb{Q}$-rational point.
(3) Let $A = k[x,y,z]/(x^2 + y^4 + z^4)$, $I = \mathfrak{m} = (x,y,z)$, and $Q = (y,z)$. Then $p_g(A) = 1$ and $\overline{\mathfrak{m}^n} = x(y,z)^{n-2} + \mathfrak{m}^n$ for every $n \ge 2$.

PROPOSITION 3.16. *Let $(A,\mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$. Assume that I is a strongly elliptic ideal.*

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Then the following conditions are equivalent:

(1) $\overline{I^2} = I^2$.
(2) $\overline{I^n} = I^n$ *for some* $n \geq 2$.
(3) $\overline{I^n} = I^n$ *for every* $n \geq 2$.

Proof. By Theorem 3.9(1), we have $\ell_A(\overline{I^2}/QI) = 1$ and $\overline{I^n} = Q\overline{I^{n-1}}$ for $n \geq 3$. Hence, if $I^2 = \overline{I^2}$, then $I^n = \overline{I^n}$ for all $n \geq 2$.

Conversely, assume that $I^2 \neq \overline{I^2}$. Since $\ell_A(\overline{I^2}/QI) = 1$, we should have $I^2 = QI$. This implies that $G(I) := \oplus_{n\geq 0} I^n/I^{n+1}$ is Cohen–Macaulay with $a(G(I)) = -1$ (see [8], [37, Proposition 2.6]) and hence

$$\ell_A(A/I^{n+1}) = e_0(I)\binom{n+2}{2} - e_1(I)\binom{n+1}{1}$$

with $e_0(I) = \bar{e}_0(I)$ and $e_1(I) = e_0(I) - \ell_A(A/I)$.

On the other hand, by Theorem 3.2 and Corollary 3.9, we have

$$\begin{aligned}\ell_A(A/\overline{I^{n+1}}) = \bar{P}_I(n) &= \overline{e}_0(I)\binom{n+2}{2} - \overline{e}_1(I)\binom{n+1}{1} + \overline{e}_2(I)\\ &= e_0(I)\binom{n+2}{2} - (e_0(I) - \ell_A(A/I) + 1)\binom{n+1}{1} + 1\\ &= \ell_A(A/I^{n+1}) - n.\end{aligned}$$

This implies that $I^n \neq \overline{I^n}$ for all $n \geq 2$. ∎

We can characterize the normal ideals in a strongly elliptic singularity. Before showing the results, let us recall some definitions and basic facts on cycles and a vanishing theorem for elliptic singularities. In the following, A is an elliptic singularity, and X is a resolution of Spec(A).

For a cycle $C > 0$ on X, we denote by $\chi(C)$ the Euler characteristic $\chi(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C)$. Then $p_a(C) := 1 - \chi(C)$ is called the *arithmetic genus* of C. By the Riemann–Roch theorem, we have $\chi(C) = -(K_X + C)C/2$, where K_X is the canonical divisor on X. From this, if $C_1, C_2 > 0$ are cycles, we have $\chi(C_1 + C_2) = \chi(C_1) + \chi(C_2) - C_1C_2$. From the exact sequence

$$0 \to \mathcal{O}_{C_2}(-C_1) \to \mathcal{O}_{C_1+C_2} \to \mathcal{O}_{C_1} \to 0,$$

we have $\chi(\mathcal{O}_{C_2}(-C_1)) = -C_1C_2 + \chi(C_2)$.

If A is elliptic, then there exists a unique cycle E_{min}, called the *minimally elliptic cycle*, such that $\chi(E_{min}) = 0$ and $\chi(C) > 0$ for all cycles $0 < C < E_{min}$ (see [15]). Moreover, we have the following (see [15, Propositions 3.1 and 3.2 and Corollary 4.2], [36, (6.4) and (6.5)], [38, p. 428]).

PROPOSITION 3.17. *Assume that A is elliptic. Then $\chi(C) \geq 0$ for any cycle $C > 0$ on X and $C \geq E_{min}$ if $\chi(C) = 0$.*

Let us recall that the fundamental cycle Z_f can be computed via a sequence of cycles:

$$C_0 := 0, \quad C_1 = E_{j_1}, \quad C_i = C_{i-1} + E_{j_i}, \quad C_m = Z_f,$$

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where E_{j_1} is an arbitrary component of E and $C_{i-1}E_{j_i} > 0$ for $2 \le i \le m$. Such a sequence $\{C_i\}$ is called a *computation sequence* for Z_f. It is known that $h^0(\mathcal{O}_{C_i}) = 1$ for $1 \le i \le m$ (see [15, p. 1,260]).

The following vanishing theorems are essential in our argument.

THEOREM 3.18 (Röhr [31, 1.7]). *Let L be a divisor on X such that $LC > -2\chi(C)$ for every cycle $C > 0$ which occurs in a computation sequence for Z_f. Then $H^1(\mathcal{O}_X(L)) = 0$. If A is rational, then the converse holds, too.*

From Theorem 3.18 and Proposition 3.17, we have the following.

COROLLARY 3.19. *Assume that A is an elliptic singularity. Let L be a nef divisor on X such that $LE_{min} > 0$. Then $H^1(\mathcal{O}_X(L)) = 0$.*

PROPOSITION 3.20. *Assume that A is an elliptic singularity and D the minimally elliptic cycle on X. Let F be a nef divisor on X. If $FD > 0$, then $H^1(\mathcal{O}_X(F - D)) = 0$, and from the exact sequence $0 \to \mathcal{O}_X(F - D) \to \mathcal{O}_X(F) \to \mathcal{O}_D(F) \to 0$, the restriction map $H^0(\mathcal{O}_X(F)) \to H^0(\mathcal{O}_D(F))$ is surjective.*

Proof. If $F - D$ is nef, since $(F - D)D > 0$, we have $H^1(\mathcal{O}_X(F - D)) = 0$ by Corollary 3.19. Assume that $F - D$ is not nef. As in [6, 1.4], we have a sequence $\{D_i\}$ of cycles such that

$$D_0 = D, \quad D_i = D_{i-1} + E_{j_i}, \; (F - D_{i-1})E_{j_i} < 0 \; (1 \le i \le s), \quad F - D_s \text{ is nef.}$$

Since $F - Z_f$ is nef, $(F - D_{i-1})E_{j_i} < 0$ implies $D_{i-1}E_{j_i} > 0$, and $D \le Z_f$, we see that $D_s \le Z_f$ and D_s occurs in a computation sequence for Z_f. Then the equalities $\chi(D) = \chi(D_s) = 0$ and $\chi(D_i) = \chi(D_{i-1}) + \chi(E_{j_i}) - D_{i-1}E_{j_i}$ imply that $FE_{j_i} = 0$, $D_{i-1}E_{j_i} = 1$, and $h^j(\mathcal{O}_{E_{j_i}}(F - D_{i-1})) = 0$ for $j = 0, 1$ and $1 \le i \le s$. Since

$$0 \le \chi(D_s + D) = \chi(D_s) + \chi(D) - DD_s = -DD_s,$$

we have $(F - D_s)D > 0$. Therefore, from the exact sequence

$$0 \to \mathcal{O}_X(F - D_i) \to \mathcal{O}_X(F - D_{i-1}) \to \mathcal{O}_{E_{j_i}}(F - D_{i-1}) \to 0,$$

we obtain $H^1(\mathcal{O}_X(F - D)) = H^1(\mathcal{O}_X(F - D_s)) = 0$. ∎

THEOREM 3.21 [5, 2.7]. *Let C be a Cohen–Macaulay projective scheme of pure dimension 1, and let $\mathcal{F}$ be a rank 1 torsion-free sheaf on C. Assume that $\deg \mathcal{F}|_W := \chi(\mathcal{F}|_W) - \chi(W) > -2\chi(W)$ for every subcurve $W \subset C$. Then $H^1(\mathcal{F}) = 0$.*

To show the normality of an ideal I, the following is essential.

PROPOSITION 3.22. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be nef invertible sheaves on the minimally elliptic cycle D such that $d_i := \deg \mathcal{L}_i \ge 3$ for $i = 1, 2$. Then the multiplication map*

$$\gamma : H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \to H^0(\mathcal{L}_1 \otimes \mathcal{L}_2)$$

is surjective.

Proof. First, note that $\chi(W) > 0$ for any cycle $0 < W \lneqq D$ by the definition of the minimally elliptic cycle. For any subscheme $\Lambda \subset D$, we denote by $I_\Lambda \subset \mathcal{O}_D$ the ideal sheaf of Λ. For any cycle $W \le D$ and any $p \in \operatorname{Supp}(W)$, we have $\deg(I_p\mathcal{L}_i)|_W = \deg \mathcal{L}_i|_W - 1$. Therefore, it follows from Theorem 3.21 that $H^1(I_p\mathcal{L}_i) = 0$ for any point $p \in \operatorname{Supp}(D)$.

Hence, $\mathcal{L}_i$ is generated by global sections. Let $s \in H^0(\mathcal{L}_1)$ be a general section and consider the exact sequence

$$0 \to \mathcal{O}_D \xrightarrow{\times s} \mathcal{L}_1 \to \mathcal{L}_1|_B \to 0, \tag{3.3}$$

where B is the zero-dimensional subscheme of D of degree $d_1 = \deg \mathcal{L}_1$ defined by s. Note that since $\mathcal{L}_1$ is generated by global sections, each point of $\mathrm{Supp}(B)$ is a nonsingular point of $\mathrm{Supp}(D)$ and there exists $s_1 \in H^0(\mathcal{L}_1)$ such that $\mathcal{L}_1|_B \cong s_1\mathcal{O}_B \cong \mathcal{O}_B$. Let $p \in \mathrm{Supp}(B)$ be any point. The following fact makes our proof easier.

Claim 7. *Let $\mathfrak{n} \subset \mathcal{O} := \mathcal{O}_{X,p}$ be the maximal ideal. Then we can take generators x, y of $\mathfrak{n}$, so that $\mathcal{O}_{D,p} = \mathcal{O}/(x^{n_p})$ with $n_p \geq 1$ and $\mathcal{O}_{B,p} \cong \mathcal{O}/(x^{n_p}, y)$. Hence, at p, the subschemes of B correspond to monomials x^ℓ with $\ell \leq n_p$.*

Proof of Claim 7. Since E is nonsingular at p, we have the generators $x, y \in \mathfrak{n}$ such that $\mathcal{O}_{D,p} = \mathcal{O}/(x^{n_p})$. Assume that $\mathcal{O}_{B,p} = \mathcal{O}/(x^{n_p}, f)$, where $f \in \mathfrak{n}$. If $\ell_{\mathcal{O}}(\mathcal{O}/(x,f)) = 1$, we can put $y = f$. Assume that $\ell_{\mathcal{O}}(\mathcal{O}/(x,f)) \geq 2$. Let $0 < W \leq D$ be any cycle, and let $\mathcal{O}_{W,p} = \mathcal{O}/(x^{m_p})$. Assume that $p \in \mathrm{Supp}(W)$. Then the cokernel of $(I_p^2\mathcal{L}_1)|_W \to \mathcal{L}_1|_W$ is isomorphic to $\mathcal{O}/\mathfrak{n}^2 + (x^{m_p})$. If $m_p = 1$, then $\deg(I_p^2\mathcal{L}_1)|_W = \deg \mathcal{L}_1|_W - 2$. If $m_p \geq 2$, then $\deg(I_p^2\mathcal{L}_1)|_W = \deg \mathcal{L}_1|_W - 3 \geq \ell_{\mathcal{O}}(\mathcal{O}/(x^{m_p}, f)) - 3 \geq 1$. Thus, we have $H^1(I_p^2\mathcal{L}_1) = 0$ by Theorem 3.21, and the map $H^0(I_p\mathcal{L}_1) \to I_p\mathcal{L}_1/I_p^2\mathcal{L}_1$ is surjective; however, this shows that we can take $f = y$. ∎

Tensoring $\mathcal{L}_2$ with the sequence (3.3), we obtain the exact sequence

$$0 \to H^0(\mathcal{L}_2) \xrightarrow{\times s} H^0(\mathcal{L}_1 \otimes \mathcal{L}_2) \to H^0(\mathcal{O}_B) \to 0,$$

since $H^1(\mathcal{L}_2) = 0$. As seen above, we have general sections $s_1 \in H^0(\mathcal{L}_1)$ and $s_2 \in H^0(\mathcal{L}_2)$ such that $s_1 s_2 \mapsto 1 \in H^0(\mathcal{O}_B)$. Thus, the sections of $H^0(\mathcal{L}_1 \otimes \mathcal{L}_2)$ which map to $1 \in H^0(\mathcal{O}_B)$ are in the image of γ. It is now sufficient to show that for any subscheme $B' \subset B$ of $\deg B' < d_1$, the image of γ contains a section $t \in H^0(\mathcal{I}_{B'}\mathcal{L}_1 \otimes \mathcal{L}_2)$ such that $\mathcal{L}_1 \otimes \mathcal{L}_2/t\mathcal{O}_D \cong \mathcal{O}_{B'} \oplus \mathcal{O}_{\overline{B}}$, where $\mathrm{Supp}(\overline{B}) \cap \mathrm{Supp}(B) = \emptyset$. To prove this, we write $\mathcal{I}_{B'} = \mathcal{I}_{B_1}\mathcal{I}_{B_2}$ $(B_1, B_2 \subset B')$, so that $\deg \mathcal{I}_{B_i}\mathcal{L}_i = d_i - \deg B_i \geq 2$ for $i = 1, 2$ (note that $\deg B_1 + \deg B_2 < d_1$). Let $0 < W \leq D$ be any cycle, and let $p \in \mathrm{Supp}(B)$. We use the notation of the proof of Claim 7. Suppose that $\mathcal{O}_{B_1,p} = \mathcal{O}/(x^{\ell_p}, y)$. Then $\deg \mathcal{L}_1|_W = \sum_W m_p$, where $\sum_W$ means the sum over $p \in \mathrm{Supp}(B) \cap \mathrm{Supp}(W)$, and the cokernel of $(I_{B_1}\mathcal{L}_1)|_W \to \mathcal{L}_1|_W$ is isomorphic to $\mathcal{O}/(x^{m_p}, x^{\ell_p}, y)$. Therefore,

$$\deg(I_{B_1}\mathcal{L}_1)|_W = \sum_W (m_p - \min(m_p, \ell_p)).$$

Since $\deg \mathcal{I}_{B_1}\mathcal{L}_1 = d_1 - \deg B_1 \geq 2$, by Theorem 3.21, we have $H^1(I_q\mathcal{I}_{B_1}\mathcal{L}_1) = 0$ for any point $q \in \mathrm{Supp}(B)$. Hence, $H^0(\mathcal{I}_{B_1}\mathcal{L}_1)$ has no base points. Clearly, the same results for $\mathcal{I}_{B_2}\mathcal{L}_2$ hold. Therefore, for each $i = 1, 2$, we have a section $t_i \in H^0(\mathcal{I}_{B_i}\mathcal{L}_i)$ such that $\mathcal{L}_i/t_i\mathcal{O}_D \cong \mathcal{O}_{B_i} \oplus \mathcal{O}_{\overline{B_i}}$, where $\mathrm{Supp}(\overline{B_i}) \cap \mathrm{Supp}(B) = \emptyset$. Then $t := t_1 t_2$ satisfies the required property.

Theorem 3.23. *Let $(A, \mathfrak{m})$ be a two-dimensional excellent normal local domain containing an algebraically closed field $k = A/\mathfrak{m}$. Assume that A is a strongly elliptic singularity. If $I = I_Z$ is an elliptic ideal (equivalently, I is not a p_g-ideal) and D is the minimally elliptic cycle on X, then I^2 is integrally closed (equivalently, I is normal) if and only if $-ZD \geq 3$ and if $-ZD \leq 2$, then $I^2 = QI$.*

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Proof. Assume that $-ZD \le 2$. Since $H^1(\mathcal{O}_X(-Z)) = 0$, by the Riemann–Roch theorem, we have $h^0(\mathcal{O}_D(-nZ)) = -nZD$ for $n \ge 1$. Hence,

$$H^0(\mathcal{O}_D(-Z)) \otimes H^0(\mathcal{O}_D(-Z)) \to H^0(\mathcal{O}_D(-2Z))$$

cannot be surjective. By Proposition 3.20, the map

$$H^0(\mathcal{O}_X(-Z)) \otimes H^0(\mathcal{O}_X(-Z)) \to H^0(\mathcal{O}_X(-2Z))$$

cannot be surjective, too. Therefore, $I^2 \ne I_{2Z}$, and hence $I^2 = QI$.

Assume that $-ZD \ge 3$. By Propositions 3.20 and 3.22, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_X(-Z)) \otimes H^0(\mathcal{O}_X(-Z)) & \xrightarrow{\alpha} & H^0(\mathcal{O}_X(-2Z)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_D(-Z)) \otimes H^0(\mathcal{O}_D(-Z)) & \longrightarrow & H^0(\mathcal{O}_D(-2Z)), \end{array}$$

where at least the maps other than α are surjective. By Proposition 3.20 and its proof, we have $I_{2Z} = I^2 + H^0(\mathcal{O}_X(-2Z - D))$, $H^0(\mathcal{O}_X(-2Z - D)) = H^0(\mathcal{O}_X(-2Z - D_s))$, and $-(Z + D_s)D \ge -ZD \ge 3$. We have as above a surjective map

$$H^0(\mathcal{O}_D(-Z)) \otimes H^0(\mathcal{O}_D(-Z - D_s)) \to H^0(\mathcal{O}_D(-2Z - D_s))$$

and $H^0(\mathcal{O}_X(-2Z - D_s)) \subset IH^0(\mathcal{O}_X(-Z - D_s)) + H^0(\mathcal{O}_X(-2Z - D_s - D))$. From these arguments, for $m > 0$, we have $I_{2Z} \subset I^2 + H^0(\mathcal{O}_X(-2Z - mD))$. We denote by $H(m)$ the minimal anti-nef cycle on X such that $H(m) \ge 2Z + mD$. Then $H^0(\mathcal{O}_X(-2Z - mD)) = H^0(\mathcal{O}_X(-H(m)))$, and for an arbitrary $n \in \mathbb{Z}_+$, there exists $m(n) \in \mathbb{Z}_+$ such that $H(m(n)) \ge nE$. Therefore, $H^0(\mathcal{O}_X(-2Z - mD)) \subset I^2$ for sufficiently large m, and we obtain $I_{2Z} = I^2$. □

REMARK 3.24. Assume that A is elliptic. It follows from Proposition 4.5 and Corollary 3.19 that $q(I) = 0$ if and only if $ZD \ne 0$. By an argument similar to the proof of Theorem 3.23, we can prove that if $ZD \ne 0$, then $I = I_Z$ is normal if and only if $-ZD \ge 3$.

For elliptic ideals in an elliptic singularity (not strongly elliptic), Remark 3.24 cannot be applied because the condition $ZD \ne 0$ does not hold in general. Next example shows that the condition $0 < -ZD < 3$ is not necessary for I_Z being not normal.

EXAMPLE 3.25. Suppose that $p \ge 1$ be an integer. Let $A = k[x, y, z]/(x^2 + y^3 + z^{6(p+1)})$, and assume that X is the minimal resolution. Then E is a chain of $p+1$ nonsingular curves $E_0, E_1, \dots, E_p$, where $g(E_0) = 1$, $E_0^2 = -1$, $g(E_i) = 0$, $E_i^2 = -2$, $E_{i-1}E_i = 1$, for $1 \le i \le p$, and $E_iE_j = 0$ if $|i - j| \ge 2$. It is easy to see that A is elliptic and E_0 is the minimally elliptic cycle. Furthermore, $\mathfrak{m}$ is a p_g-ideal and $p_g(A) = p+1$ by [26, 3.1 and 3.10]. Since A is not strongly elliptic, there is a non-p_g-ideal I_Z such that $-ZE_0 = 0$ (see Theorem 3.14 and Proposition 4.5). Let $W = \sum_{i=0}^{p}(p+1-i)E_i$. Then $-W \sim K_X$ and the exceptional part of the divisors $\mathrm{div}_X(x)$, $\mathrm{div}_X(y)$, and $\mathrm{div}_X(z)$ are $3W$, $2W$, and E, respectively. For $1 \le n \le p+1$, let $D_n = \gcd(nE, W) := \sum_{i=0}^{p} \min(n, p+1-i)E_i$. (Our cycle D_n coincides with C_{n-1} in [22, 2.6].) Then $\mathcal{O}_X(-2D_n)$ is generated (cf. [22, 3.6(4)]) and $D_n^2 = -n$. Let $I_n = I_{2D_n}$. Since the cohomological cycle of $(D_n)^\perp$ is $W - D_n$, we have $q(I_n) = p_g(A) - n$ by Proposition 4.5; note that $-(W - D_n) \sim K_X$ on a neighborhood of $\mathrm{Supp}(W - D_n) = E_0 \cup \cdots \cup E_{p-n}$. Then $I_n = (x, y, z^{2n})$. We have $D_nE_0 = 0$ for $1 \le n \le p$ and $D_{p+1}E_0 = E_0^2 = -1$. Therefore, it follows from Remark 3.24 that $\overline{I_{p+1}^2} \neq I_{p+1}^2$, since $-2D_{p+1}E_0 = 2$. However, the condition $0 < -ZE_0 \leq 2$ is not necessary for I_Z being not normal. In fact, we have $\overline{I_n^2} \neq I_n^2$ for all $1 \leq n \leq p+1$ because $xz^n \notin I_n^2$ and $(xz^n)^2 \in I_n^4$.

§4. The existence of strongly elliptic ideals

Motivated by the fact that in every two-dimensional excellent normal local domain which is not a rational singularity elliptic ideals always exist, it is natural to ask if it is also true for strongly elliptic ideals. We need some more preliminaries for proving that the answer is negative, in particular there are two-dimensional excellent normal local domains with no integrally closed $\mathfrak{m}$-primary ideals I with $\bar{e}_2(I) = 1$. Assume $(A, \mathfrak{m})$ is a two-dimensional excellent normal local domain over an algebraically closed field.

Let $\pi\colon X \to \operatorname{Spec} A$ be a resolution of singularity with exceptional set $E = \bigcup E_i$.

Definition 4.1. Let $D \geq 0$ be a cycle on X, and let

$$h(D) = \max\left\{ h^1(\mathcal{O}_B) \,\middle|\, B \in \sum \mathbb{Z}E_i,\ B \geq 0,\ \operatorname{Supp}(B) \subset \operatorname{Supp}(D) \right\}.$$

We put $h^1(\mathcal{O}_B) = 0$ if $B = 0$. There exists a unique minimal cycle $C \geq 0$ such that $h^1(\mathcal{O}_C) = h(D)$ (cf. [30, 4.8]). We call C the *cohomological cycle* of D. The cohomological cycle of E is denoted by C_X.

Note that $p_g(A) = h(E)$, and that if A is Gorenstein and π is the minimal resolution, then the canonical cycle $Z_{K_X} = C_X$ (see [30, 4.20]). Clearly, the minimally elliptic cycle is the cohomological cycle of itself.

Remark 4.2. (1) If C_1 and C_2 are cohomological cycles of some cycles on X such that $C_1 \leq C_2$ and $h^1(\mathcal{O}_{C_1}) < h^1(C_2)$, then $\operatorname{Supp}(C_1) \neq \operatorname{Supp}(C_2)$.

(2) In general, for $q < p_g(A)$, cohomological cycle C with $h^1(\mathcal{O}_C) = q$ is not unique. For example, there exists a singularity whose minimal good resolution has two minimally elliptic cycles (e.g., [20]).

The following result is a generalization of [25, 2.6].

Proposition 4.3. *Assume that $p_g(A) > 0$, and let $D \geq 0$ be a reduced cycle on X. Then the cohomological cycle C of D is the minimal cycle such that $H^0(X \setminus D, \mathcal{O}_X(K_X)) = H^0(X, \mathcal{O}_X(K_X + C))$. Therefore, if $g\, X' \to X$ is the blowing-up at a point in* $\operatorname{Supp} C$ *and E' the exceptional set for g, then the cohomological cycle C' of g^*D satisfies that $g_*^{-1}C \leq C' \leq g^*C - E'$ and $h^1(\mathcal{O}_{C'}) = h^1(\mathcal{O}_C)$; we have $C' = g^*C - E'$ if $\mathcal{O}_X(K_X + C)$ is generated at the center of the blowing-up.*

Proof. Let $F > 0$ be an arbitrary cycle with $\operatorname{Supp}(F) \subset D$. By the duality, we have $h^1(\mathcal{O}_F) = h^0(\mathcal{O}_F(K_X + F))$. From the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + F) \to \mathcal{O}_F(K_X + F) \to 0$$

and the Grauert–Riemenschneider vanishing theorem, we have

$$h^1(\mathcal{O}_F) = \ell_A(H^0(X, \mathcal{O}_X(K_X + F))/H^0(X, \mathcal{O}_X(K_X))). \tag{4.1}$$

On the other hand, we have the inclusion

$$H^0(X, \mathcal{O}_X(K_X + F)) \subset H^0(X \setminus D, \mathcal{O}_X(K_X)),$$

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where the equality holds if F is sufficiently large; if the equality holds, we obtain $h^1(\mathcal{O}_F) = h(D)$, because the upper bound $\ell_A(H^0(X \setminus D, \mathcal{O}_X(K_X))/H^0(X, \mathcal{O}_X(K_X)))$ for $h^1(\mathcal{O}_F)$ depends only on $\mathrm{Supp}(D)$. Clearly, the minimum of such cycles F exists as the maximal poles of rational forms in $H^0(X \setminus D, \mathcal{O}_X(K_X))$. Let $D' = g^{-1}(D)$. Since $K_{X'} + g^*C - E' = g^*(K_X + C)$, we have

$$H^0(X', \mathcal{O}_{X'}(K_{X'} + g^*C - E')) = H^0(X' \setminus D', \mathcal{O}_X(K_{X'})).$$

Hence, $C' \le g^*C - E'$. The inequality $g_*^{-1}C \le C'$ is clear. From (4.1), we have $h^1(\mathcal{O}_{C'}) = h^1(\mathcal{O}_C)$. If $\mathcal{O}_X(K_X + C)$ is generated at the center of the blowing-up, then $\mathcal{O}_{X'}(K_{X'} + g^*C - E')$ has no fixed components, and the minimality of the cycle $g^*C - E'$ follows. ∎

DEFINITION 4.4. We define a reduced cycle $Z^\perp$ to be the sum of the components $E_i \subset E$ such that $ZE_i = 0$.

From [24, 3.4], we have the following.

PROPOSITION 4.5. *Let $I = I_Z$ be represented by a cycle Z on X and denote by C the cohomological cycle of $Z^\perp$. Assume $\bar{r}(I) = 2$, then $\mathcal{O}_C(-Z) \cong \mathcal{O}_C$ and $h^1(\mathcal{O}_C) = q(I)$.*

The converse of the result above is described as follows.

PROPOSITION 4.6. *If C is the cohomological cycle of a cycle on X with $h^1(\mathcal{O}_C) = q > 0$, then there exist a resolution $Y \to \mathrm{Spec}\, A$ and a cycle $Z > 0$ on Y such that $\mathcal{O}_Y(-Z)$ is generated and $q(I_Z) = q(\infty I_Z) = q$.*

Proof. There exists a cycle W on X such that $WE_i < 0$ for all E_i and $\mathcal{O}_X(-W)$ is generated (cf. the proof of [23, 4.5]). Let $h \in I_W$ be a general element. First, we show that there exist a resolution $Y \to \mathrm{Spec}\, A$ and a cohomological cycle D on Y with $h^1(\mathcal{O}_D) = q$ such that if Z_h is the exceptional part of $\mathrm{div}_Y(h)$, then $Z_h^\perp = D_{red}$. We obtain the resolution Y from X by taking blowing-ups appropriately as follows. Let $H \subset X$ be an irreducible component of the proper transform of $\mathrm{div}_{\mathrm{Spec}\, A}(h)$ intersecting C at a point p, and let $g\, X' \to X$ be the blowing-up at p. Let C' be the cohomological cycle of g^*C. Then $h^1(\mathcal{O}_{C'}) = q$ by Proposition 4.3. If the intersection number $C'(g_*^{-1}H)$ is positive, then we take again the blowing-up at the intersection point. By the property of the intersection number of curves and Proposition 4.3, taking blowing-ups in this manner, we obtain a resolution $Y \to \mathrm{Spec}\, A$ and a cohomological cycle D which satisfy the conditions described above; in fact, for an exceptional prime divisor F on Y, we have that $F \le Z_h^\perp$ if and only if F does not intersect the proper transform of $\mathrm{div}_{\mathrm{Spec}\, A}(h)$. Thus, it follows from [22, 3.6] (cf. [24, 3.4]) that $\mathcal{O}_Y(-nZ_h)$ is generated and $h^1(\mathcal{O}_Y(-nZ_h)) = q$ for $n \ge p_g(A)$. Then the cycle $Z := p_g(A)Z_h$ satisfies the assertion. ∎

COROLLARY 4.7. *There exists a strongly elliptic ideal in A if and only if there exists a cohomological cycle C of a cycle on a resolution $Y \to \mathrm{Spec}\, A$ such that $h^1(\mathcal{O}_C) = p_g(A) - 1$.*

EXAMPLE 4.8. Let C be a nonsingular curve of genus $g \ge 2$ and D an divisor on C with $\deg D > 0$. Let $A = \bigoplus_{n \ge 0} H^0(C, \mathcal{O}_C(nD))$, and assume that $a(A) = 0$. Then $p_g(A) = g$ and A has no strongly elliptic ideals because any cycle F on any resolution has $h^1(\mathcal{O}_F) = 0$ or g. More precisely, if $ZE_0 = 0$, where $E_0 \subset E$ denotes the curve of genus g, then I_Z is a p_g-ideal; otherwise, $q(\infty I_Z) = 0$.

Next example shows that there are local normal Gorenstein domains that always have strongly elliptic ideals.

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EXAMPLE 4.9. Let C be a nonsingular curve of genus $g \geq 2$, and put

$$A = \bigoplus_{n \geq 0} H^0(\mathcal{O}_C(nK_C)).$$

Then A is a normal Gorenstein ring by [39]. Suppose that $f : X \to \operatorname{Spec} A$ is the minimal resolution. We have

$$p_g(A) = \sum_{n \geq 0} h^1(\mathcal{O}_C(nK_C)) = g+1$$

by Pinkham's formula [28], $E \cong C$, $\mathcal{O}_E(-E) \cong \mathcal{O}_E(K_E)$, and $K_X = -2E$ (cf. [23, 4.6]). Let $Y \to X$ be the blowing-up at a point $p \in E$, and let E_1 be the fiber of p and E_0 the proper transform of E. By Proposition 4.3, we have $C_Y = 2E_0 + E_1$. It follows from (b) of the theorem in [30, 4.8] that $h^1(\mathcal{O}_{E_0}) \leq h^1(\mathcal{O}_{nE_0}) < p_g(A)$ for every $n \geq 1$. Hence, the cohomological cycle of E_0 is E_0 and $h^1(\mathcal{O}_{E_0}) = g = p_g(A) - 1$. Therefore, A has a strongly elliptic ideal by Corollary 4.7.

Next, we construct a strongly elliptic ideal. Take a general linear form $L \in A_1 \subset A$, and suppose that $\sup(\operatorname{div}_X(L) - E) \cap E$ consists of $\deg K_C$ points $p_1, \ldots, p_{2g-2} \in E$. Let $\phi : X' \to X$ be the blowing-up at $\{p_1, \ldots, p_{2g-2}\}$, and let F_0 be the proper transform of E and $F_i = \pi^{-1}(p_i)$. Let $Z = F_0 + 2(F_1 + \cdots + F_{2g-2})$. Then $\mathcal{O}_{X'}(-Z)$ is generated, $ZC_{X'} \neq 0$, and $ZF_0 = 0$. Thus, $q(I_Z) = g = p_g(A) - 1$ (cf. [24, 3.4]). Moreover, we have that $\ell_A(A/I_Z) = g$ by Theorem 2.2 and $I_Z = \overline{\mathfrak{m}^2} + (L)$. Note that if C is not hyperelliptic, then $\mathfrak{m}^2$ is normal, because A is a standard graded ring.

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REFERENCES

[1] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.
[2] A. Corso, C. Polini, and M. E. Rossi, *Depth of associated graded rings via Hilbert coefficients of ideals*, J. Pure Appl. Algebra **201** (2005), 126–141.
[3] A. Corso, C. Polini, and M. E. Rossi, *Bounds on the normal Hilbert coefficients*, Proc. Amer. Math. Soc. **144** (2016), 1919–1930.
[4] S. D. Cutkosky, *A new characterization of rational surface singularities*, Invent. Math. **102** (1990), 157–177.
[5] M. Franciosi and E. Tenni, *On Clifford's theorem for singular curves*, Proc. Lond. Math. Soc. (3) **108** (2014), 225–252.
[6] J. Giraud, *Improvement of Grauert–Riemenschneider's theorem for a normal surface*, Ann. Inst. Fourier (Grenoble) **32** (1982), 13–23.
[7] S. Goto and K. Nishida, *The Cohen–Macaulay and Gorenstein Rees Algebras Associated to Filtrations*, Mem. Amer. Math. Soc. **526**, Amer. Math. Soc., Providence, RI, 1994.
[8] S. Goto and Y. Shimoda, "On the Rees algebras of Cohen–Macaulay local rings" in *Commutative Algebra (Fairfax, VA, 1979)*, Lect. Notes Pure Appl. Math. **68**, Dekker, New York, 1982, 201–231.
[9] S. Goto and K. Watanabe, On graded rings. I, J. Math. Soc. Japan **30** (1978), 179–213.
[10] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293–318.
[11] S. Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988), 390–401.
[12] S. Itoh, *Coefficients of normal Hilbert polynomials*, J. Algebra **150** (1992), 101–117.
[13] S. Itoh, *Hilbert coefficients of integrally closed ideals*, J. Algebra **176** (1995), 638–652.

https://doi.org/10.1017/nmj.2022.5 Published online by Cambridge University Press

[14] M. Kato, *Riemann–Roch theorem for strongly pseudoconvex manifolds of dimension 2*, Math. Ann. **222** (1976), 243–250.
[15] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** (1975), 1257–1295.
[16] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 195–279.
[17] S. Masuti, K. Ozeki, and M. E. Rossi, *A filtration of the Sally module and the first normal Hilbert coefficient*, J. Algebra **571** (2021), 376–392 (Volume in honor of Craig Huneke).
[18] S. Masuti, K. Ozeki, M. E. Rossi, and H. Le Truong, *On the structure of the Sally module and the second normal Hilbert coefficient*, Proc. Amer. Math. Soc. **148** (2020), 2757–2771.
[19] S. Masuti, P. Sarkar, and J. Verma, *Variations on the Grothendieck–Serre formula for Hilbert functions and their applications*, Algebra and its applications, Springer Proc. Math. Stat., vol. **174**, Springer, Singapore, 2016, pp. 123–158.
[20] A. Némethi and T. Okuma, *Analytic singularities supported by a specific integral homology sphere link*, Methods Appl. Anal. **24** (2017), 303–320.
[21] D. G. Northcott and D. Rees, *Reduction of ideals in local rings*, Math. Proc. Cambridge Philos. Soc. **50** (1954), 145–158.
[22] T. Okuma, *Cohomology of ideals in elliptic surface singularities*, Ill. J. Math. **61** (2017), 259–273.
[23] T. Okuma, K. -i. Watanabe, and K. Yoshida, *Good ideals and p_g-ideals in two-dimensional normal singularities*, Manuscripta Math. **150** (2016), 499–520.
[24] T. Okuma, K. Watanabe, and K. Yoshida, *Rees algebras and* $\mathrm{p_g}$*-ideals in a two-dimensional normal local domain*, Proc. Amer. Math. Soc. **145** (2017), 39–47.
[25] T. Okuma, K. Watanabe, and K. Yoshida, *A characterization of 2-dimensional rational singularities via core of ideals*, J. Algebra **499** (2018), 450–468.
[26] T. Okuma, K. Watanabe, and K. Yoshida, *Normal reduction numbers for normal surface singularities with application to elliptic singularities of Brieskorn type*, Acta Math. Vietnam. **44** (2019), 87–100.
[27] T. Okuma, K.-i. Watanabe, and K. Yoshida, *The normal reduction number of two-dimensional cone-like singularities*, Proc. Amer. Math. Soc. **149** (2021), 4569–4581.
[28] H. Pinkham, *Normal surface singularities with* C^* *action*, Math. Ann. **227** (1977), 183–193.
[29] D. Rees, *A note on analytically unramified local rings*, J. Lond. Math. Soc. **36** (1961), 24–28.
[30] M. Reid, "Chapters on algebraic surfaces" in *Complex Algebraic Geometry*, IAS/Park City Math. Ser. **3**, Amer. Math. Soc., Providence, RI, 1997, 3–159.
[31] A. Röhr, *A vanishing theorem for line bundles on resolutions of surface singularities*, Abh. Math. Semin. Univ. Hambg. **65** (1995), 215–223.
[32] M. E. Rossi, *A bound on the reduction number of a primary ideal*, Proc. Amer. Math. Soc. **128** (1999), 1325–1332.
[33] M. E. Rossi and G. Valla, *Hilbert Functions of Filtered Modules*, Lect. Notes Unione Mat. Ital. **9**, Springer, Bologna (Italy), 2010.
[34] J. Sally, Cohen–Macaulay local ring of embedding dimension $\mathrm{e}+\mathrm{d}-2$, J. Algebra **83** (1983), 393–408.
[35] J. Sally, Ideals whose Hilbert function and Hilbert polynomial agree at $\mathrm{n}=1$, J. Algebra **157** (1993), 534–547.
[36] M. Tomari, *Ap*$_\mathrm{g}$*-formula and elliptic singularities*, Publ. Res. Inst. Math. Sci. **21** (1985), 297–354.
[37] P. Valabrega and G. Valla, *Form rings and regular sequence*, Nagoya Math. J. **72** (1978), 93–101.
[38] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math. **92** (1970), 419–454.
[39] K.-i. Watanabe, *Some remarks concerning Demazure's construction of normal graded rings*, Nagoya Math. J. **83** (1981), 203–211.
[40] K.-i. Watanabe and K. Yoshida, *Hilbert–Kunz multiplicity, McKay correspondence and good ideals in 2-dimensional rational singularities*, Manuscripta Math. **104** (2001), 275–294.
[41] S. S. T. Yau, *On strongly elliptic singularities*, Amer. J. Math. **101** (1979), 855–884.

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