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GRAPHS SUPPRESSIBLE TO AN EDGE

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An application of graph theory to automatic traffic control [2] gave rise to the problem of deciding which connected graphs have points of degree 2 which can be successively suppressed until only a single edge remains. We characterize such graphs in terms of four forbidden subgraphs (Fig. 1) illustrated by the complete

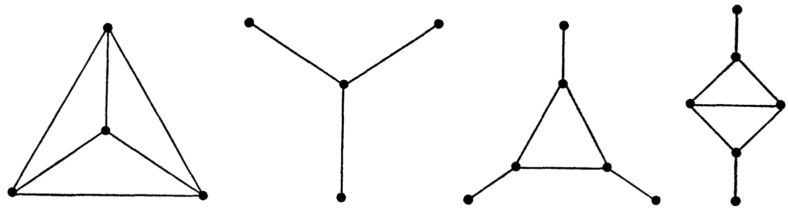

FIGURE 1. The simplest examples of the four types of graphs not suppressible to K_2.

graph K_4, the star $K_{1,3}$, a triangle with one endline at each point, and the graph obtained from K_4 by cutting one of its lines in half thus obtaining two endlines.

In general, we follow the notation in [3] with the following additions: If a point v in G has degree 2, we say that v is *suppressible*. We say the graph itself is *suppressible* if it has at least one such point. Let v_1 and v_2 be the two points adjacent to v. We write $G \sim v$ (read this "G suppress v") for the graph obtained from G by suppressing v. It is defined by $G \sim v = G - v \cup v_1v_2$. Observe that a new line v_1v_2 is added only if v_1 and v_2 are not already adjacent. We call $G \sim v$ an *elementary suppression* of G. Any sequence of elementary suppressions is a *suppression* of G. A suppression of G to a nonsuppressible graph is a *total suppression* of G. This notation is demonstrated in Fig. 2.

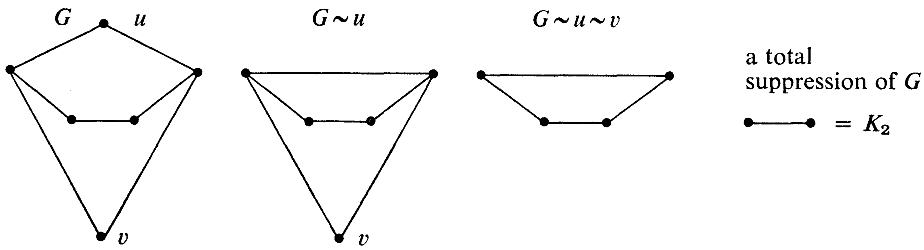

FIGURE 2. A graph suppressible to an edge.

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The following result is similar to one for suppressed digraphs in [4, p. 335].

LEMMA. *Any two total suppressions of G are isomorphic.* (Consequently, we may refer to *the* total suppression of G, which we denote by G^0.)

Proof. This result is instant for $p \leq 2$. Let $p \geq 3$ and assume uniqueness has been proven for all graphs on fewer than p points. We proceed to prove that G has a unique total suppression.

If G has no point of degree 2, then G is not suppressible, so $G^0 = G$. If G has just one point v of degree 2, then the only possible elementary suppression of G is $G \sim v$. But since $G \sim v$ has $p-1$ points, $G \sim v$ has a unique total suppression, $(G \sim v)^0$. Consequently, G has a unique total suppression, and $G^0 = (G \sim v)^0$. Assume G has at least two points u and v of degree 2. We consider two possible elementary suppressions, $G \sim u$ and $G \sim v$.

Case 1. If u is adjacent to v, we observe that $G \sim u \cong G \sim v$. Consequently, $(G \sim u)^0 \cong (G \sim v)^0$.

Case 2. If u is not adjacent to v, we observe that v has degree 2 in $G \sim u$, and u has degree 2 in $G \sim v$. We can now form $G \sim u \sim v$ and $G \sim v \sim u$. These two graphs are obviously isomorphic. Thus

$$(G \sim u)^0 \cong (G \sim u \sim v)^0 \cong (G \sim v \sim u)^0 \cong (G \sim v)^0.$$

In both cases, the total suppression of G is the same regardless of which initial elementary suppression is chosen. This completes the induction.

We can now demonstrate the main result of this note.

THEOREM. *Let G be a connected graph with $p \geq 2$. Then $G^0 = K_2$ if and only if G contains none of the following*:

(1) *a subgraph homeomorphic to K_4,*
(2) *a cutpoint lying in three blocks,*
(3) *a block containing three cutpoints,*
(4) *a subgraph homeomorphic to $K_4 - x$ in which the two nonadjacent points are cutpoints of G.*

[By *property* (n) for $n = 1, 2, 3, 4$, we mean that G does not contain (n) above.]

Proof of necessity. Assume $G^0 = K_2$, and suppose G does contain one of the four types of subgraphs. Then it is a simple observation that each suppression of G contains that same type of subgraph. Consequently, $G^0 = K_2$ contains that subgraph, which is absurd, as K_2 has only two points.

Proof of sufficiency. We demonstrate the contrapositive. Assume $G^0 \neq K_2$, and suppose G, and hence G^0, does not contain (1), (2), or (3) as above. We now show that G^0, and consequently G, contains a subgraph (4).

If a graph G has blocks $\{B_i\}$ and cutpoints $\{c_j\}$, the *block-cutpoint graph* $\text{bc}(G)$ is defined [3, p. 36] as that graph having point set $\{B_i\} \cup \{c_j\}$ with two points adjacent if one is a block B_i, the other is a cutpoint c_j, and c_j is in B_i.

Properties (2) and (3) imply that each point of $\text{bc}(G^0)$ has degree at most 2. Since every block-cutpoint graph is known to be a tree [3, p. 37], $\text{bc}(G^0)$ is a path of the form $B_1c_1B_2c_2 \ldots c_{n-1}B_n$, as illustrated for $n=4$ in Fig. 3.

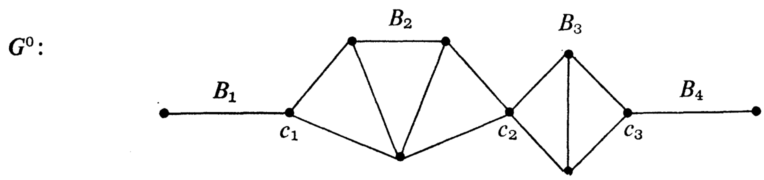

FIGURE 3. A totally suppressed graph.

By a result of Dirac [1, Theorem 5], every connected graph in which at most one point has degree ≤ 2 contains a homeomorph of K_4.

But by property (1), G^0 has no such subgraph, so G^0 must have at least two points of degree at most 2. But being totally suppressed, G^0 has no points of degree 2. Hence, these points must have degree 1! Knowing that $\text{bc}(G^0)$ is a path, we conclude that there are just two such endpoints, and these lie in B_1 and B_n respectively. Since H has only one point of degree at most 2, Dirac's theorem guarantees that H contains a homeomorph of K_4. By property (1), we conclude that the corresponding subgraph of G^0 is of the type shown in Fig. 4.

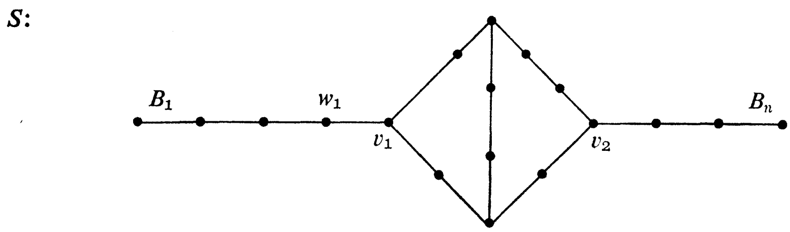

FIGURE 4. A subgraph of G^0.

There are two possibilities concerning the points v_1 and v_2 in the figure. If they are both cutpoints, there is nothing more to prove since we already have a subgraph of type (4) as desired. If v_1 is not a cutpoint, then there is a path from w_1 to w_2 avoiding v_1. This path must contain some first point in the portion of S between v_1 and v_2 because w_2 itself is such a point. Any such path is seen to produce either a homeomorph of K_4 (violating property (1)) or a new subgraph of this type in which w_1 and v_2 are the critical vertices. By an induction argument, we must obtain a subgraph of type (4), completing the proof.

References

1. G. A. Dirac, *In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterteilungen*, Math. Nachr. **22** (1960), 61–85.

2. F. Harary and J. Krarup, *On outerplanar graphs and a class of discrete optimization programs* (unpublished manuscript).

3. F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass., 1969.

4. F. Harary, R. Norman, and D. Cartwright, *Structural models: an introduction to the theory of directed graphs*, Wiley, New York, 1965.

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